ON SEMIDEFINITE LINEAR FRACTIONAL OPTIMIZATION PROBLEMS

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Abstract. Recently, semidefinite optimization problems have been intensively studied since many optimization problems can be changed into the problems and the problems are very tractable ([5]). In this paper, we review the duality results for semidefinite linear fractional optimization problems in the paper ([3] On duality theorems for semidefinite linear fractional optimization problems, Journal of Nonliner and Convex Analysis, volume 20, 2019, 1907-1912).

1. Introduction

B. D. Craven [1, 2] considered the following linear fractional program:

$$\text{(LF)} \quad \text{Maximize } \frac{c^T x + \alpha}{d^T x + \beta} \text{ subject to } x \geqq 0, Ax = b,$$

where $c, d \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are given, and formulated the dual problem for (LF) as follows:

$$\text{(LD)} \quad \text{Minimize } \gamma \text{ subject to } A^T s + d\gamma \geqq c, \quad \beta\gamma - b^T s \geqq \alpha,$$

and then studied the duality theorems which holds between (LF) and (LD).

Recently, semidefinite optimization problems have been intensively studied since many optimization problem can be changed into the problems and the problems are very tractable ([5]) . In the paper [3], we considered a semidefinite linear fractional optimization problem (SLP) and formulated its dual problem (SLD), and then obtained the duality theorems which hold between (SLF) and (SLD). By using the B. D. Craven's approaches in [1, 2], we can prove the duality theorems which hold between (SLF) and (SLD). But we proved the duality theorems by a direct methods in the paper [3]. In this paper, we review the duality results for semidefinite linear fractional optimization problems in the paper [3].

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We considered the semidefinite linear fractional optimization problem in the paper [3] :

$$\text{(SLF)} \qquad \text{Maximize} \quad \frac{Tr(CX)+\alpha}{Tr(DX)+\beta}$$
$$\text{subject to} \quad X \succeq 0,\ Tr(A_iX) = b_i, i = 1, \cdots, m$$

where C, D are given symmetric $n \times n$ matrices, α, β are given real numbers, $A_i, i = 1, \cdots, m$ are given symmetric $n \times n$ matrices and $b_i, i = 1, \cdots, m$ are given real numbers.

TrA is the trace of $n \times n$ matrix A. For a symmetric $n \times n$ matrix A, $A \succeq 0$ means that A is positive semidefinite and $A \succ 0$ means that A is positive definite. Let S^n be the space of $n \times n$ symmetric matrices. Then $Tr(\cdot\cdot)$ is the inner product on S^n and S^n is the finite-dimensional Banach space with the norm $\|\cdot\|$ induced by the inner product $Tr(\cdot\cdot)$ ([4]).

Moreover we formulated the dual problem for (SLF) in the paper [3] as follows;

$$\text{(SLD)} \qquad \text{Minimize} \quad \gamma$$
$$\text{subject to} \quad \sum_{i=1}^{m} y_iA_i + \gamma D \succeq C$$
$$\beta\gamma - b^Ty \geqq \alpha.$$

2. Duality Theorems

Let $\triangle = \{X \in S^n |\ X \succeq 0,\ Tr(A_iX) = b_i, i = 1, \cdots, m\}$ and $F = \{(\gamma, y) \in \mathbb{R} \times \mathbb{R}^m \mid \sum_{i=1}^{m} y_iA_i + rD \succeq C,\ \beta\gamma - b^Ty \geqq \alpha\}$.

Now we stste the dulaity theorems in the paper [3] .

Theorem 2.1. [3] **(Weak duality)** *Assume that* $\Big[X \in \triangle\Big] \Rightarrow \Big[Tr(DX) + \beta > 0\Big]$. *Let* X *be feasible (SLF) and let* (γ, y) *be feasible (SLD). Then*

$$\frac{Tr(CX)+\alpha}{Tr(DX)+\beta} \leqq \gamma.$$

Theorem 2.2. [3] **(Strong duality)** *Assume that* $\Big[X \in \triangle\Big] \Rightarrow \Big[Tr(DX) + \beta > 0\Big]$ *and that* $\Big[S - \sum_{i=1}^{m} \mu_iA_i = 0\Big] \Rightarrow \Big[S = 0,\ \mu = 0\Big]$. *If* $\bar{X}$ *is a solution of (SLF), then there exists* $(\bar{\gamma}, \bar{y}) \in F$ *such that* $(\bar{\gamma}, \bar{y})$ *is a solution of (SLD) and*

$$\frac{Tr(CX)+\alpha}{Tr(DX)+\beta} = \bar{\gamma}.$$

Theorem 2.3. [3] **(Converse duality)** *Assume that* $\Big[X \in \triangle\Big] \Rightarrow \Big[Tr(DX)+\beta > 0\Big]$ *and that* $\triangle$ *is bounded. Further assume that there exist* $y_0 \in \mathbb{R}^m$ *and* $\gamma^0 \in \mathbb{R}$ *such* *that* $\sum_{i=1}^m y_0^i A_i + \gamma^0 D - C \succ 0$ *and* $\beta\gamma^0 - b^T y > \alpha$. *If* $(\bar{\gamma}, \bar{y}) \in F$ *is a solution of (SLD) then there exists a solution* $\bar{X}$ *of (SLF) such that*

$$\bar{\gamma} = \frac{Tr(C\bar{X}) + \alpha}{Tr(D\bar{X}) + \beta}.$$

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