ESTIMATING THE CONVERGENCE RATE OF FUNCTIONAL ITERATIONS FOR SOLVING QUADRATIC MATRIX EQUATIONS ARISING IN HYPERBOLIC QUADRATIC EIGENVALUE PROBLEMS

JONG-HYEON SEO AND HYUN-MIN KIM

Abstract. We consider Bernoulli's method for solving quadratic matrix equations (QMEs) having form $Q(X) = AX^2 + BX + C = 0$ arising in hyperbolic quadratic eigenvalue problems (QEPs) and quasi-birth-death problems (QBDs) where $A, B, C \in \mathbb{R}^{m\times m}$ satisfy Esenfeld's condition [8]. First, we analyze the exsistence of a solution and the convergence of the methods. Second, we sharpen bounds of the rates of convergence. Finally, in numerical experimentations, we show that the modified bounds give appropriate estimations of the numbers of iterations.

1. Introduction

In this paper, we consider the *quadratic matrix equation* (QME)

$$Q(X) := AX^2 + BX + C = 0, \tag{1.1}$$

where $A, B, C \in \mathbb{R}^{m\times m}$ satisfy Esenfeld's condition

$$4\|B^{-1}A\|\|B^{-1}C\| < 1 \tag{1.2}$$

[8,39]. If $S \in \mathbb{R}^{m\times m}$ satisfies $Q(S) = 0$ in (1.1) then we call that S is a solvent [6]. Note that in [8] Esenfeld assumed that all coefficient matrices are nonsingular, but our condition needs that only B is nonsingular.

From Bernoulli's iteration [7,13,21], we construct the functional iterative methods

$$\begin{cases} X_0 = 0 \\ X_{k+1} = \mathcal{F}_1(X_k) = -B^{-1}\left(AX_k^2 + C\right), \end{cases} \quad k = 0,1,2,\ldots \tag{1.3}$$

and

$$\begin{cases} X_0 = 0 \\ X_{k+1} = \mathcal{F}_2(X_k) = -\left(B + AX_k\right)^{-1} C, \end{cases} \quad k = 0,1,2,\ldots. \tag{1.4}$$

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Higham and Kim [21] showed that the iteration (1.4) converges to minmal solvents of the QMEs (1.1) if they have such solvents. Bai and Gao [3] modified the iteration by techniques of the Gauss-Seidal iteration. In stochastic areas, it is well-known that the iterations (1.3) and (1.4) converge to elementwise minimal nonnegative solvents of QMEs (1.1) arising in QBDs (for more details, see Favati and Meini [11].). Definitions of minimal and elementwise minimal nonnegative solvents are found in [21] and [36] respectively.

Those implies that the convergence of the iterations (1.3) and (1.4) are depend on the existence of such extreme solvents. There are well-known two conditions guaranteeing the existence of such extreme sovlents. One is from quadratic eigenvalue problems (QEPs) and the other is from quasi-birth-death processes (QBDs).

Let coefficient matrices A, B and C are Hermitian positive definite. Then QEPs satisfying

$$\min_{\|x\|=1} \left((x^*Bx)^2 - 4\,(x^*Ax)\,(x^*Cx) \right) > 0, \tag{1.5}$$

are called *hyperbolic* [26,37], and (1.5) guarantees the exisitence of minimal solvents of QMEs (1.1). Those problems have been widely studied together with *elliptic* problems [18, 23, 24, 34, 37]. Besides, let A, C, $I + B := N$ be real and have no negative entries. Then QMEs satisfying

$$-B\mathbf{1}_m = (A + C)\,\mathbf{1}_m \quad \text{and} \quad A + N + C \text{ is irreducible} \tag{1.6}$$

have elementwise minimal nonnegative solvents and transition matrices from the associated QBDs have the unique stationary vectors where $\mathbf{1}_m$ is the m-column vector of which entries are 1. Those solvents of QMEs (1.1) play key roles in stochastic areas [5, 9, 17, 27].

On the other hand, the existence of solvents of QMEs (1.1) in view point of functional analysis on Banach spaces were also discussed by some authors [2,8,21,25,30]. From the results, we found that Esenfeld's condition (1.2) is much involved in not only the existence of solvents but also the semi-local convergence of functional iterative methods for special strarting matrices (note the relation to the condition that the discriminant of a scalar quadratic be positive [21]). Moreover, many examples and applications of QMEs arising in hyperbolic QEPs including over-damped mass-spring systems referred in [3, 12, 16, 18, 21, 22, 24, 32, 34, 37–39] and stochastic problems including QBDs referred in [1, 4, 5, 9–11, 14, 15, 17, 19–21, 27–29, 31, 35] satisfy (1.2) as well as (1.5) and (1.6) respectively.

2. Related Definitions and Theorems

For $S \in \mathbb{R}^{m\times m}$ and $\delta > 0$, $\mathcal{B}(S, \delta)$ denotes an open ball centered by S with a radius δ such as

$$\mathcal{B}(S, \delta) := \left\{ X \in \mathbb{R}^{m\times m} \,:\, \|X - S\| < \delta \right\},$$

and the spectral radius of $A \in \mathbb{R}^{m\times m}$ is defined by

$$\rho(A) := \max\{|\lambda| : \det(A - \lambda I) = 0\}.$$

Since $\mathcal{F}_1$ in (1.3) is well-defined on $\mathbb{R}^{m\times m}$, from the properties of matrix norms we have

$$\|\mathcal{F}_1(Y) - \mathcal{F}_1(X)\| \leq \left(\|B^{-1}AY\| + \|B^{-1}A\|\|X\|\right)\|Y - X\|. \tag{2.1}$$

Define

$$\Gamma_1(X, Y) := \|B^{-1}AY\| + \|B^{-1}A\|\|X\|. \tag{2.2}$$

Since $\mathcal{F}_2$ in (1.4) has the matrix inverse operations, we need the following lemmas.

Lemma 2.1 (Neumann Lemma, see [33, 2.3.1]). *For $A, B \in \mathbb{C}^{m\times m}$, if A be nonsingular and $\rho(A^{-1}B) < 1$, then $A - B$ is also nonsingular and represented by*

$$(A - B)^{-1} = A^{-1} + A^{-1}B(A - B)^{-1}. \tag{2.3}$$

$\|A^{-1}B\| < 1$ instead of $\rho\left(A^{-1}B\right) < 1$ also leads to the same conclusion in Lemma 2.1.

Lemma 2.2. (See [33, 2.3.1].) *For $L, M, N \in \mathbb{C}^{m\times m}$, if L is nonsingular and $\|L^{-1}M\| < 1$ then $L - M$ is also nonsingular and*

$$\|(L - M)^{-1}N\| \leq \frac{\|L^{-1}N\|}{1 - \|L^{-1}M\|}$$

For $X \in \mathbb{R}^{m\times m}$ if $\mathcal{F}_2(X)$ in (1.4) is well-defined, then from the Neumann lemma for a sufficiently small perturbation matrix $H \in \mathbb{R}^{m\times m}$ we have

$$\begin{aligned}
\mathcal{F}_2(X + H) &= ((B - AX) - AH)^{-1}C \\
&= (B - AX)^{-1}C + (B - AX)^{-1}AH((B - AX) - AH)^{-1}C \\
&= \mathcal{F}_2(X) + (B - AX)^{-1}AH(B - AX - AH)^{-1}C.
\end{aligned}$$

From the above expression $H := Y - X$ yields

$$\mathcal{F}_2(Y) - \mathcal{F}_2(X) = (B - AX)^{-1}A(Y - X)(B - Y)^{-1}C$$

and

$$\begin{aligned}
\|\mathcal{F}_2(Y) - \mathcal{F}_2(X)\| &= \|(B - AX)^{-1}A(Y - X)(B - AY)^{-1}C\| \\
&\leq \|(B - AX)^{-1}A\|\|(B - AY)^{-1}C\|\|Y - X\|.
\end{aligned}$$

Define

$$\Gamma_2(X, Y) := \|(B - AX)^{-1}A\|\|(B - AY)^{-1}C\|. \tag{2.4}$$

We show that the iterations (1.3) and (1.4) always converge to a solvent of which the existence is guaranteed. The followings are used in the analysis.

One-step stationary iterations have the form

$$X_{k+1} = F(X_k), \quad k = 0, 1, 2, \ldots, \tag{2.5}$$

where $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$. This include Newton's method and some of minimization methods.

Theorem 2.3 (Contraction Mapping Theorem, see [33, Thm. 12. 1. 2])**.** *Let $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$, and suppose that F maps a closed set $\mathbf{D}_0 \subset \mathbf{D}$ into itself and that*

$$\|F(X) - F(Y)\| \le \alpha \|X - Y\|, \quad \forall X, Y \in \mathbf{D}_0$$

for some $\alpha < 1$. Then, for any $X_0 \in \mathbf{D}_0$, the sequence $\{X_k\}$ generated by (2.5) converges to the unique fixed point S of F in $\mathbf{D}_0$ and

$$\|X_k - S\| \le \frac{\alpha}{1-\alpha}\|X_k - X_{k-1}\|, \quad k = 1, 2, \ldots.$$

Definition 2.4. (See [33, Def. 12.3.1].) The matrix function $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ is an *iterated contraction* on the set $\mathbf{D}_0 \subset \mathbf{D}$ if there is an $\alpha < 1$ such that

$$\|F(F(X)) - F(X)\| \le \alpha \|F(X) - X\|$$

whenever X and $F(X)$ are in $\mathbf{D}_0$.

Theorem 2.5. (See [33, Thm. 12.3.2].) *Suppose that $F : \mathbf{D} \subset \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ is an iterated contraction on the closed set $\mathbf{D}_0 \subset \mathbf{D}$ and that for some $X_0 \in \mathbf{D}_0$ the sequence*

$$X_{k+1} = F(X_k), \quad k = 0, 1, 2, \ldots,$$

remains in $\mathbf{D}_0$. Then, $\lim_{k\to\infty} X_k := S \in \mathbf{D}_0$, and the estimation

$$\|X_k - S\| \le \frac{\alpha}{1-\alpha}\|X_k - X_{k-1}\|, \quad k = 1, 2, \ldots \tag{2.6}$$

holds. Moreover, if F is continuous at S, then $S = F(S)$.

3. Analysis of the Exsistence and the Convergence

3.1. **Local Convergence of $X_{k+1} = \mathcal{F}_1(X_k)$ for $X_0 = 0$.** We show that $\mathcal{F}_1$ in (1.3) is invariant and iterated contractive on some closed balls.

Lemma 3.1. *Suppose that B is nonsingular,*

$$\|B^{-1}A\| \le a \quad \text{and} \quad \|B^{-1}C\| \le b$$

where a and b are positive constants. If

$$\gamma := 4ab < 1$$

then $\mathcal{F}_1$ in (1.3) is invariant and iterated contractive on $\bar{\mathcal{B}} := \bar{\mathcal{B}}(0, 1/2a)$.

Proof. Let $X \in \bar{\mathcal{B}}$. Then from

$$b = \frac{\gamma}{4a} < \frac{1}{4a} \text{ and } \|X\| \le \frac{1}{2a},$$

we have

$$\begin{aligned} \|\mathcal{F}_1(X)\| &\leq \|B^{-1}AX^2\| + b \leq a\left(\frac{1}{2a}\right)^2 + \frac{\gamma}{4a} \\ &\leq \frac{1+\gamma}{4a} < \frac{1}{2a} \end{aligned} \tag{3.1}$$

Therefore $\mathcal{F}_1$ is invariant on $\bar{\mathcal{B}}$.

From (3.1) we get

$$\begin{aligned} \Gamma_1\left(\mathcal{F}_1\left(\mathcal{F}_1(X)\right), \mathcal{F}_1(X)\right) &= \|B^{-1}A\mathcal{F}_1(X)\| + a\|X\| \\ &\leq a \cdot \frac{1+\gamma}{4a} + a \cdot \frac{1}{2a} = \frac{3+\gamma}{4} < 1 \end{aligned} \tag{3.2}$$

where Γ_1 is in (2.2). From (3.2) we obtain

$$\|\mathcal{F}_1\left(\mathcal{F}_1(X)\right) - \mathcal{F}_1(X)\| \leq \frac{3+\gamma}{4}\|\mathcal{F}_1(X) - X\|. \tag{3.3}$$

□

Since $\mathcal{F}_1$ is Fréchet differentiable and iterated contractive on $\bar{\mathcal{B}}(0, 1/2a)$, from Theorem 2.5 the next theorem follows.

Theorem 3.2. *Under the assumption of Lemma 3.1, if*

$$\gamma := 4ab < 1$$

then the sequence $\{X_k\}$ *generated by* (1.3) *has a limit* S *in* $\bar{\mathcal{B}}$. *Moreover* S *is a solvent of QME* (1.1) *and the estimation*

$$\|X_k - S\| \leq \frac{3+\gamma}{1-\gamma}\|X_k - X_{k-1}\|, \quad k = 1, 2, \ldots$$

holds.

Proof. From (3.3) in Lemma 3.3 the estimation is given by

$$\frac{\alpha}{1-\alpha} = \frac{\frac{3+\gamma}{4}}{1 - \frac{3+\gamma}{4}} = \frac{3+\gamma}{1-\gamma}.$$

□

3.2. **Local Convergence of** $X_{k+1} = \mathcal{F}_2(X_k)$ **for** $X_0 = 0$**.** We show that $\mathcal{F}_2$ in (1.4) is invariant and contractive on some closed balls.

Lemma 3.3. *Under the assumption of Lemma 3.1, If*

$$\gamma := 4ab < 1$$

then $\mathcal{F}_2$ *in* (1.4) *is invariant and contractive on* $\bar{\mathcal{B}} := \bar{\mathcal{B}}(0, \delta)$ *where*

$$\delta := \frac{\sqrt{\gamma}}{2a}. \tag{3.4}$$

Proof. Since

$$\sqrt{\gamma} = 2\sqrt{ab} < 1, \tag{3.5}$$

we have

$$1 > 1 - \sqrt{ab} > \sqrt{ab} > 0$$

Let $X \in \bar{\mathcal{B}}$. Then from

$$\|X\| \leq \frac{\sqrt{\gamma}}{2a} = \frac{\sqrt{ab}}{a}$$

we get

$$\|B^{-1}AX\| \leq a\|X\| < 1.$$

Therefore from Lemma 2.2, $B - AX$ is nonsingular and

$$\begin{aligned} \|\mathcal{F}_2(X)\| &\leq \left\|(B - AX)^{-1} C\right\| \\ &\leq \frac{b}{1 - a\|X\|} < \frac{b}{1 - \sqrt{ab}} \\ &< \frac{b}{\sqrt{ab}} = \frac{\sqrt{ab}}{a} = \frac{\sqrt{\gamma}}{2a}. \end{aligned} \tag{3.6}$$

From (3.6), $\mathcal{F}_2$ is invariant on $\bar{\mathcal{B}}$.

Let $Y, Z \in \bar{\mathcal{B}}$. Then from Lemma 2.2 and (3.6) we get

$$\begin{aligned} \Gamma_2(Y, Z) &= \|(B - AY)^{-1}A\|\|(B - AZ)^{-1}C\| \\ &\leq \frac{a}{1 - a\|Y\|} \cdot \frac{b}{1 - a\|Z\|} \\ &\leq \frac{a}{1 - \sqrt{ab}} \cdot \frac{b}{1 - \sqrt{ab}} \\ &< \frac{a}{\sqrt{ab}} \cdot \frac{b}{\sqrt{ab}} = 1 \end{aligned} \tag{3.7}$$

where Γ_2 is in (2.4). From (3.7) we obtain

$$\|\mathcal{F}_2(Y) - \mathcal{F}_2(Z)\| \leq \frac{\gamma}{\left(2 - \sqrt{\gamma}\right)^2}\|Y - Z\|. \tag{3.8}$$

□

Since $\mathcal{F}_2$ is Fréchet differentiable and iterated contractive on $\bar{\mathcal{B}}(0, \delta)$, from the contraction mapping theorem we have the next theorem.

Theorem 3.4. *Under the assumption of Lemma 3.1, if*

$$\gamma := 4ab < 1$$

then the sequence $\{X_k\}$ *generated by* (1.4) *has a limit* S *in* $\bar{\mathcal{B}}$. *Moreover* S *is the unique solvent of QME* (1.1) *in* $\bar{\mathcal{B}}$ *and the estimation is given by*

$$\|X_k - S\| \leq \frac{\gamma}{4\left(1 - \sqrt{\gamma}\right)}\|X_k - X_{k-1}\|, \quad k = 1, 2, \ldots$$

where δ *and* γ *are in* (3.4) *and* (3.5) *in Lemma 3.3 respectively.*

Proof. From (3.3) in Lemma 3.3 the estimation is given by

$$\frac{\alpha}{1-\alpha} = \frac{\frac{\gamma}{(2-\sqrt{\gamma})^2}}{1-\frac{\gamma}{(2-\sqrt{\gamma})^2}} = \frac{\gamma}{(2-\sqrt{\gamma})^2-\gamma} = \frac{\gamma}{4-4\sqrt{\gamma}}.$$

□

4. Sharpening Bounds of the Rates of Convergence

4.1. The Rate of Convergence of $X_{k+1} = \mathcal{F}_1(X_k)$ for $X_0 = 0$.

Theorem 4.1. *Let $A, B, C \in \mathbb{R}^{m\times m}$ and supoose that B is nonsingular and*

$$0 < \gamma := 4\|B^{-1}A\|\|B^{-1}C\| < 1.$$

Then, for the sequence $\{X_k\}$ generated by (1.3), we have

$$\frac{\|X_{k+1}-X_k\|}{\|X_k - X_{k-1}\|} \le 1-\sqrt{1-\gamma}, \quad k = 1, 2, \dots .$$

Proof. Since

$$\|X_1\| = \|B^{-1}C\| \le \frac{\gamma}{4\|B^{-1}A\|},$$

we get

$$\|X_{k+1}\| \le \|B^{-1}A\|\|X_k\|^2 + \frac{\gamma}{4\|B^{-1}A\|}, \quad \forall k \in \mathbb{N}. \tag{4.1}$$

Since the sequence of the right side of (4.1) is increasing and bounded above by $1/2a$, we have

$$\|X_k\| \le \frac{1-\sqrt{1-\gamma}}{2\|B^{-1}A\|}, \quad \forall k \in \mathbb{N}. \tag{4.2}$$

From (4.2), we obtain for all $k \in \mathbb{N}$

$$\begin{aligned} \|X_{k+1}-X_k\| &\le \Gamma_1(X_k, X_{k-1})\|X_k - X_{k-1}\| \\ &\le \|B^{-1}A\| \left(\|X_k\| + \|X_{k-1}\|\right) \|X_k - X_{k-1}\| \\ &\le \|B^{-1}A\| \left(\frac{1-\sqrt{1-\gamma}}{\|B^{-1}A\|}\right) \|X_k - X_{k-1}\| \\ &\le \left(1-\sqrt{1-\gamma}\right) \|X_k - X_{k-1}\|. \end{aligned}$$

□

From Theorem 2.5 we have the next corollary.

Corollary 4.2. *Let $A, B, C \in \mathbb{R}^{m\times m}$ and supoose that B is nonsingular and*

$$0 < \gamma := 4\|B^{-1}A\|\|B^{-1}C\| < 1.$$

Then,, the sequence $\{X_k\}$ *generated by* (1.3) *has a limit* S *in* $\bar{\mathcal{B}}$*. Moreover* S *is a solvent of QME* (1.1) *and estimation*

$$\|X_k - S\| \leq \frac{1-\sqrt{1-\gamma}}{\sqrt{1-\gamma}}\|X_k - X_{k-1}\|, \quad k = 0, 1, 2, \ldots$$

holds.

4.2. **The Rate of Convergence of** $X_{k+1} = \mathcal{F}_2(X_k)$ **for** $X_0 = 0$**.**

Theorem 4.3. *Let* $A, B, C \in \mathbb{R}^{m\times m}$ *and supoose that* B *is nonsingular and*

$$0 < \gamma := 4\|B^{-1}A\|\|B^{-1}C\| < 1.$$

Then, for the sequence $\{X_k\}$ *generated by* (1.4)*, we have*

$$\frac{\|X_{k+1} - X_k\|}{\|X_k - X_{k-1}\|} \leq \frac{1-\sqrt{1-\gamma}}{1+\sqrt{1-\gamma}}, \quad k = 1, 2, \ldots.$$

Proof. Since

$$\|X_1\| = \|B^{-1}C\| \leq \frac{\gamma}{4\|B^{-1}A\|},$$

we obtain

$$\|X_{k+1}\| = \left\|(B - AX_k)^{-1}C\right\| \leq \frac{\gamma}{4\left(1 - \|B^{-1}A\|\|X_k\|\right)}, \quad \forall k \in \mathbb{N}. \tag{4.3}$$

By the mathematical induction, we easily prove that the sequence of the right side of the inequality (4.3) is increasing and bounded above by $\sqrt{\gamma}/2\|B^{-1}A\|$. So we have

$$\|X_k\| \leq \frac{1-\sqrt{1-\gamma}}{2\|B^{-1}A\|}, \quad k = 0, 1, 2, \ldots.$$

From Lemma 2.2 and

$$\Gamma_2\left(\mathcal{F}_2(X), X\right) = \left\|(B - A\mathcal{F}_2(X))^{-1}A\right\|\|\mathcal{F}_2(X)\| \leq \frac{\|B^{-1}A\|\|\mathcal{F}_2(X)\|}{1 - \|B^{-1}A\|\|\mathcal{F}_2(X)\|},$$

we get

$$\begin{aligned}
\|X_{k+1} - X_k\| &= \|\mathcal{F}_2(X_k) - \mathcal{F}_2(X_{k-1})\| \\
&\leq \Gamma_2\left(\mathcal{F}_2(X_k), X_k\right)\|X_k - X_{k-1}\| \\
&\leq \frac{\|B^{-1}A\|\|X_k\|}{1 - \|B^{-1}A\|\|X_k\|}\|X_k - X_{k-1}\| \\
&\leq \frac{\frac{1-\sqrt{1-\gamma}}{2}}{1 - \frac{1-\sqrt{1-\gamma}}{2}} = \frac{1-\sqrt{1-\gamma}}{1+\sqrt{1-\gamma}}
\end{aligned}$$

where Γ_2 is in (2.4). □

From Theorem 2.3 we have the next corollary.

Corollary 4.4. *Let* $A, B, C \in \mathbb{R}^{m\times m}$ *and supoose that* B *is nonsingular and*

$$0 < \gamma := 4\|B^{-1}A\|\|B^{-1}C\| < 1.$$

Then, the sequence $\{X_k\}$ *generated by* (1.3) *has the unique solvent solvent* S *in* $\bar{\mathcal{B}}$ *and estimation*

$$\|X_k - S\| \leq \frac{1-\sqrt{1-\gamma}}{2\sqrt{1-\gamma}}\|X_k - X_{k-1}\|, \quad k = 0, 1, 2, \ldots$$

holds and

$$\|S\| \leq \frac{1-\sqrt{1-\gamma}}{2\|B^{-1}A\|}.$$

In Corollary 4.4, since S is the unique solvent on neighborhood of the zero matrix, S must be the minimal solvent in view point of matrix norms and the spectral radius (i.e. $\rho(S) \leq \rho(S')$ and $\|S\| \leq \|S'\|$ if S' is arbitrary solvent of QME (1.1)). Therefore we have the following theorem.

Theorem 4.5. *Let* $A, B, C \in \mathbb{R}^{m\times m}$ *and supoose that* B *is nonsingular and*

$$0 < \gamma := 4\|B^{-1}A\|\|B^{-1}C\| < 1.$$

Then, QME (1.1) *has the minimal solvents* S *on* $\mathcal{B}\left(0, \frac{\sqrt{\gamma}}{2a}\right)$ *in view point of spectral radius. Furthermore, if* S *is nonsingular and*

$$1 < \sqrt{1-\gamma} + 2\|B^{-1}A\|, \tag{4.4}$$

then the inverse of S *is a solvent of* $Q_2(X) = CX^2 + BX + A$ *satisfying* $|\lambda_{\min}(S)| > 1$.

In Theroem 4.4, since $1 < \sqrt{1-\gamma} + 2\|B^{-1}C\|$ implies that there exists the minimal solvent of $Q_2(X) = 0$ in view point of the spectral radius such that $\rho(S) \leq \|S\| < 1$, if S is nonsingular then S^{-1} is also a solvent of $Q(X) = AX^2 + BX + C = 0$ such that $|\lambda_{\min}(S^{-1})| > 1$. That is S^{-1} could be dominant solvent of $Q(X) = 0$ (see [21, 37] for more details of minimal and dominant solvents.).

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Department of Mathematics, Pusan National University, Busan , Republic of Korea

Email address: hyunmin@pusan.ac.kr