CONVERGENCE THEOREMS TO COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this article, we present methods for finding common fixed points of nonexpansive mappings. First, Mann type weak convergence theorems are proved. As a corollary, we obtain an alternative method to Mann's type iteration for finding a fixed point of a nonexpansive mapping. Also, a strong convergence theorem of Halpern type iterations is presented. The results base on those of Kondo and Takahashi [6].

1. INTRODUCTION

Let H be a real Hilbert space, let C be a nonempty subset of H, and let S be a mapping from C into H. The set of fixed points of S is denoted by $F(S) = \{x \in C : Sx = x\}$. A mapping $S : C \to H$ is called *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. For nonexpansive mappings, many approximation methods for finding fixed points have been intensively studied. The problem is as follows:

$$\text{Find an element } \widehat{x} \in F(S).$$

The following iteration is called Mann's type [8]:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) Sx_n \quad \text{for all } n \in \mathbb{N}, \tag{1.1}$$

where $x_1 \in C$ is given. In (1.1), $\mathbb{N}$ is the set of natural numbers, and $\{\lambda_n\}$ is a sequence of real numbers in the interval $[0, 1]$. It is known that under the iteration scheme (1.1), the sequence $\{x_n\}$ converges weakly to a fixed point of S; see, for example, Reich [10] and Takahashi [13]. The next iteration is called Halpern's type [2]:

$$x_{n+1} = \lambda_n x + (1 - \lambda_n) Sx_n \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

where $x_1 = x \in C$ is given. Under the iteration scheme (1.2), $\{x_n\}$ converges strongly to a fixed point of S; see, for example, Wittmann [16].

For two mappings S and T, consider a problem as

$$\text{Find an element } \overline{x} \in F(S) \cap F(T),$$

which is called a common fixed point problem. There are many studies for finding common fixed points of nonlinear mappings; see, for example, Lions

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[7], Shimizu and Takahashi [11], Atsushiba and Takahashi [1], Iemoto and Takahashi [4], Takahashi [14], and Kondo and Takahashi [6].

In this short article, we present methods for finding common fixed points of two nonexpansive mappings. The results base on those of Kondo and Takahashi [6]. First, Mann type weak convergence theorems are obtained (Theorems 3.1). As a corollary, we obtain an alternative method to Mann's type iteration (1.1) for finding a fixed point of a nonexpansive mapping. Strong convergence theorem of Halpern type iterations is also presented (Theorems 4.1 and Corollary 4.1).

2. Preliminaries

This section provides background information and results. Let H be a real Hilbert space with inner product $\langle \cdot,\ \cdot \rangle$ and norm $\|\cdot\|$. For four elements $x, y, z, w \in H$, it holds that

$$2\langle x-y,\ z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 . \tag{2.1}$$

We can easily prove the equation (2.1) as follows:

$$\begin{aligned} & \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \\ = \;& \|x\|^2 - 2\langle x,\ w\rangle + \|w\|^2 + \|y\|^2 - 2\langle y,\ z\rangle + \|z\|^2 \\ & - \left(\|x\|^2 - 2\langle x,\ z\rangle + \|z\|^2\right) - \left(\|y\|^2 - 2\langle y,\ w\rangle + \|w\|^2\right) \\ = \;& -2\langle x,\ w\rangle - 2\langle y,\ z\rangle + 2\langle x,\ z\rangle + 2\langle y,\ w\rangle \\ = \;& -2\langle x,\ w-z\rangle - 2\langle y,\ z-w\rangle \\ = \;& 2\langle x,\ z-w\rangle - 2\langle y, z-w\rangle \\ = \;& 2\langle x-y,\ z-w\rangle . \end{aligned}$$

The strong and weak convergence of a sequence $\{x_n\}$ in H to an element $x\,(\in H)$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Regarding weak and strong convergence, the following are well-known:

(A) a closed and convex subset of a real Hilbert space is weakly closed, that is, $\{x_n\} \subset C$ and $x_n \rightharpoonup u \Longrightarrow u \in C$;

(B) if $x_n \to x$ and $y_n \rightharpoonup y$, then $\langle x_n,\ y_n\rangle \to \langle x,\ y\rangle$; see Problem 5.4.1 in Takahashi [13].

(C) $x_n \rightharpoonup \overline{x}$ if and only if for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \rightharpoonup \overline{x}$.

Let C be a nonempty, closed and convex subset of H. If a mapping $S : C \to H$ is nonexpansive, $F(S)$ is closed and convex in C. A mapping $S : C \to H$ with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Sx - u\| \leq \|x - u\|$ for all $x \in C$ and $u \in F(S)$. It is easily ascertained that a nonexpansive mapping with $F(S) \neq \emptyset$ is quasi-nonexpansive.

Let F be a nonempty, closed, and convex subset of H. For any $x \in H$, there exists a unique nearest point $p \in F$, that is, $\|x - p\| = \inf_{v \in F} \|x - v\|$. This correspondence is called a *metric projection* from H onto F, and is denoted by P_F. For the metric projection P_F from H onto F, it holds that

$$\langle x - P_F x,\ P_F x - u \rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $u \in F$. For more details, see Takahashi [12] and [13].

For existence of a common fixed point, we know the following theorem that guarantees the existence of a common fixed point of commutative nonexpansive mappings; For its proof, see, for example, Hojo [3].

Theorem 2.1. *Let C be a nonempty, closed and convex subset of H, and let S and T be nonexpansive mappings of C into itself such that $ST = TS$. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N}\}$ is bounded. Then, $F(S) \cap F(T)$ is nonempty.*

From Theorem 2.1, we know a set of sufficient conditions for the existence of a common fixed point of nonexpansive mappings. In the main theorems of this article, we assume the existence. The following lemma will be used in the proof of a main theorem. For completeness, we present a proof for each lemma.

Lemma 2.1 ([15]). *Let F be a nonempty, closed, and convex subset of H, let P_F be the metric projection from H onto F, and let $\{x_n\}$ be a sequence in H. If*

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{2.3}$$

for all $q \in F$ and $n \in \mathbb{N}$, then $\{P_F x_n\}$ is convergent in F.

Proof. Since H is complete and F is closed in H, it holds that F is complete. Thus, it suffices to show that $\{P_F x_n\}$ is a Cauchy sequence in F. Let $m, n \in \mathbb{N}$ such that $m \geq n$. Since $P_F x_n \in F$, we have from (2.2) that

$$2 \langle x_m - P_F x_m,\ P_F x_m - P_F x_n \rangle \geq 0.$$

By using (2.1), we obtain

$$\|x_m - P_F x_n\|^2 - \|x_m - P_F x_m\|^2 - \|P_F x_m - P_F x_n\|^2 \geq 0.$$

Since $m \geq n$, it follows from the assumption (2.3) that

$$\begin{aligned} \|x_m - P_F x_m\|^2 + \|P_F x_m - P_F x_n\|^2 &\leq \|x_m - P_F x_n\|^2 \\ &\leq \|x_n - P_F x_n\|^2 . \end{aligned} \tag{2.4}$$

Since $\|P_F x_m - P_F x_n\|^2 \geq 0$, we have from (2.4) that

$$\|x_m - P_F x_m\|^2 \leq \|x_n - P_F x_n\|^2$$

for all $m, n \in \mathbb{N}$ such that $m \geq n$. This means that $\left\{ \|x_n - P_F x_n\|^2 \right\}$ is monotone decreasing, and thus, it is convergent. It holds from (2.4) that

$$\|P_F x_m - P_F x_n\|^2 \leq \|x_n - P_F x_n\|^2 - \|x_m - P_F x_m\|^2 .$$

Since the right-hand side converges to 0 as $m, n \to \infty$, we have that $P_F x_m - P_F x_n \to 0$. Thus, $\{P_F x_n\}$ is a Cauchy sequence. This completes the proof. □

Lemma 2.2 ([9]). *Let $x, y, z \in H$, and let $a, b, c \in \mathbb{R}$ such that $a+b+c=1$, where $\mathbb{R}$ stands for the set of real numbers. Then,*

$$\begin{aligned}&\|ax+by+cz\|^2\\&= a\|x\|^2+b\|y\|^2+c\|z\|^2-ab\|x-y\|^2-bc\|y-z\|^2-ca\|z-x\|^2.\end{aligned}$$

Proof. By easy calculations, we have the following:

$$\begin{aligned}\|ax+by+cz\|^2 &= \langle ax+by+cz,\ ax+by+cz\rangle\\&= a^2\|x\|^2+ab\langle x,\ y\rangle+ac\langle x,\ z\rangle\\&\quad +ba\langle y,\ x\rangle+b^2\|y\|^2+bc\langle y,\ z\rangle\\&\quad +ca\langle z,\ x\rangle+cb\langle z,\ y\rangle+c^2\|z\|^2\\&= a^2\|x\|^2+b^2\|y\|^2+c^2\|z\|^2\\&\quad +2ab\langle x,\ y\rangle+2bc\langle y,\ z\rangle+2ca\langle z,\ x\rangle.\end{aligned}$$

Using the relationship $2\langle u,\ v\rangle=\|u\|^2+\|v\|^2-\|u-v\|^2$, we have that

$$\begin{aligned}&\|ax+by+cz\|^2\\=\ &a^2\|x\|^2+b^2\|y\|^2+c^2\|z\|^2\\&+ab\left(\|x\|^2+\|y\|^2-\|x-y\|^2\right)+bc\left(\|y\|^2+\|z\|^2-\|y-z\|^2\right)\\&+ca\left(\|z\|^2+\|x\|^2-\|z-x\|^2\right)\\=\ &a(a+b+c)\|x\|^2+b(a+b+c)\|y\|^2+c(a+b+c)\|z\|^2\\&-ab\|x-y\|^2-bc\|y-z\|^2-ca\|z-x\|^2\end{aligned}$$

Since $a+b+c=1$, we obtain the desired result. □

Letting $c=0$ in Lemma 2.2, we obtain

$$\|ax+by\|^2=a\|x\|^2+b\|y\|^2-ab\|x-y\|^2, \tag{2.5}$$

where $a+b=1$. For the equation (2.5), see Theorem 6.1.2 in Takahashi [13]. Substituting $a=b=1/2$ into (2.5), we have the parallelogram law $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$. Therefore, Lemma 2.2 and (2.5) are generalization of the parallelogram law. It is known that more general equalities than Lemma 2.2 hold; see [6].

The next lemma is also mentioned in Takahashi [13] as Problem 6.2.3. The proof has been developed by many existing studies; see, for example, Kocourek et al. [5].

Lemma 2.3. *Let C be a nonempty, closed and convex subset of H, and let S be a nonexpansive mapping from C into H. Let $\{x_n\}$ be a sequence in C such that $x_n-Sx_n\to 0$ and $x_n\rightharpoonup u$. Then, $u\in F(S)$.*

Proof. First, note that since C is closed and convex, it holds from (A) in Section 2 that it is weakly closed. Since $\{x_n\}\subset C$ and $x_n\rightharpoonup u$, we have that $u \in C$. Since S is a mapping from C into H, there exists an element Su of H. We prove that $Su = u$. Since S is nonexpansive, it holds that

$$\|Sx_n - Su\|^2 \leq \|x_n - u\|^2$$

for all $n \in \mathbb{N}$. Then, we have that

$$\|Sx_n - x_n + x_n - Su\|^2 \leq \|x_n - u\|^2 ,$$

and hence,

$$\|Sx_n - x_n\|^2 + 2\langle Sx_n - x_n,\ x_n - Su\rangle + \|x_n - Su\|^2 \leq \|x_n - u\|^2 .$$

Similarly, we have

$$\|Sx_n - x_n\|^2 + 2\langle Sx_n - x_n,\ x_n - Su\rangle + \|x_n - u\|^2 + 2\langle x_n - u,\ u - Su\rangle + \|u - Su\|^2 \leq \|x_n - u\|^2 .$$

We obtain

$$\|Sx_n - x_n\|^2 + 2\langle Sx_n - x_n,\ x_n - Su\rangle + 2\langle x_n - u,\ u - Su\rangle + \|u - Su\|^2 \leq 0.$$

Since $x_n - Sx_n \to 0$ and $x_n \rightharpoonup u$, we have from (B) in Section 2 that $\langle Sx_n - x_n,\ x_n - Su\rangle \to \langle 0,\ u - Su\rangle = 0$. Thus, it holds in the limit as $n \to \infty$ that $\|u - Su\|^2 \leq 0$, which implies that $u = Su$. □

To use Lemma 2.3, crucial steps we need to show are as follows: (a) a sequence $\{x_n\}\,(\subset C)$ is bounded; and (b) $x_n - Sx_n \to 0$. Once (a) and (b) are demonstrated, we can conclude from (a) that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $u \in H$ such that $x_{n_i} \rightharpoonup u$. Then, we have from (b) and Lemma 2.3 that $u = Su$.

3. Weak Convergence

In this section, we prove a weak convergence theorem, which is a simple version of that of Kondo and Takahashi [6]. As a corollary, we obtain a method to approximate weakly to fixed points of a nonexpansive mapping, which is an alternative to the iteration scheme (1.1).

Theorem 3.1 ([6]). *Let C be a nonempty, closed and convex subset of H, and let S and T be nonexpansive mappings from C into itself. Suppose that $F(S) \cap F(T)$ is nonempty. Let $\alpha, \beta \in (0,1)$ such that $\alpha \leq \beta$, and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $(0,1)$ such that $a_n + b_n + c_n = 1$ and $0 < \alpha \leq a_n, b_n, c_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = a_n x_n + b_n S x_n + c_n T x_n \quad \textit{for all } n \in \mathbb{N},$$

where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to a common fixed point $\overline{x} \equiv \lim_{n\to\infty} P_F x_n \in F(S) \cap F(T)$, where P_F is the metric projection from H onto $F(S) \cap F(T)$.

Note that when sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are constant coefficients of a convex combination, the required conditions on the sequences are satisfied.

Proof. First, let us verify that there exists the metric projection P_F from H onto $F(S) \cap F(T)$. Since S and T be nonexpansive, $F(S)$ and $F(T)$ are closed and convex subsets of C. Thus, the intersection $F(S) \cap F(T)$ is also closed and convex in C. Since $F(S) \cap F(T) \neq \emptyset$ is assumed, there exists the metric projection P_F from H onto $F(S) \cap F(T)$.

Next, we show that a sequence $\{\|x_n - q\|\}$ is monotone decreasing for all $q \in F(S) \cap F(T)$. Indeed, since $a_n + b_n + c_n = 1$, S and T are quasi-nonexpansive and $q \in F(S) \cap F(T)$, we obtain

$$\begin{aligned} \|x_{n+1} - q\| &\equiv \|a_n x_n + b_n S x_n + c_n T x_n - q\| \\ &= \|a_n x_n + b_n S x_n + c_n T x_n - (a_n + b_n + c_n) q\| \\ &= \|a_n (x_n - q) + b_n (S x_n - q) + c_n (T x_n - q)\| \\ &\leq a_n \|x_n - q\| + b_n \|S x_n - q\| + c_n \|T x_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\| \end{aligned}$$

for all $n \in \mathbb{N}$. This means that $\{\|x_n - q\|\}$ is monotone decreasing for all $q \in F(S) \cap F(T)$. As consequences, we obtain the following: (i) The sequence $\{\|x_n - q\|\}$ is convergent in $\mathbb{R}$ for all $q \in F(S) \cap F(T)$. (ii) From Lemma 2.1, $\{P_F x_n\}$ is convergent in $F(S) \cap F(T)$. We denote the limit by $\overline{x}$, that is, $\overline{x} \equiv \lim_{n\to\infty} P_F x_n$. (iii) The sequence $\{x_n\}$ is bounded since $\{\|x_n - q\|\}$ is convergent.

The following inequality is necessary to complete the proof:

$$\begin{aligned} &a_n b_n \|x_n - S x_n\|^2 + b_n c_n \|S x_n - T x_n\|^2 + c_n a_n \|T x_n - x_n\|^2 \\ &\leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \end{aligned} \tag{3.1}$$

for any $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, since $a_n + b_n + c_n = 1$, it holds from Lemma 2.2 that

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ \equiv\ &\|a_n x_n + b_n S x_n + c_n T x_n - q\|^2 \\ =\ &\|a_n (x_n - q) + b_n (S x_n - q) + c_n (T x_n - q)\|^2 \\ =\ &a_n \|x_n - q\|^2 + b_n \|S x_n - q\|^2 + c_n \|T x_n - q\|^2 \\ &-a_n b_n \|x_n - S x_n\|^2 - b_n c_n \|S x_n - T x_n\|^2 - c_n a_n \|T x_n - x_n\|^2 \\ \leq\ &a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\ &-a_n b_n \|x_n - S x_n\|^2 - b_n c_n \|S x_n - T x_n\|^2 - c_n a_n \|T x_n - x_n\|^2 \\ =\ &\|x_n - q\|^2 \\ &-a_n b_n \|x_n - S x_n\|^2 - b_n c_n \|S x_n - T x_n\|^2 - c_n a_n \|T x_n - x_n\|^2 . \end{aligned}$$

Therefore, we obtain (3.1).

Since $\{\|x_n - q\|\}$ is convergent and it is assumed that $0 < \alpha \leq a_n, b_n, c_n, d_n \leq \beta < 1$ for all $n \in \mathbb{N}$, we obtain from (3.1) that

$$x_n - S x_n \to 0 \quad \text{and} \quad T x_n - x_n \to 0. \tag{3.2}$$

Our aim is to show that $x_n \rightharpoonup \overline{x}\,(\equiv \lim_{n\to\infty} P_F x_n)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and $u \in H$ such that $x_{n_j} \rightharpoonup u$. Since the mapping S and T are nonexpansive, we have from (3.2) and Lemma 2.3 that $u \in F(S) \cap F(T)$.

We prove that $u\left(= \text{the weak limit of } \{x_{n_j}\}\right) = \overline{x}\,(\equiv \lim_{n\to\infty} P_F x_n)$. Since $u \in F(S) \cap F(T)$, it holds from (2.2) that

$$\langle x_n - P_F x_n,\ P_F x_n - u\rangle \geq 0$$

for all $n \in \mathbb{N}$. Therefore,

$$\langle x_n - P_F x_n,\ P_F x_n - \overline{x} + \overline{x} - u\rangle \geq 0.$$

By using Schwarz's inequality, we have that

$$\begin{aligned} \langle x_n - P_F x_n,\ u - \overline{x}\rangle &\leq \langle x_n - P_F x_n,\ P_F x_n - \overline{x}\rangle \\ &\leq \|x_n - P_F x_n\| \,\|P_F x_n - \overline{x}\|\,. \end{aligned} \tag{3.3}$$

Since the sequence $\{x_n\}$ is bounded and P_F is nonexpansive, $\{P_F x_n\}$ is also bounded. Indeed, it holds that

$$\|P_F x_n - q\| \leq \|P_F x_n - P_F q\| \leq \|x_n - q\|$$

for any $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This means that $\{P_F x_n\}$ is bounded since $\{x_n\}$ is bounded. Define $L \equiv \sup_{n\in\mathbb{N}} \|x_n - P_F x_n\|$. Then, L is a real number. From (3.3), we have that

$$\langle x_n - P_F x_n,\ u - \overline{x}\rangle \leq L\,\|P_F x_n - \overline{x}\|$$

for all $n \in \mathbb{N}$. Thus,

$$\left\langle x_{n_j} - P_F x_{n_j},\ u - \overline{x}\right\rangle \leq L\left\|P_F x_{n_j} - \overline{x}\right\|$$

for all $j \in \mathbb{N}$. Since $x_{n_j} \rightharpoonup u$ and $P_F x_n \to \overline{x}$, we obtain $\langle u - \overline{x},\ u - \overline{x}\rangle \leq 0$, which means that $u = \overline{x}$. From (C) in Section 2, we have that $x_n \rightharpoonup \overline{x}$. This completes the proof. □

Let $T = S^2$ in Theorem 3.1. Since T is nonexpansive and $F(S) \cap F(T) = F(S) \cap F(S^2) = F(S)$, we obtain the following corollary:

Corollary 3.1. *Let C be a nonempty, closed and convex subset of H, and let S be nonexpansive mappings from C into itself. Suppose that $F(S)$ is nonempty. Let $\alpha, \beta \in (0,1)$ such that $\alpha \leq \beta$, and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $(0,1)$ such that $a_n + b_n + c_n = 1$ and $0 < \alpha \leq a_n, b_n, c_n \leq \beta < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = a_n x_n + b_n S x_n + c_n S^2 x_n \quad \textit{for all } n \in \mathbb{N}, \tag{3.4}$$

where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to a fixed point $\widehat{x} \equiv \lim_{n\to\infty} P_F x_n \in F(S)$, where P_F is the metric projection from H onto $F(S)$.

Similarly, (3.4) can be replaced by

$$x_{n+1} = a_n x_n + b_n S x_n + c_n S^k x_n, \ \text{ where } k \in \mathbb{N} \cup \{0\}\,.$$

4. Strong Convergence

This section presents a strong convergence theorem, which is a simple version of that of Kondo and Takahashi [6].

Theorem 4.1 ([6]). *Let C be a nonempty, closed and convex subset of H, and let S and T be nonexpansive mappings from C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\alpha, \beta \in (0,1)$ such that $\alpha \leq \beta$, and let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$a_n + b_n + c_n = 1, \quad 0 < \alpha \leq a_n, b_n, c_n \leq \beta < 1 \quad \text{for all } n \in \mathbb{N}.$$

Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n x + (1 - \lambda_n)(a_n x_n + b_n S x_n + c_n T x_n) \in C \quad \text{for all } n \in \mathbb{N},$$

where $x_1 = x \in C$ is given. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point $\overline{x} \equiv P_F x \in F(S) \cap F(T)$, where P_F is the metric projection from H onto $F(S) \cap F(T)$.

As a corollary, we obtain a method to approximate strongly to fixed points of a nonexpansive mapping, which is an alternative method to Halpern's type iteration (1.2).

Corollary 4.1. *Let C be a nonempty, closed and convex subset of H, and let S be a nonexpansive mapping from C into itself such that $F(S) \neq \emptyset$. Let $\alpha, \beta \in (0,1)$ such that $\alpha \leq \beta$, and let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$a_n + b_n + c_n = 1, \quad 0 < \alpha \leq a_n, b_n, c_n \leq \beta < 1 \quad \text{for all } n \in \mathbb{N}.$$

Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n x + (1 - \lambda_n)\left(a_n x_n + b_n S x_n + c_n S^2 x_n\right) \in C \quad \text{for all } n \in \mathbb{N}, \tag{4.1}$$

where $x_1 = x \in C$ is given. Then, the sequence $\{x_n\}$ converges strongly to a fixed point $\widehat{x} \equiv P_F x \in F(S)$, where P_F is the metric projection from H onto $F(S)$.

As Corollary 3.1, (4.1) can be replaced by

$$x_{n+1} = \lambda_n x + (1 - \lambda_n)\left(a_n x_n + b_n S x_n + c_n S^k x_n\right), \quad \text{where } k \in \mathbb{N} \cup \{0\}.$$

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