VARIATIONAL RELATION PROBLEMS IN ABSTRACT CONVEX SPACES

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Abstract. In 2008, Luc initiated the study of variational relations, which is a unifying approach to various models of equilibrium theory and variational inclusions. In 2018, we generalized some of Luc's results by reflecting recent development of the KKM theory on abstract convex spaces. Moreover in 2019, we obtain some abstract space versions of known results on generalized KKM maps and variational relation problems appeared in the papers of Park and Lee; Balaj and Luc; Luc, Sarabi and Soubeyran; Lin; and Balaj, in the chronological order. In this talk, we introduce some contents of our papers in 2018 and 2019.

1. Introduction

In 2008, Luc [5] initiated the study of variational relations, which is a unifying approach to various models of equilibrium theory and variational inclusions. There, a simple condition was established for the existence of solutions of variational relations and was applied to a number of variational problems. In 2018 [13], some of Luc's results are generalized by reflecting recent development of the KKM theory on abstract convex spaces.

Moreover in 2019 [14], we obtain some abstract space versions of known results on generalized KKM maps and variational relation problems appeared in the papers of Park and Lee [15], Balaj and Luc [2], Luc, Sarabi and Soubeyran [6], Lin [4], and Balaj [1], in the chronological order.

In this talk, we introduce some contents of [13] and [14].

The definitions, some basic facts, and some of typical examples of abstract convex spaces are shown in our previous talks at RIMS or [13] and [14].

In Sections 2-4, we follow the pioneering work of Luc [5] and show that some of his results can be extended to our abstract convex spaces. In fact, Section 2 deals with Luc's condition linking the existence of solutions to the variational relation problem (VR) and the intersection property of a certain multimap. Sections 3 and 4 are concerned with sufficient conditions for existence of solutions of a broad class of models, respectively, in which conditions based on intersection theorems and fixed point theorems are derived.

Section 5 deals with Luc, Sarabi and Soubeyran [6] on the existence of solutions in variational relation problems without convexity. Finally, Section 6 concerns with a generalization of a basic result of Balaj [1] on three types of variational relation problems.

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Let $\langle D \rangle$ denote the class of nonempty finite subsets of a nonempty set D.

2. Variational relation problem

According to Luc [5], we assume: $A,\ B$, and Y are nonempty sets, $S_1 : A \multimap A$, $S_2 : A \multimap B$ and $T : A \times B \multimap Y$ are multimaps with nonempty values. Let $R(a, b, y)$ be a relation linking $a \in A,\ b \in B$ and $y \in Y$. We consider the following problem, denoted **(VR)**:

Find $\bar{a} \in A$ such that:

(1) $\bar{a}$ is a fixed point of S_1, that is $\bar{a} \in S_1(\bar{a})$;
(2) $R(\bar{a}, b, y)$ holds for every $b \in S_2(\bar{a})$ and $y \in T(\bar{a}, b)$.

This problem is called a *variational relation problem* in which the multimaps $S_1,\ S_2,\ T$ are constraints and R is a variational relation. The relation R is often determined by equalities and inequalities of real functions or by inclusions and intersections of set-valued maps. Typical instances of variational relation problems are the following as shown by Luc [5]:

(i) Optimization Problem
(ii) Equilibrium Problem
(iii) Variational Inclusion Problem
(iv) Differential Inclusion

To study the variational relation problem (VR), Luc [5] defined a multimap $P : B \multimap A$ by

$$P(b) = [A \setminus S_2^-(b)] \cup \{a \in A : a \in S_1(a),\ R(a, b, y) \text{ holds } \forall y \in T(a, b)\}.$$

The following main theorem of [5] expresses the existence of solutions of (VR) by an intersection relation:

Theorem 2.1. ([5]) *A point $\bar{a} \in A$ is a solution of the variational relation problem* (VR) *if and only if it belongs to the set $\bigcap_{b \in B} P(b)$.*

The following corollary is useful in establishing sufficient conditions for the existence of solutions via fixed point theorems.

Corollary 2.2. ([5]) *A point $\bar{a} \in A$ is a solution of* (VR) *if and only if the set $B \setminus P^-(\bar{a})$ is empty. In particular, if $A = B$, then* (VR) *has a solution under the following conditions:*

(i) *The map $a \mapsto A \setminus P^-(a),\ a \in A$, has a fixed point whenever it has nonempty values.*
(ii) *For each $a \in A,\ S_2(a) \subset S_1(a)$.*
(iii) *For each fixed point a of S_1, the relation $R(a, a, y)$ holds for all $y \in T(a, a)$.*

3. Criteria Based on Intersections

In this section, we derive two sufficient conditions for the existence of solutions of (VR) as in Section 3 of Luc [5]:

Definition 3.1. ([5]) We say that the problem (VR) is *finitely solvable* if, for every finite subset $N \in \langle B \rangle$, there is some $a_0 \in A$ such that, for each $b \in N$, either $b \notin S_2(a_0)$ or $a_0 \in S_1(a_0)$ and $R(a_0, b, y)$ holds for all $y \in T(a_0, b)$.

Proposition 3.2. ([5]) *Assume that A is a compact set. Then, the variational relation problem* (VR) *has a solution if and only if it is finitely solvable.*

From now on, we assume that $A = B$, a nonempty subset of a partial KKM space $(X;\Gamma)$, and that $(Y;\Lambda)$ is another partial KKM space.

Definition 3.3. ([5]) We say that the relation R is T-KKM (or KKM for short) if, for every finite subset $N = \{a_1, \ldots, a_k\}$ of A and for every $a \in \Gamma_N$, one can find some index i such that $R(a, a_i, y)$ holds for all $y \in T(a, a_i)$.

This definition is an adaptation of the KKM maps to variational relations. We recall that a multimap $G : A \multimap A$ is said to be KKM if for every finite subset $N = \{a_1, \ldots, a_k\}$ of A, we have $\Gamma_N \subset G(N) = \bigcup_{i=1}^{k} G(a_i)$.

The following intersection theorem of the KKM-Fan type for our abstract convex space theory will be needed: If A is a nonempty compact and Γ-convex subset of a partial KKM space and if $G : A \multimap A$ is a KKM map with nonempty closed values, then $\bigcap_{a\in A} G(a) \neq \emptyset$.

Theorem 3.4. *The following conditions are sufficient for* (VR) *to have a solution:*
(i) $(A;\Gamma)$ *is a compact partial KKM space.*
(ii) *The map P has closed values.*
(iii) *For every $a \in A$, the Γ-convex hull of $S_2(a)$ is contained in $S_1(a)$.*
(iv) *The relation R is KKM.*

Proof. Consider the map P on A. We start with proving that for each $a \in A$, the set $P(a)$ is nonempty. Indeed, if not, say $P(a_0)$ is empty for some $a_0 \in A$. By the definition of P, every $S_2(a)$ contains a_0. In particular, $a_0 \in S_2(a_0) \subset S_1(a_0)$. Since R is KKM, we deduce $a_0 \in P(a_0)$, a contradiction. We show next that P is KKM. To this purpose, let $N = \{a_1, \ldots, a_k\} \in \langle A\rangle$ and let $a \in \Gamma_N$. If a belongs to the set $A \setminus S_2^-(a_i)$ for some i, then we are done because a belongs to $P(a_i)$ as well. If not, in view of (iii), a belongs to the $\mathrm{co}_\Gamma S_2(a)$, and hence $a \in S_1(a)$. As R is KKM, there is some index i such that $R(a, a_i, y)$ holds for all $y \in T(a, a_i)$. This implies $a \in P(a_i)$, and P is KKM. It remains to apply our KKM-Fan theorem [7] and Theorem 2.1 to conclude. □

When A is a nonempty convex compact subset of a (not necessarily Hausdorff) topological vector spaces, Theorem 3.4 reduces to Luc [5, Theorem 3.1].

In order to develop sufficient conditions for (ii) and (iv), it is recalled some definitions of continuity of multimaps. Let G be a multimap between two topological spaces X and Z. It is *closed* (resp. *open*) if its graph is a closed (resp. open) set in $X \times Z$; it is *upper semicontinuous* if for $x \in X$ and an open set $V \subset Z$ containing $G(x)$, there is some open neighborhood $U \subset X$ of x such that $G(U) \subset V$; and it is *lower semicontinuous* if for $x \in X$ and an open set $V \subset Z$ with $V \cap G(x) \neq \emptyset$, there is some open neighborhood $U \subset X$ of x such that $G(x') \cap V \neq \emptyset$ when $x' \in U$.

Definition 3.5. Let $b \in A$ be given. We say that the relation $R(., b, .)$ is *closed* in the first and the third variables if, for every net $\{(a_\alpha, y_\alpha)\}$ converging to some (a, y), and if $R(a_\alpha, b, y_\alpha)$ holds for all α, the relation $R(a, b, y)$ holds too.

We set

$$Z := \{a \in A : a \in S_1(a)\},$$
$$P_R(b) := \{x \in A : R(x,b,y) \text{ holds for all } y \in T(x,b)\}.$$

It is clear that $P(b)$ is the union of the sets $A \setminus S_2^{-1}(b)$ and $Z \cap P_R(b)$. Therefore, $P(b)$ is closed if these two latter sets are closed. Moreover, the set Z of all fixed points of S_1 on A is closed if the map S_1 is closed. The converse is evidently not always true.

Lemma 3.6. ([5]) *Let $b \in A$. Assume that:*

(i) *The set A and the set Z of all fixed points of S_1 are closed.*

(ii) *The inverse value $S_2^-(b)$ is open in A.*

(iii) *$T(.,b)$ is lower semicontinuous in the first variable.*

(iv) *$R(.,b,.)$ is closed in the first and the third variables.*

Then, the set $P(b)$ is closed.

Corollary 3.7. *The following conditions are sufficient for* (VR) *to have a solution:*

(i) *$(A;\Gamma)$ is a compact partial KKM space.*

(ii) *The set of all fixed points of S_1 is closed.*

(iii) *The map S_2 has open inverse values and, for every $b \in A$, the Γ-convex hull of $S_2(b)$ is contained in $S_1(b)$.*

(iv) *For every given $b \in A$ fixed, $T(.,b)$ is lower semicontinuous in the first variable.*

(v) *The relation R is KKM and, for every given $b \in A$, $R(.,b,.)$ is closed in the first and the third variables.*

Proof. Apply Lemma 3.6 and Theorem 3.4. □

When A is a nonempty convex compact subset of a (not necessarily Hausdorff) topological vector spaces, Corollary 3.7 reduces to Luc [5, Corollary 3.1].

The concept of KKM relations can be found in the majority of papers on variational inequalities in one or another form. Luc [5] mentioned some of them as follows:

(i) Diagonally Quasiconvex Maps.
(ii) Properly Quasimonotone Maps.
(iii) Quasiconvex Inclusions.

4. Criteria Based on Fixed Points

The criteria that we are going to establish in this section are based on Corollary 2.2, in which fixed point theorems are involved. As before, it is assumed that $A = B$ is a nonempty subset of a partial KKM space $(X;\Gamma)$ and that $(Y;\Lambda)$ is another partial KKM space. Consider the map $Q : A \multimap A$ defined by

$$Q(a) = \{x \in A : \; R(a,x,y) \text{ does not hold for some } y \in T(a,x)\}.$$

It can be seen that

$$A \setminus P^-(a) = \begin{cases} S_2(a), & \text{if } a \notin S_1(a); \\ S_2(a) \cap Q(a), & \text{else.} \end{cases}$$

The next result gives a relationship between R, P_R and Q.

Lemma 4.1. *The following assertions hold:*

(i) *For $a \in A$, the relation $R(a, a, y)$ holds for all $y \in T(a, a)$ if and only if a is not a fixed point of Q. In particular, if R is KKM, then Q has no fixed points.*

(ii) *If $Q(a)$ is Γ-convex for all $a \in A$ and if Q does not have fixed points, then R is KKM.*

(iii) *For $a \in A$, one has $A \setminus Q^-(a) = P_R(a)$.*

Consequently, the map Q has open inverse values if and only if the map P_R has closed values.

Proof. The first assertion is clear. For the second assertion, suppose to the contrary that R is not KKM. Then, there exist $N = \{a_1, \ldots, a_k\} \in \langle A \rangle$ and $a \in \Gamma_N$ such that, for each i, $R(a, a_i, y_i)$ does not hold for some $y_i \in T(a, a_i)$. In other words, all a_i's belong to $Q(a)$. Under the convexity hypothesis, a is a fixed point of Q, a contradiction. In the last assertion, the equality is obtained by direct calculation. □

When A is a nonempty convex compact subset of X, and X and Y are (not necessarily Hausdorff) topological vector spaces, Lemma 4.1 reduces to Luc [5, Lemma 3.1].

The next result is a consequence of Theorem 3.4 and Lemma 4.1, but we shall give another proof based on the Fan-Browder fixed point theorem in our abstract convex space theory, which states that, if $(A; \Gamma)$ is a compact partial KKM space and if $G : A \multimap A$ is a multimap with $A = \bigcup_{a \in A} \mathrm{int} G^-(a)$, then there is some $a \in A$ belonging to the Γ-convex hull of $G(a)$.

Theorem 4.2. *The problem* (VR) *has a solution if the following conditions hold:*

(i) *$(A; \Gamma)$ is a compact partial KKM space.*

(ii) *The set of all fixed points of S_1 on A is closed.*

(iii) *The map S_2 has Γ-convex values and open inverse values, and $S_2(a) \subset S_1(a)$ for every $a \in A$.*

(iv) *The map Q has Γ-convex values, open inverse values and no fixed points.*

Proof. We recall that Z denotes the set of all fixed points of S_1 on A. Consider the multimap $A \setminus P^-$ on A. If, for some point $a \in A$, the set $A \setminus P^-(a)$ is empty, then a is a solution of (VR) (Corollary 2.2). Assume that this map has nonempty values. It follows that $A = \bigcup_{a \in A} (A \setminus P^-)^-(a)$. Moreover, one has

$$[A \setminus P^-]^-(a) = \{x \in A \setminus E : a \in S_2(x)\} \cup \{x \in E : a \in S_2(x) \cap Q(x)\}$$
$$= \{(A \setminus Z) \cup Q^-(a)\} \cap S_2^-(a).$$

By the hypotheses (ii)-(iv), this set is open in A. Hence, $A = \bigcup_{a \in A} \mathrm{int}(A \setminus P^-)^-(a)$. Apply the Fan-Browder theorem to find a fixed point $\bar{a} \in A$ of $A \setminus P^-$. In particular, this point belongs to $S_2(\bar{a})$, hence to Z as well. By this, $\bar{a} \in Q(\bar{a})$, which contradicts (iv). □

When A is a nonempty convex compact subset of a (not necessarily Hausdorff) topological vector space, Theorem 4.2 reduces to Luc [5, Theorem 4.1].

Until now, all proofs are imitations of corresponding ones of Luc [5], and some of other results of him not mentioned here also can be extended to abstract convex spaces.

5. Comments on Luc, Sarabi, and Soubeyran 2010 [6]

In [6], two main existence conditions for solutions of variational relation problems are established without convexity. The first one is based on a finite solvability property and the second one on generalized KKM map. These conditions unify and strengthen several existing results in the literature on the topic. A model of satisficing process by rejection is considered which gives an economic interpretation of the introduced concepts.

In Section 4 of [6], the authors establish existence conditions for variational relation problems that share certain properties of the so-called KKM maps.

We will use generalized KKM maps in the sense of [15]. Now we assume that A and B are nonempty subsets of a abstract convex space $(X;\Gamma)$.

Definition 5.1. The relation R is said to be *generalized KKM* if for every finite subset $\{b_1,\dots,b_m\}$ of B there exists a corresponding subset $\{a_1,\dots,a_m\}$ of A such that $\mathrm{co}_\Gamma\{a_1,\dots,a_m\}\subset A$, for any subset $I\subseteq\{1,\dots,m\}$ and any $\bar a\in\mathrm{co}_\Gamma\{a_j : j\in I\}$, one can find some index $i\in I$ such that $R(\bar a,b_i,y)$ holds for all $y\in T(\bar a,b_i)$.

As for multimaps, KKM relations in [1, 13] are generalized KKM, but the converse is not true in general.

Here is the main result of this section on existence of solutions of (VR) when the relation R is generalized KKM.

Theorem 5.2. *The following conditions are sufficient for* (VR)*to have a solution:*
(i) *A is a nonempty compact set;*
(ii) *The multimap $P(.)$ is intersectionally closed on B;*
(iii) *$S_1(a)=A$ for every $a\in A$;*
(iv) *The relation R is generalized KKM.*

Proof. We first prove that P is generalized KKM. Consider a finite subset $\{b_1,\dots,b_m\}$ of B. Using (iv), we can find a corresponding subset $\{a_1,\dots,a_m\}$ of A such that for any subset $I\subset\{1,\dots,m\}$ and any $\bar a\in\mathrm{co}_\Gamma\{a_j : j\in I\}$, one can find some index $j\in I$ such that $R(\bar a,b_j,y)$ holds for all $y\in T(\bar a,b_j)$. This yields $\bar a\in P(b_j)$ which shows that P is generalized KKM. Since P is generalized KKM, for each $b\in B$ there is some $a\in A$ such that $a\in P(b)$. In particular $P(b)$ is nonempty for each $b\in B$. Now consider the multimap $b\mapsto \mathrm{cl}(P(b))$. It is a generalized KKM map too. Similarly to Lemma 4.1 of [5] the family $\{\mathrm{cl}(P(b)) : b\in B\}$ has the finite intersection property. By the abstract convex space version of the 1961 KKM-Fan lemma, that family has a common point, and so does the family $\{P(b) : b\in B\}$ in view of (ii). By Theorem 2.1 of [5] problem (VR) has a solution. □

The above theorem generalizes Theorem 3.1 of [5] in three aspects; for details, see [6].

6. Comments on Balaj 2013 [1]

Balaj [1] investigates the existence of solutions for three types of variational relation problems which encompass several generalized equilibrium problems, variational inequalities and variational inclusions studied in a long list of papers in the field.

Let X, Y and Z be nonempty sets. A nonempty subset R of the product $X \times Y \times Z$ determines a relation $R(x, y, z)$ in a natural manner: we say that $R(x, y, z)$ holds if and only if $(x, y, z) \in R$. When Z is a parameter set, then R is called a variational relation.

Now we generalize Balaj's three types as follows:

Assume that $(X; \Gamma)$ is an abstract convex space and Y and Z are two sets, endowed for each problem with an adequate topological and/or algebraic structure. Let $T : X \to 2^Y$, $P : X \to 2^Z$ be two multimaps and $R(x, y, z)$ be a relation linking elements $x \in X$, $y \in Y$, $z \in Z$.

(VRP1a) Find $\bar{x} \in X$ such that $R(\bar{x}, y, z)$ holds for all $y \in T(\bar{x})$ and all $z \in P(\bar{x})$.

(VRP1b) Find $\bar{x} \in X$ such that for each $y \in T(\bar{x})$ there exists $z \in P(\bar{x})$ such that $R(\bar{x}, y, z)$ holds.

(VRP2) Find $\bar{x} \in X$ and $\bar{z} \in P(\bar{x})$ such that $R(\bar{x}, y, \bar{z})$ holds for all $y \in T(\bar{x})$.

These problems encompass several generalized equilibrium problems, variational inequalities and variational inclusions studied in a long list of papers in the field. Actually, Balaj [1] listed a few typical examples.

In order to study the solution existence of problems (VRP1a) and (VRP1b), Balaj established the inclusion result [1, Theorem 3.1], which can be generalized as follows:

Theorem 6.1. *Let $(X; \Gamma)$ be a partial KKM space, and Y be a nonempty set. Assume that $T, S : X \to 2^Y$ are two multimaps with nonempty values satisfying:*
(i) *T has open fibers and $X \setminus T^-(y)$ is compact for at least one $y \in Y$;*
(ii) *S has closed fibers;*
(iii) *the set $Z = \{x \in X : x \in (S^- T)(x)\}$ is compact;*
(iv) *S^- is a generalized KKM map.*
Then there exists $\bar{x} \in X$ such that $T(\bar{x}) \subseteq S(\bar{x})$.

Proof. Consider the map $Q : Y \to 2^X$ defined by $Q(y) = (X \setminus T^-(y)) \cup (Z \cap S^-(y))$. We show that Q is a generalized KKM map as in [15]. If $\{y_0, ..., y_n\}$ is a finite subset of Y , by (iv), there exists a subset $\{x_0, ..., x_n\}$ of X such that for each subset of indices $I \subseteq \{0, ..., n\}$,

$$\mathrm{co}_\Gamma\{x_i : i \in I\} \subset \bigcup_{i \in I} S^-(y_i). \tag{1}$$

Let $x \in \mathrm{co}_\Gamma\{x_i : i \in I\}$. We prove that (1) implies

$$x \in \bigcup_{i \in I} Q(y_i). \tag{2}$$

If $x \in Z$, by (1) one has

$$x \in Z \cap (\bigcup_{i \in I} S^-(y_i)) = \bigcup_{i \in I}(Z \cap S^-(y_i)) \subset \bigcup_{i \in I} Q(y_i).$$

If $x \in X \setminus Z$, we claim that $x \in X \setminus T^-(y_i)$ for some index $i \in I$. Suppose on the contrary that $y_i \in T(x)$ for all $i \in I$. Then $S^-(y_i) \subset S^-(T(x))$. In view of (1), we have

$$x \in \mathrm{co}_\Gamma\{x_i : i \in I\} \subset \bigcup_{i \in I} S^-(y_i) \subset S^-(T(x));$$

a contradiction. Hence, $x \in \bigcup_{i \in I}(X \setminus T^-(y_i)) \subset \bigcup_{i \in I} Q(y_i)$. Since Q has closed values and $Q(y)$ is compact for at least one $y \in Y$, by a KKM type result in [11], there exists $\bar{x} \in \bigcap_{y \in Y} Q(y)$. For each $y \in T(\bar{x})$, i.e. $\bar{x} \notin X \setminus T^-(y)$, since $\bar{x} \in Q(y)$, we have $\bar{x} \in S^-(y)$, that is $y \in S(\bar{x})$. Thus $T(\bar{x}) \subset S(\bar{x})$ and this means exactly the conclusion of the theorem. □

Remark 6.2. Let us observe that

$$Z = \{x \in X : \exists y \in Y \text{such that } x \in T^-(y) \cap S^-(y)\} = (T^- \cap S^-)(Y).$$

Hence condition (iii) in Theorem 6.1 means actually the compactness of the range of the map $T^- \cap S^-$.

Actually Theorem 6.1 is due to Balaj [1, Theorem 3.1] for a convex set X in a topological vector space, and we followed his proof. He applied his theorem to study the solution existence of certain variational relation problems (VRP1a) and (VRP1b).

Other results in [1] might be extended to abstract convex spaces.

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