MULTI-OBJECTIVE OPTIMIZATION WITH SOS-CONVEX POLYNOMIALS OVER A POLYNOMIAL MATRIX INEQUALITY†

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ABSTRACT. This paper is concerned with a multi-objective optimization problem, where the objective functions are sum of square convex polynomials and the constraint is a polynomial matrix inequality. We propose methods for finding (exactly) efficient solutions to the considered multi-objective optimization problem.

1. INTRODUCTION

In this paper, we are interested in a multi-objective optimization problem, where the objective functions are SOS-convex polynomials (see Definition 2.2) and the constraint is an SOS-concave polynomial matrix (see Definition 2.4). Note that the notion of SOS-convex polynomial has been proposed as a *tractable* sufficient condition for convexity based on semidefinite programming; see e.g., [1, 2, 8, 12] and the references therein. Note also that, for an SOS-convex optimization problem, its optimal value as well as optimal solution(s) can be found by solving a single semidefinite programming problem [13].

It is well-known that one of the most famous methods for studying multi-objective optimization problems is the scalarization approach [3, 4, 5, 6, 7, 9, 10, 15]. It consists in solving one or several parametrized single-objective optimization problems to find an optimal solution to the original multi-objective problem.

In this paper, we are concerned with the study of finding efficient solutions to the multi-objective optimization problem. To this end, we first establish strong

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duality results for the formulated scalar objective problem, its semidefinite programming relaxation dual problem, and the dual problem of the semidefinite programming problem, for a mentioned scalarization approach. Then, according to the results, we propose methods of finding (exactly) efficient solutions to the considered problems.

The rest of the paper is organized as follows. Section 2 provides some basic notations, preliminaries and several auxiliary results. Section 3 states the problems formulation and solution concepts and gives the main results of the paper, which are methods for finding efficient solutions to problem (MP). Finally, conclusions are given in Sect. 4.

2. Preliminaries

We begin this section by fixing the notations and preliminaries. By $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, we denote, respectively, the inner product and the norm in the Euclidean space with dimension n. The non-negative orthant of $\mathbb{R}^n$ is defined by $\mathbb{R}^n_+ := \{x := (x_1, \dots, x_n) \colon x_i \geq 0, i = 1, \dots n\}$. The space of all real polynomials in the variable x is denoted by $\mathbb{R}[x]$. Moreover, the space of all real polynomials in the variable x with degree at most d is denoted by $\mathbb{R}[x]_d$. The degree of a polynomial f is denoted by $\deg f$.

We say that a real polynomial f is sum of squares (SOS) if there exist real polynomials f_l, $l = 1, \dots, r$, such that $f = \sum_{l=1}^r f_l^2$. The set consisting of all sum of squares real polynomial is denoted by Σ^2. In addition, the set consisting of all sum of squares real polynomial with degree at most d is denoted by Σ^2_d. For a multi-index $\alpha \in \mathbb{N}^n$, let $|\alpha| := \sum_{i=1}^n \alpha_j$, and let $\mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d\}$. The notation x^α stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The canonical basis of $\mathbb{R}[x]_d$ is denoted by

$$v_d(x) := (x^\alpha)_{\alpha \in \mathbb{N}^n_d} = (1, x_1, \dots, x_n, x_1^2, x_1 x_2, \dots, x_n^2, \dots, x_1^d, \dots, x_n^d)^T, \tag{2.1}$$

which has dimension $s(d) := \binom{n+d}{d}$.

Let S^n be the set of $n \times n$ symmetric matrices. For $M, N \in S^n$, $\langle M, N \rangle := \mathrm{tr}(MN)$, where "tr" denotes the trace (sum of diagonal elements) of a matrix. For $X \in S^n$, $X \succeq 0$ (resp., $X \succ 0$) means that X is positive semidefinite (resp., positive definite) matrix. Let $S^n_+ := \{X \in S^n \colon X \succeq 0\}$ be the set of $n \times n$ symmetric positive semidefinite matrices. The gradient and the Hessian of a polynomial $f \in \mathbb{R}[x]$ at a point $\bar{x}$ are denoted by $\nabla f(\bar{x})$ and $\nabla^2 f(\bar{x})$, respectively.

Below, we first recall a useful and celebrated result, i.e., an SOS polynomial can be written as a sum of squares via positive semidefinite programming.

Proposition 2.1. [14] *A polynomial $f \in \mathbb{R}[x]_{2d}$ has a sum of squares decomposition if and only if there exists a real symmetric and positive semidefinite matrix $X \in \mathbb{R}^{s(d)\times s(d)}$ such that $f(x) = v_d(x)^T X v_d(x)$ for all $x \in \mathbb{R}^n$.*

We now recall a very interesting subclass of convex polynomials in $\mathbb{R}[x]$ introduced by Helton and Nie [8]; see also [1, 2].

Definition 2.2. [8] A polynomial $f \in \mathbb{R}[x]$ is called *SOS-convex* if there exists a matrix polynomial $F(x)$ such that $\nabla^2 f(x) = F(x)F(x)^T$, equivalently,

$$f(x) - f(y) - \nabla f(y)^T(x-y)$$

is a sum of squares polynomial in $\mathbb{R}[x; y]$ (with respect to variables x and y).

Observe that, an SOS-convex polynomial is convex; the converse is not true, i.e., there exists a convex polynomial which is not SOS-convex [1, 2].

In what follows, we recall a result, which is an extension of Jensen's inequality to a class of linear functionals that are not necessarily probability measures when one restricts its application to the class of SOS-convex polynomials [13].

Lemma 2.3. [13] *Let $f \in \mathbb{R}[x]_{2d}$ be SOS-convex, and let $y = (y_\alpha)_{\alpha\in\mathbb{N}^n_{2d}}$ satisfy $y_0 = 1$ and $\sum_{\alpha\in\mathbb{N}^n_{2d}} y_\alpha B_\alpha \succeq 0$. Let $L_y : \mathbb{R}[x] \to \mathbb{R}$ be a linear functional defined by $L_y(f) := \sum_{\alpha\in\mathbb{N}^n_{2d}} f_\alpha y_\alpha$, where $f(x) = \sum_{\alpha\in\mathbb{N}^n_{2d}} f_\alpha x^\alpha$. Then*

$$L_y(f) \geq f(L_y(x)),$$

where $L_y(x) := (L_y(x_1), \ldots, L_y(x_n)) = (y)_{|\alpha|=1}$.

We close this section by recalling the concept of the matrix SOS-concavity.

Definition 2.4. [16] An $m \times m$ symmetric polynomial matrix $G(x)$ is called *matrix SOS-concave* if for every $\xi \in \mathbb{R}^m$, there exists a polynomial matrix $F_\xi(x)$ in x such that

$$-\nabla^2_x(\xi^T G(x)\xi) = F_\xi(x)F_\xi(x)^T,$$

equivalently, $-\xi^T G(x)\xi$ is SOS-convex for all $\xi \in \mathbb{R}^m$.

3. Problems Formulation and Solution Concepts

Consider the following multi-objective optimization problem with polynomial objective functions over a polynomial matrix SOS-concave constraint,

$$\text{(MP)} \qquad \begin{aligned} \min_{x\in\mathbb{R}^n} \ & (f_1(x), \ldots, f_p(x)) \\ \text{s.t.} \ & G(x) \succeq 0, \end{aligned}$$

where $f_j\colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, p$ are SOS-convex polynomials and $G(x)$ is a polynomial $m \times m$ matrix SOS-concave constraint. We denote the feasible set of (MP) by

$$K := \{x \in \mathbb{R}^n : G(x) \succeq 0\}, \tag{3.1}$$

Next, we recall the notions of efficient solutions to (MP).

Definition 3.1. A point $\bar{x} \in K$ is said to be an *efficient solution* to problem (MP), if

$$f(x) - f(\bar{x}) \notin -\mathbb{R}^p_+ \backslash \{0\},\ \forall x \in K.$$

Recall the following scalar optimization problem [4, 7], which is transformed from (MP) by the hybrid method:

$$
\begin{aligned}
(\mathrm{P}(z)) \qquad & \min_{x\in\mathbb{R}^n} \ \sum_{j=1}^{p} \rho_j f_j(x) \\
& \text{s.t.} \quad f_j(x) \le f_j(z),\ j = 1, \ldots, p, \\
& \qquad\ \ G(x) \succeq 0,
\end{aligned}
$$

where $\rho \in \operatorname{int} \mathbb{R}^p_+$ is fixed and $z \in \mathbb{R}^n$ is a parameter. Then, the feasible set of problem $(\mathrm{P}(z))$ is given by

$$K(z) := \{x \in \mathbb{R}^n : G(x) \succeq 0,\ f_j(x) \le f_j(z),\ j = 1, \ldots, p\}. \tag{3.2}$$

It is worth mentioning that the feasible set $K(z)$ as in (3.2) is nonempty when the parameter z is chosen in the feasible set of (MP).

Now we formulate the semidefinite programming relaxation dual problem for $(\mathrm{P}(z))$ as follows:

$$
\begin{aligned}
(\mathrm{Q}(z)) \qquad & \sup_{\gamma, X, \Lambda, \lambda_j} \ \gamma \\
& \text{s.t.} \ \sum_{j=1}^{p} \left(\rho_j (f_j)_0 + \lambda_j \left((f_j)_0 - f_j(z)\right)\right) - \langle \Lambda, G_0\rangle - \gamma = \langle X, B_0\rangle, \\
& \qquad \sum_{j=1}^{p} \left(\rho_j (f_j)_\alpha + \lambda_j (f_j)_\alpha\right) - \langle \Lambda, G_\alpha\rangle = \langle X, B_\alpha\rangle,\ \alpha \in \mathbb{N}^{\alpha}_{2d} \backslash \{0\}, \\
& \qquad \gamma \in \mathbb{R},\ X \in S^{s(d)}_+,\ \Lambda \in S^m_+,\ \lambda_j \ge 0,\ j = 1, \ldots, p.
\end{aligned}
$$

In addition, the dual problem of (Q(z)) is the following semidefinite programming problem

$$(\widehat{\mathrm{Q}}(z)) \qquad \begin{aligned} \inf_{y\in\mathbb{R}^{s(2d)}} \quad & \sum_{j=1}^{p}\sum_{\alpha\in\mathbb{N}^n_{2d}} \rho_j (f_j)_\alpha y_\alpha \\ \text{s.t.} \quad & \sum_{\alpha\in\mathbb{N}^n_{2d}} (f_j)_\alpha y_\alpha \leq f_j(z),\ j=1,\dots,p, \\ & \sum_{\alpha\in\mathbb{N}^n_{2d}} y_\alpha G_\alpha \succeq 0,\ \sum_{\alpha\in\mathbb{N}^n_{2d}} y_\alpha B_\alpha \succeq 0,\ y_0 = 1. \end{aligned}$$

We would mention here that the weak duality between ($\widehat{\mathrm{Q}}(z)$) and (Q(z)) always holds if the two problems have nonempty feasible set, i.e., $\inf(\widehat{\mathrm{Q}}(z)) \geq \sup(\mathrm{Q}(z))$ (see, e.g., [17]).

Now, we first give a strong duality result for (P(z)), (Q(z)), and ($\widehat{\mathrm{Q}}(z)$). To do this, we need the following assumption.

(H3) For a given $z \in K$ (K is as in (3.1)), the Slater-type condition holds for the problem (P(z)), that is, there exists $\hat{x} \in \mathbb{R}^n$ such that

$$G(\hat{x}) \succ 0 \text{ and } f_j(\hat{x}) - f_j(z) < 0,\ j=1,\dots,p.$$

Theorem 3.2. *Let $z \in K$ be given. If* (H3) *holds, then*

$$\inf(\mathrm{P}(z)) = \max(\mathrm{Q}(z)) = \inf(\widehat{\mathrm{Q}}(z)).$$

Proof. Let $z \in K$ be given. We first show that $\inf(\mathrm{P}(z)) \leq \max(\mathrm{Q}(z))$. Since the feasible set $K(z)$ is nonempty, we may assume that $\bar{\gamma} := \inf(\mathrm{P}(z)) \in \mathbb{R}$. Indeed, if $\inf(\mathrm{P}(z)) = -\infty$, the desired inequality holds trivially. Note that for any $\Lambda \in S^m_+$, $-\langle \Lambda, G(x)\rangle$ is convex and for each $j=1,\dots,p$, f_j is also convex. Since the assumption (H3) holds, it follows from the Lagrangian-type duality for convex programming that

$$\bar{\gamma} = \max_{\lambda_j\geq 0, \Lambda\in S^m_+} \inf_{x\in\mathbb{R}^n} \left\{ \sum_{j=1}^{p} \left(\rho_j f_j(x) + \lambda_j \left(f_j(x) - f_j(z)\right)\right) - \langle \Lambda, G(x)\rangle \right\}.$$

It means that there exist $\bar{\lambda} \in \mathbb{R}^p_+$ and $\overline{\Lambda} \in S^m_+$ such that

$$\bar{\gamma} = \inf_{x\in\mathbb{R}^n} \left\{ \sum_{j=1}^{p} \left(\rho_j f_j(x) + \bar{\lambda}_j \left(f_j(x) - f_j(z)\right)\right) - \langle \overline{\Lambda}, G(x)\rangle \right\}.$$

Let $h(x) := \sum_{j=1}^{p} \left(\rho_j f_j(x) + \bar{\lambda}_j \left(f_j(x) - f_j(z)\right)\right) - \langle \overline{\Lambda}, G(x)\rangle - \bar{\gamma}$. Note that f_j, $j=1,\dots,p$, and $-\langle \overline{\Lambda}, G(\cdot)\rangle$ are SOS-convex polynomials. This implies that

h is a non-negative SOS-convex polynomial. It follows from [11, Remark 3.2] that h is sum of square with degree at most $2d$, i.e.,

$$\sum_{j=1}^{p}\left(\rho_j f_j+\bar{\lambda}_j\left(f_j-f_j(z)\right)\right)-\langle\overline{\Lambda},G(\cdot)\rangle-\bar{\gamma}\in\Sigma^2_{2d}.$$

By Proposition 2.1, there exists $\overline{X}\in S_+^{s(d)}$ such that for all $x\in\mathbb{R}^n$,

$$\sum_{j=1}^{p}\left(\rho_j f_j(x)+\bar{\lambda}_j\left(f_j(x)-f_j(z)\right)\right)-\langle\overline{\Lambda},G(x)\rangle-\bar{\gamma}=\left\langle\overline{X},v_d(x)v_d(x)^T\right\rangle.$$

Let $v_d(x)v_d(x)^T=\sum_{\alpha\in\mathbb{N}^n_{2d}}x^\alpha B_\alpha$. Then we have

$$\sum_{j=1}^{p}\left(\rho_j (f_j)_0+\bar{\lambda}_j\left((f_j)_0-f_j(z)\right)\right)-\langle\overline{\Lambda},G_0\rangle-\bar{\gamma}=\langle\overline{X},B_0\rangle,$$

$$\sum_{j=1}^{p}\left(\rho_j (f_j)_\alpha+\bar{\lambda}_j(f_j)_\alpha\right)-\langle\overline{\Lambda},G_\alpha\rangle=\langle\overline{X},B_\alpha\rangle,\ \alpha\in\mathbb{N}^\alpha_{2d}\backslash\{0\}.$$

Hence, $\left(\bar{\gamma},\overline{X},\overline{\Lambda},\bar{\lambda}_1,\ldots,\bar{\lambda}_p)\right)$ is a feasible solution of $(\mathrm{Q}(z))$, and so, we get

$$\inf(\mathrm{P}(z))=\bar{\gamma}\le\max(\mathrm{Q}(z)).$$

Now, we prove that $\inf(\mathrm{P}(z))\ge\inf(\widehat{\mathrm{Q}}(z))$. Let x be any feasible solution of $(\mathrm{P}(z))$. and let $\gamma:=\sum_{j=1}^{p}\rho_j f_j(x)$. Letting $y:=v_d(x)$ as in (2.1), we have

$$\begin{cases}\sum\limits_{\alpha\in\mathbb{N}^n_{2d}}(f_j)_\alpha y_\alpha=f_j(x)\le f_j(z),\ j=1,\ldots,p,\ i=0,1,\ldots,t_j,\\ \sum\limits_{\alpha\in\mathbb{N}^n_{2d}}y_\alpha G_\alpha=G(x)\succeq 0,\\ yy^T=\sum\limits_{\alpha\in\mathbb{N}^n_{2d}}y_\alpha B_\alpha\succeq 0.\end{cases}$$

So, y is a feasible solution of $(\widehat{\mathrm{Q}}(z))$. Moreover, since x is an arbitrary feasible solution of $(\mathrm{P}(z))$, we obtain

$$\gamma=\sum_{j=1}^{p}\rho_j f_j(x)=\sum_{j=1}^{p}\sum_{\alpha\in\mathbb{N}^n_{2d}}\rho_j(f_j)_\alpha y_\alpha\ge\inf(\widehat{\mathrm{Q}}(z)).$$

Note that the feasible sets of $(\widehat{\mathrm{Q}}(z))$ and $(\mathrm{Q}(z))$ are nonempty. Hence, by the usual weak duality theorem for semidefinite programming, we see that $\inf(\widehat{\mathrm{Q}}(z))\ge\sup(\mathrm{Q}(z))$. Thereby, we get the desired conclusion. □

The following proposition suggests a way to obtain an efficient solution to problem (MP) by solving problem $(\mathrm{P}(z))$.

Proposition 3.3. [7, Propsition 12] *Let $z \in K$. If $\bar{x}$ is an optimal solution to problem* (P(z)), *then $\bar{x}$ is also an optimal solution to problem* (P($\bar{x}$)); *in addition, $\bar{x}$ is an efficient solution to problem* (MP).

The following theorem shows that an efficient solution to problem (MP) can be obtained by solving a single semidefinite programming problem.

Theorem 3.4. *Let $z \in K$ be given. Suppose that the assumption* (H3) *holds. If $\bar{y}$ is an optimal solution to problem* ($\widehat{\mathrm{Q}}(z)$), *then $\bar{x} := (\bar{y})_{|\alpha|=1}$ is an efficient solution to problem* (MP).

Proof. Suppose that $\bar{y}$ is an optimal solution to the problem ($\widehat{\mathrm{Q}}(z)$). Then, we have

$$\sum_{\alpha \in \mathbb{N}^n_{2d}} (f_j)_\alpha \bar{y}_\alpha \leq f_j(z), \; j = 1, \dots, p, \tag{3.3}$$

$$\sum_{\alpha \in \mathbb{N}^n_{2d}} \bar{y}_\alpha G_\alpha \succeq 0,$$

$$\sum_{\alpha \in \mathbb{N}^n_{2d}} \bar{y}_\alpha B_\alpha \succeq 0, \; y_0 = 1. \tag{3.4}$$

Now, let $\bar{x} := (L_{\bar{y}}(x_1), \dots, L_{\bar{y}}(x_n)) = (\bar{y})_{|\alpha|=1}$. Note that each f_j is an SOS-convex polynomial and $\bar{y}$ satisfies (3.4). Then, by Lemma 2.3, we see that

$$\sum_{\alpha \in \mathbb{N}^n_{2d}} (f_j)_\alpha \bar{y}_\alpha = L_{\bar{y}}(f_j) \geq f_j\left(L_{\bar{y}}(x_1), \dots, L_{\bar{y}}(x_n)\right), \; j = 1, \dots, p.$$

This, together with (3.3), yields that

$$f_j(\bar{x}) \leq f_j(z), \; j = 1, \dots, p. \tag{3.5}$$

www Note that, for each $H \in S^m_+$,

$$0 \geq \sum_{\alpha \in \mathbb{N}^n_{2d}} \bar{y}_\alpha \langle H, -G_\alpha \rangle = L_{\bar{y}} \langle H, -G(\cdot) \rangle.$$

Since for each $H \in S^m_+$, $\langle H, -G(\cdot) \rangle$ is SOS-convex, according to Lemma 2.3 again, we have

$$0 \geq L_{\bar{y}} \langle H, -G(\cdot) \rangle \geq \langle H, -G(L_{\bar{y}}(x)) \rangle = \langle H, -G(\bar{x}) \rangle, \; \forall H \in S^m_+,$$

i.e., $G(\bar{x}) \succeq 0$. So, $\bar{x}$ is a feasible solution of (P(z)).

Furthermore, by a similar argument as (3.5), we see that

$$\sum_{j=1}^p \sum_{\alpha \in \mathbb{N}^n_{2d}} \rho_j (f_j)_\alpha \bar{y}_\alpha \geq \sum_{j=1}^p \rho_j f_j(\bar{x}).$$

So, from Theorem 3.2, we easily see that $\bar{x}$ is an optimal solution to the problem (P(z)), and thus, by Proposition 3.3, the desired result follows. □

4. Conclusions

In this paper, we studied a multi-objective optimization problem admiring the form (MP). We solve the problem (MP) by employing the hybrid method. In conclusion, we proved that finding (exactly) efficient solutions to a multi-objective optimization problem of the form (MP) is tractable by using the hybrid method.

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