

# On invariants for handlebody-knots and spatial surfaces

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## Part I

# Handlebody-knots

In this part, we introduce the  $f$ -twisted Alexander invariant for handlebody-links. A *handlebody-link* [7] is a disjoint union of handlebodies embedded in the 3-sphere  $S^3$ . A *handlebody-knot* is a one component handlebody-link. In this paper, we assume that every component of a handlebody-link is of genus at least 1. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of  $S^3$  which sends one to the other. The  $f$ -twisted Alexander invariant is an invariant for handlebody-links derived from a linear extension of a multiple conjugation quandle, which is an algebra whose axioms are motivated from Reidemeister moves for handlebody-links. As an application, we demonstrate that our invariant detects 4-move equivalence classes of handlebody-links. This is a joint work with Atsushi Ishii.

## 1 Multiple conjugation quandles and MCQ Alexander pairs

A *quandle* [12, 14] is a non-empty set  $Q$  equipped with a binary operation  $\triangleleft : Q \times Q \rightarrow Q$  satisfying the following axioms:

(Q1) For any  $a \in Q$ ,  $a \triangleleft a = a$ .

(Q2) For any  $a \in Q$ , the map  $\triangleleft a : Q \rightarrow Q$  defined by  $\triangleleft a(x) = x \triangleleft a$  is bijective.

(Q3) For any  $a, b, c \in Q$ ,  $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ .

We denote  $(\triangleleft a)^n : Q \rightarrow Q$  by  $\triangleleft^n a$  for  $n \in \mathbb{Z}$ .

Let  $G$  be a group and  $n$  an integer. We define a binary operation  $\triangleleft$  on  $G$  by  $a \triangleleft b = b^{-1}ab$ . Then,  $\text{Conj } G := (G, \triangleleft)$  is a quandle, called the *conjugation quandle* of  $G$ . We define another binary operation  $\triangleleft$  on  $G$  by  $a \triangleleft b = ba^{-1}b$ . Then,  $\text{Core } G := (G, \triangleleft)$  is a quandle, called the *core quandle* of  $G$ . For a positive integer  $n$ , we denote by  $\mathbb{Z}_n$  the cyclic group  $\mathbb{Z}/n\mathbb{Z}$  of order  $n$ . We define a binary operation  $\triangleleft$  on  $\mathbb{Z}_n$  by  $a \triangleleft b = 2b - a$ . Then,  $R_n := (\mathbb{Z}_n, \triangleleft)$  is a quandle, called the *dihedral quandle* of order  $n$ . Let  $M$  be an  $R[t^{\pm 1}]$ -module for a commutative ring  $R$ . We define a binary operation  $\triangleleft$  on  $M$  by  $a \triangleleft b = ta + (1 - t)b$ . Then  $M$  is a quandle, called an *Alexander quandle*.

We define the *type* of a quandle  $Q$  by

$$\text{type } Q = \min\{n \in \mathbb{Z}_{>0} \mid x \triangleleft^n y = x \text{ (for any } x, y \in Q)\},$$

where we set  $\min \emptyset := \infty$  for the empty set  $\emptyset$ . We note that  $(Q, \triangleleft^i)$  is also a quandle for any  $i \in \mathbb{Z}$ , and any finite quandle is of finite type.

Let  $(Q, \triangleleft)$  be a quandle and  $R$  a ring. Throughout this paper, we assume that every ring has the multiplicative identity  $1 \neq 0$ . The pair of maps  $f_1, f_2 : Q \times Q \rightarrow R$  is *Alexander pair* [10] if  $f_1$  and  $f_2$  satisfy the following conditions:

- For any  $a \in Q$ ,  $f_1(a, a) + f_2(a, a) = 1$ .
- For any  $a, b \in Q$ ,  $f_1(a, b)$  is invertible.
- For any  $a, b, c \in Q$ ,

$$\begin{aligned} f_1(a \triangleleft b, c) f_1(a, b) &= f_1(a \triangleleft c, b \triangleleft c) f_1(a, c), \\ f_1(a \triangleleft b, c) f_2(a, b) &= f_2(a \triangleleft c, b \triangleleft c) f_1(b, c), \\ f_2(a \triangleleft b, c) &= f_1(a \triangleleft c, b \triangleleft c) f_2(a, c) + f_2(a \triangleleft c, b \triangleleft c) f_2(b, c). \end{aligned}$$

An Alexander pair is a dynamical cocycle [1] corresponding to a linear extension of a quandle. Many examples of Alexander pairs are given in [10].

**Definition 1.1** ([8]). A *multiple conjugation quandle (MCQ)*  $X$  is a disjoint union of groups  $G_\lambda (\lambda \in \Lambda)$  with a binary operation  $\triangleleft : X \times X \rightarrow X$  satisfying the following axioms:

- For any  $a, b \in G_\lambda$ ,  $a \triangleleft b = b^{-1}ab$ .
- For any  $x \in X$  and  $a, b \in G_\lambda$ ,  $x \triangleleft e_\lambda = x$  and  $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$ , where  $e_\lambda$  is the identity of  $G_\lambda$ .
- For any  $x, y, z \in X$ ,  $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ .
- For any  $x \in X$  and  $a, b \in G_\lambda$ ,  $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$ , where  $a \triangleleft x, b \triangleleft x \in G_\mu$  for some  $\mu \in \Lambda$ .

We remark that an MCQ itself is a quandle. For two MCQs  $X_1 = \bigsqcup_{\lambda \in \Lambda} G_\lambda$  and  $X_2 = \bigsqcup_{\mu \in M} G_\mu$ , an *MCQ homomorphism*  $f : X_1 \rightarrow X_2$  is defined to be a map from  $X_1$  to  $X_2$  satisfying  $f(x \triangleleft y) = f(x) \triangleleft f(y)$  for any  $x, y \in X_1$  and  $f(ab) = f(a)f(b)$  for any  $\lambda \in \Lambda$  and  $a, b \in G_\lambda$ . An MCQ homomorphism  $\rho : X_1 \rightarrow X_2$  is also called an *MCQ representation* of  $X_1$  to  $X_2$ . We denote by  $\text{Hom}(X_1, X_2)$  the set of MCQ homomorphisms from  $X_1$  to  $X_2$ . We call a bijective MCQ homomorphism an *MCQ isomorphism*. When there exists an MCQ isomorphism from  $X_1$  to  $X_2$ , we call that  $X_1$  and  $X_2$  are *isomorphic*, denoted by  $X_1 \cong X_2$ . Let  $\rho_1 : X_1 \rightarrow Y$  and  $\rho_2 : X_2 \rightarrow Y$  be MCQ representations. We say  $(X_1, \rho_1)$  and  $(X_2, \rho_2)$  are *isomorphic*, denoted by  $(X_1, \rho_1) \cong (X_2, \rho_2)$ , if there exists an MCQ isomorphism  $f : X_1 \rightarrow X_2$  such that  $\rho_1 = \rho_2 \circ f$ .

Let  $Q$  be a quandle. Then  $Q \times \mathbb{Z}_{\text{type } Q} = \bigsqcup_{x \in Q} (\{x\} \times \mathbb{Z}_{\text{type } Q})$  is an MCQ, called the *associated MCQ* of  $Q$ , with

$$(x, a) \triangleleft (y, b) := (x \triangleleft^b y, a), \quad (x, a)(x, b) := (x, a + b)$$

for any  $x, y \in Q$  and  $a, b \in \mathbb{Z}_{\text{type } Q}$ , where we put  $\mathbb{Z}_\infty := \mathbb{Z}$ .

**Definition 1.2** ([15]). Let  $X = \bigsqcup_{\lambda \in \Lambda} G_\lambda$  be an MCQ and  $R$  a ring. The pair  $(f_1, f_2)$  of maps  $f_1, f_2 : X \times X \rightarrow R$  is an *MCQ Alexander pair* if  $f_1$  and  $f_2$  satisfy the following conditions:

- For any  $a, b \in G_\lambda$ ,

$$f_1(a, b) + f_2(a, b) = f_1(a, a^{-1}b). \quad (1-i)$$

- For any  $a, b \in G_\lambda$  and  $x \in X$ ,

$$f_1(a, x) = f_1(b, x), \quad (2-i)$$

$$f_2(ab, x) = f_2(a, x) + f_1(b \triangleleft x, a^{-1} \triangleleft x) f_2(b, x). \quad (2-ii)$$

- For any  $x \in X$  and  $a, b \in G_\lambda$ ,

$$f_1(x, e_\lambda) = 1, \quad (3-i)$$

$$f_1(x, ab) = f_1(x \triangleleft a, b) f_1(x, a), \quad (3-ii)$$

$$f_2(x, ab) = f_1(x \triangleleft a, b) f_2(x, a). \quad (3-iii)$$

- For any  $x, y, z \in X$ ,

$$f_1(x \triangleleft y, z) f_1(x, y) = f_1(x \triangleleft z, y \triangleleft z) f_1(x, z), \quad (4-i)$$

$$f_1(x \triangleleft y, z) f_2(x, y) = f_2(x \triangleleft z, y \triangleleft z) f_1(y, z), \quad (4-ii)$$

$$f_2(x \triangleleft y, z) = f_1(x \triangleleft z, y \triangleleft z) f_2(x, z) + f_2(x \triangleleft z, y \triangleleft z) f_2(y, z). \quad (4-iii)$$

By using the following proposition, we can construct MCQ Alexander pairs from Alexander pairs.

**Proposition 1.3.** *Let  $(Q, \triangleleft)$  be a quandle and assume  $k := \text{type } Q < \infty$ . Let  $R$  be a ring and let  $(f_1, f_2)$  be an Alexander pair of maps  $f_1, f_2 : Q \times Q \rightarrow R$  satisfying*

$$\prod_{i=1}^k f_1(x \triangleleft^{k-i} y, y) = 1 \quad \text{and} \quad \sum_{i=1}^k f_1(x, x)^i = 0$$

for any  $x, y \in Q$ . Let  $X := Q \times \mathbb{Z}_k$  be the associated MCQ of  $Q$ . We define maps  $\tilde{f}_1, \tilde{f}_2 : X \times X \rightarrow R$  by

$$\begin{aligned} \tilde{f}_1((x, a), (y, b)) &= \prod_{i=1}^{\bar{b}} f_1(x \triangleleft^{b-i} y, y), \\ \tilde{f}_2((x, a), (y, b)) &= \left( \prod_{i=1}^{\overline{b-1}} f_1(x \triangleleft^{b-i} y, y) \right) \sum_{j=1}^{\bar{a}} f_1(x \triangleleft y, x \triangleleft y)^{j-a} f_2(x, y), \end{aligned}$$

where for any  $l \in \mathbb{Z}$ , we denote by  $\bar{l}$  the integer such that  $1 \leq \bar{l} \leq k$  and  $l \equiv \bar{l} \pmod{k}$ . Then the pair  $(\tilde{f}_1, \tilde{f}_2)$  is an MCQ Alexander pair.

## 2 MCQ presentations and the fundamental MCQ of a handlebody-link

In this section, we review the notions of MCQ presentations and the fundamental MCQ of a handlebody-link. For details see [9].

We denote by  $F_{\text{Grp}}(S)$  the free group on a set  $S$ . Let  $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$  be a set of pairwise disjoint sets, and put  $S := \bigcup_{\lambda \in \Lambda} S_\lambda$ . For  $(a, x), (b, y) \in \bigcup_{\lambda \in \Lambda} F_{\text{Grp}}(S_\lambda) \times F_{\text{Grp}}(S)$ , we write  $(a, x) \sim (b, y)$  if there exists  $c \in \bigcup_{\lambda \in \Lambda} F_{\text{Grp}}(S_\lambda)$  such that  $b = cac^{-1}$  and  $y = cx$ . Then  $\sim$  is an equivalence relation on  $\bigcup_{\lambda \in \Lambda} F_{\text{Grp}}(S_\lambda) \times F_{\text{Grp}}(S)$ . We define  $F_{\text{MCQ}}(S_\Lambda) := \bigcup_{\lambda \in \Lambda} F_{\text{Grp}}(S_\lambda) \times F_{\text{Grp}}(S) / \sim$  and

$$F_{\text{Grp}}(S; S_\Lambda) := \left\{ a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \in F_{\text{Grp}}(S) \left| \begin{array}{l} n \geq 0, \varepsilon_1, \dots, \varepsilon_n \in \{1, -1\}, \\ a_1 \in S - S_\lambda, a_2, \dots, a_n \in S, \\ \text{If } a_i = a_{i+1}, \text{ then } \varepsilon_i = \varepsilon_{i+1} \text{ for each } i. \end{array} \right. \right\}.$$

Set  $\bar{\Lambda} := \bigcup_{\lambda \in \Lambda} (\{\lambda\} \times F_{\text{Grp}}(S; S_\Lambda))$ , and we define

$$F_{\text{Grp}}(S_\lambda) \triangleleft x := \{[(a, x)] \in F_{\text{MCQ}}(S_\Lambda) \mid a \in F_{\text{Grp}}(S_\lambda)\}$$

for  $x \in F_{\text{Grp}}(S)$ . Then we have  $F_{\text{MCQ}}(S_\Lambda) = \bigsqcup_{(\lambda, x) \in \bar{\Lambda}} F_{\text{Grp}}(S_\lambda) \triangleleft x$ , which is an MCQ with

$$\begin{aligned} [(a, x)] \triangleleft [(b, y)] &:= [(a, xy^{-1}by)] && ([ (a, x), [(b, y)] \in F_{\text{MCQ}}(S_\Lambda)), \\ [(a, x)][(b, x)] &:= [(ab, x)] && (a, b \in F_{\text{Grp}}(S_\lambda), x \in F_{\text{Grp}}(S)). \end{aligned}$$

We call  $F_{\text{MCQ}}(S_\Lambda)$  the *free multiple conjugation quandle (free MCQ)* on  $S_\Lambda$ . By the injection  $\iota : S \rightarrow F_{\text{MCQ}}(S_\Lambda)$  defined by  $\iota(a) = [(a, 1_{F_{\text{Grp}}(S)})]$ , we regard  $S \subset F_{\text{MCQ}}(S_\Lambda)$  and often denote  $[(a, 1_{F_{\text{Grp}}(S)})]$  by  $a$ . Then any element  $[(a, a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n})]$  in  $F_{\text{MCQ}}(S_\Lambda)$  can be represented as  $(\cdots (a \triangleleft a_1^{\varepsilon_1}) \triangleleft \cdots) \triangleleft a_n^{\varepsilon_n}$ .

For any MCQ  $X$ , there exist a set of pairwise disjoint sets  $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$  and  $R \subset F_{\text{MCQ}}(S_\Lambda) \times F_{\text{MCQ}}(S_\Lambda)$  such that  $X$  is isomorphic to  $\langle S_\Lambda \mid R \rangle$ , which is also denoted  $\langle S_\lambda \ (\lambda \in \Lambda) \mid R \rangle$  [9]. We then call  $\langle S_\Lambda \mid R \rangle$  a *presentation* of  $X$ ,  $S_\Lambda$  the *generating set* of  $\langle S_\Lambda \mid R \rangle$  and an element of  $R$  a *relator* of  $\langle S_\Lambda \mid R \rangle$ . A relator  $(a, b)$  is also written as  $a = b$ . A presentation  $\langle S_\Lambda \mid R \rangle$  is called a *finite presentation* if  $\bigcup_{\lambda \in \Lambda} S_\lambda$  and  $R$  are finite. For a finitely presented MCQ, we often write

$$\begin{aligned} &\langle x_{1,1}, \dots, x_{1,n_1}; \dots; x_{l,1}, \dots, x_{l,n_l} \mid r_1, \dots, r_m \rangle \\ &:= \langle \{x_{1,1}, \dots, x_{1,n_1}\}, \dots, \{x_{l,1}, \dots, x_{l,n_l}\} \mid \{r_1, \dots, r_m\} \rangle. \end{aligned}$$

In the following, we recall the fundamental MCQ of a handlebody-link and its Wirtinger presentation. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph whose regular neighborhood is the handlebody-link, where a spatial trivalent graph is a finite trivalent graph embedded in  $S^3$ . In this paper, a trivalent graph may contain circle components. Let  $D$  be a diagram of a handlebody-link. A *Y-orientation* of  $D$  is a collection of orientations of all edges of  $D$  without sources and sinks with respect to the orientation as shown in Figure 1, where an edge of  $D$  is a piece of a curve each of whose endpoints is a vertex. In this paper, a circle component of  $D$  is also regarded as an



Figure 1: Y-orientations.

edge of  $D$ . We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation counterclockwise by  $\pi/2$  on a diagram. It is known that every diagram has a Y-orientation.

Let  $H$  be a handlebody-link represented by a Y-oriented diagram  $D$ . We denote by  $C(D)$ ,  $V(D)$  and  $\mathcal{A}(D)$  the sets of crossings, vertices and arcs of  $D$ , respectively. We denote by  $u_c, v_c, w_c, u_\tau, v_\tau, w_\tau$  the arcs around  $c \in C(D)$  and  $\tau \in V(D)$  as illustrated in Figure 2, respectively. We denote by  $\mathcal{A}^\sqcup(D)$  the quotient set of  $\mathcal{A}(D)$  by the equivalence relation generated by  $\bigcup_{\tau \in V(D)} \{u_\tau, v_\tau, w_\tau\}^2$ . For each  $c \in C(D)$  and  $\tau \in V(D)$ , we denote by  $r_c$  and  $r_\tau$  the relators  $(u_c \triangleleft v_c, w_c)$  and  $(u_\tau v_\tau, w_\tau)$ , respectively. Then we define

$$MCQ(D) := \langle \mathcal{A}^\sqcup(D) \mid r_c, r_\tau \ (c \in C(D), \tau \in V(D)) \rangle.$$

The isomorphism class of  $MCQ(D)$  does not depend on the choice of a diagram  $D$  of  $H$  and its Y-orientation [9]. We then define  $MCQ(H) := MCQ(D)$  and call it the *fundamental MCQ* of  $H$ . This presentation is called the *Wirtinger presentation of  $MCQ(H)$  with respect to  $D$* .

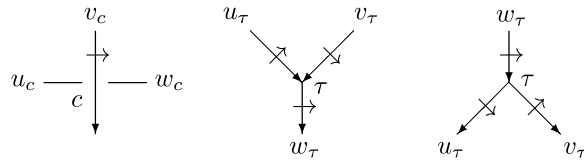


Figure 2: Notations of arcs.

Let  $D$  be a Y-oriented diagram of a handlebody-link  $H$  and let  $X$  be an MCQ. An  $X$ -coloring of  $D$  is a map  $C : \mathcal{A}(D) \rightarrow X$  satisfying the conditions

$$C(u_c) \triangleleft C(v_c) = C(w_c) \quad \text{and} \quad C(u_\tau)C(v_\tau) = C(w_\tau)$$

for each  $c \in C(D)$  and  $\tau \in V(D)$ . We denote by  $\text{Col}_X(D)$  the set of  $X$ -colorings of  $D$ . An  $X$ -coloring of  $D$  can be regarded as an MCQ representation of  $MCQ(H)$  to  $X$ , that is, we can identify  $\text{Col}_X(D)$  with  $\text{Hom}(MCQ(H), X)$ .

Let  $D$  be a Y-oriented diagram of a handlebody-link  $H$  and  $D'$  a Y-oriented diagram of  $H$  obtained by changing the Y-orientation of  $D$ . We then obtain the MCQ isomorphism  $f_{(D,D')} : MCQ(D) \rightarrow MCQ(D')$  sending  $x$  into  $x^{\varepsilon(x)}$  for any  $x \in \mathcal{A}(D)$ , where  $\varepsilon(x) = 1$  if the Y-orientations of  $D$  and  $D'$  coincide on  $x$ ; otherwise  $\varepsilon(x) = -1$ . Moreover, let  $D''$  a Y-oriented diagram of  $H$  obtained by applying one of Reidemeister moves preserving the Y-orientation to  $D$  once. We then obtain a unique MCQ isomorphism  $f_{(D,D'')} : MCQ(D) \rightarrow MCQ(D'')$  sending  $x$  into  $x$  for any  $x \in \mathcal{A}(D \cap D'')$ , where  $\mathcal{A}(D \cap D'')$  denotes the set of arcs in the outside of the disk where the move is applied. Let  $H$  and  $H'$  be handlebody-links represented by Y-oriented diagrams  $D$  and  $D'$ , respectively. Let  $\rho : MCQ(D) \rightarrow X$

and  $\rho' : MCQ(D') \rightarrow X$  be MCQ representations. Then  $(H, \rho)$  and  $(H', \rho')$  are *equivalent*, denoted by  $(H, \rho) \cong (H', \rho')$ , if there exists a sequence  $D = D_1 \leftrightarrow \dots \leftrightarrow D_n = D'$  of Reidemeister moves and Y-orientation changes such that  $\rho' = \rho \circ f_{(D_1, D_2)}^{-1} \circ \dots \circ f_{(D_{n-1}, D_n)}^{-1}$ . If  $H$  and  $H'$  are equivalent, then for any MCQ representation  $\rho : MCQ(D) \rightarrow X$ , there exists a unique MCQ representation  $\rho' : MCQ(D') \rightarrow X$  such that  $(H, \rho) \cong (H', \rho')$ .

### 3 $f$ -twisted Alexander invariants for handlebody-links

Let  $S_\Lambda = \{S_\lambda \mid \lambda \in \Lambda\}$  be a finite set of pairwise disjoint finite sets, and put  $S := \bigcup_{\lambda \in \Lambda} S_\lambda = \{x_1, \dots, x_n\}$ . Let  $X = \langle S_\Lambda \mid \{r_1, \dots, r_m\} \rangle$  be a finitely presented MCQ. Let  $F_{MCQ}(S_\Lambda)$  be the free MCQ on  $S_\Lambda$  and  $\text{pr} : F_{MCQ}(S_\Lambda) \rightarrow X$  be the canonical projection. We often omit ‘‘pr’’ to represent  $\text{pr}(x)$  as  $x$ . Let  $f = (f_1, f_2)$  be an MCQ Alexander pair of maps  $f_1, f_2 : X \times X \rightarrow R$ , and put  $\bar{\Lambda} := \bigcup_{\lambda \in \Lambda} (\{\lambda\} \times F_{\text{Grp}}(S; S_\lambda))$ . For any  $(\lambda, x) \in \bar{\Lambda}$ , we set  $G_{\lambda, x} := F_{\text{Grp}}(S_\lambda) \triangleleft x$ , that is,  $F_{MCQ}(S_\Lambda) = \bigsqcup_{(\lambda, x) \in \bar{\Lambda}} G_{\lambda, x}$ .

**Definition 3.1.** For  $j \in \{1, \dots, n\}$ , the  $f$ -derivative with respect to  $x_j$  is a map  $\frac{\partial f}{\partial x_j} : F_{MCQ}(S_\Lambda) \rightarrow R$  satisfying

$$\begin{aligned} \frac{\partial f}{\partial x_j}(x \triangleleft y) &= f_1(x, y) \frac{\partial f}{\partial x_j}(x) + f_2(x, y) \frac{\partial f}{\partial x_j}(y), \\ \frac{\partial f}{\partial x_j}(ab) &= \frac{\partial f}{\partial x_j}(a) + f_1(a, a^{-1}) \frac{\partial f}{\partial x_j}(b), \\ \frac{\partial f}{\partial x_j}(x_i) &= \delta_{ij} \end{aligned}$$

for any  $x, y \in F_{MCQ}(S_\Lambda)$ ,  $a, b \in G_{\lambda, x}$  and  $i \in \{1, \dots, n\}$ , where  $\delta_{ij}$  denotes the Kronecker delta.

Let  $R$  be a ring. We denote by  $M(m, n; R)$  the set of  $m \times n$  matrices over  $R$ . We say that two matrices  $A_1$  and  $A_2$  over  $R$  are *equivalent*, denoted by  $A_1 \sim A_2$ , if they are related by a finite sequence of the following transformations:

- $(\mathbf{a}_1, \dots, \mathbf{a}_i, \dots, \mathbf{a}_j, \dots, \mathbf{a}_n) \leftrightarrow (\mathbf{a}_1, \dots, \mathbf{a}_i + \mathbf{a}_j r, \dots, \mathbf{a}_j, \dots, \mathbf{a}_n) \quad (r \in R),$
- $\begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_i \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \leftrightarrow \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_i + r \mathbf{a}_j \\ \vdots \\ \mathbf{a}_j \\ \vdots \\ \mathbf{a}_n \end{pmatrix} \quad (r \in R),$
- $A \leftrightarrow \begin{pmatrix} A \\ \mathbf{0} \end{pmatrix},$
- $A \leftrightarrow \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix}.$

Let  $R$  be a commutative ring, and let  $A \in M(m, n; R)$ . A  $k$ -minor of  $A$  is the determinant of a  $k \times k$  submatrix of  $A$ . For any  $d \in \mathbb{Z}_{\geq 0}$ , the  $d$ -th elementary ideal  $E_d(A)$  of

$A$  is the ideal of  $R$  generated by all  $(n - d)$ -minors of  $A$  if  $n - m \leq d < n$ , and

$$E_d(A) := \begin{cases} 0 & \text{if } d < n - m, \\ R & \text{if } n \leq d. \end{cases}$$

Suppose that  $R$  is a GCD domain. Then the  $d$ -th Alexander invariant  $\Delta_d(A)$  of  $A$  is the greatest common divisor of all  $(n - d)$ -minors of  $A$  if  $n - m \leq d < n$ , and

$$\Delta_d(A) := \begin{cases} 0 & \text{if } d < n - m, \\ 1 & \text{if } n \leq d. \end{cases}$$

We remark that  $\Delta_d(A)$  coincides with the greatest common divisor of generators of  $E_d(A)$  and is determined up to unit multiple. If  $A \sim B$ , then  $E_d(A) = E_d(B)$  and  $\Delta_d(A) \doteq \Delta_d(B)$ , where “ $\doteq$ ” means “is equal to, up to multiplication by a unit”. See [6] for more details.

For an MCQ representation  $\rho : X \rightarrow Y$  and an MCQ Alexander pair  $f = (f_1, f_2)$  of maps  $f_1, f_2 : Y \times Y \rightarrow R$ , we set  $f \circ (\rho \times \rho) := (f_1 \circ (\rho \times \rho), f_2 \circ (\rho \times \rho))$ . Then  $f \circ (\rho \times \rho)$  is also an MCQ Alexander pair. For a relator  $r = (r_1, r_2)$ , we define

$$\frac{\partial f}{\partial x_j}(r) := \frac{\partial f}{\partial x_j}(r_1) - \frac{\partial f}{\partial x_j}(r_2).$$

**Definition 3.2.** Let  $H$  be a handlebody-link and

$$MCQ(D) = \langle x_1, \dots, x_k; \dots; x_l, \dots, x_n \mid r_1, \dots, r_m \rangle$$

the Wirtinger presentation of  $MCQ(H)$  with respect to a  $Y$ -oriented diagram  $D$  of  $H$ . Let  $\rho : MCQ(H) = MCQ(D) \rightarrow X$  be an MCQ representation, which can be regarded as an  $X$ -coloring of  $D$ . Let  $f = (f_1, f_2)$  be an MCQ Alexander pair of maps  $f_1, f_2 : X \times X \rightarrow R$ . Then we define the  $f$ -twisted Alexander matrix of  $(H, \rho)$  (with respect to  $D$ ) by

$$A(H, \rho; f_1, f_2) = \begin{pmatrix} \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_1) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_1}(r_m) & \cdots & \frac{\partial_{f \circ (\rho \times \rho)}}{\partial x_n}(r_m) \end{pmatrix},$$

the  $d$ -th  $f$ -twisted elementary ideal of  $(H, \rho)$  by

$$E_d(H, \rho; f_1, f_2) = E_d(A(H, \rho; f_1, f_2))$$

if  $R$  is a commutative ring, and the  $d$ -th  $f$ -twisted Alexander invariant of  $(H, \rho)$  by

$$\Delta_d(H, \rho; f_1, f_2) = \Delta_d(A(H, \rho; f_1, f_2))$$

if  $R$  is a GCD domain.

**Theorem 3.3.** Let  $H$  and  $H'$  be handlebody-links. Let  $\rho : MCQ(H) \rightarrow X$  and  $\rho' : MCQ(H') \rightarrow X$  be MCQ representations. Let  $(f_1, f_2)$  be an MCQ Alexander pair of maps  $f_1, f_2 : X \times X \rightarrow R$ . If  $(H, \rho) \cong (H', \rho')$ , then we have

$$A(H, \rho; f_1, f_2) \sim A(H', \rho'; f_1, f_2).$$

Especially, we have

$$E_d(H, \rho; f_1, f_2) = E_d(H', \rho'; f_1, f_2)$$

if  $R$  is a commutative ring, and we have

$$\Delta_d(H, \rho; f_1, f_2) \doteq \Delta_d(H', \rho'; f_1, f_2)$$

if  $R$  is a GCD domain.

## 4 Examples and applications

In this section, we give a calculation example of the  $f$ -twisted Alexander invariant of the genus  $g$  trivial handlebody-knot for any MCQ Alexander pair  $f$ , where a handlebody-knot is *trivial* if its exterior is a handlebody. Furthermore, we introduce  $k$ -moves for handlebody-links and show that an  $f$ -twisted Alexander invariant detects 4-move equivalence classes of handlebody-links for some MCQ Alexander pair  $f$ .

**Proposition 4.1.** *Let  $O_g$  be the trivial handlebody-knot of genus  $g$ . For any MCQ representation  $\rho : MCQ(O_g) \rightarrow X$  and MCQ Alexander pair  $(f_1, f_2)$  of maps  $f_1, f_2 : X \times X \rightarrow R$ , we have*

$$A(O_g, \rho; f_1, f_2) \sim (0 \ \cdots \ 0) \in M(1, g; R).$$

Especially, we have

$$E_d(O_g, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < g, \\ R & \text{if } g \leq d \end{cases}$$

if  $R$  is a commutative ring, and we have

$$\Delta_d(O_g, \rho; f_1, f_2) = \begin{cases} 0 & \text{if } d < g, \\ 1 & \text{if } g \leq d \end{cases}$$

if  $R$  is a GCD domain.

A  $k$ -move is a local move on handlebody-links as illustrated in Figure 3. Two handlebody-links are *k-move equivalent* if they are related by a finite sequence of  $k$ -moves and isotopies of  $S^3$ . In this section, we focus on 4-moves for handlebody-links. Behavior of 4-moves for classical links has been studied in, for example, [2, 3, 4, 11, 16, 17, etc.].

**Proposition 4.2.** *Let  $R_4$  be the dihedral quandle and  $X := R_4 \times \mathbb{Z}_2$  the associated MCQ of  $R_4$ , where we regard  $R_4$  as the core quandle  $\text{Core}(t|t^4)$ . Let  $f = (f_1, f_2)$  be the MCQ Alexander pair of maps  $f_1, f_2 : X \times X \rightarrow \mathbb{Z}_4[t^{\pm 1}]/(t^2 + 1)$  defined by*

$$f_1((x, a), (y, b)) = \begin{cases} 1 & \text{if } b = 0, \\ -yx^{-1} & \text{otherwise,} \end{cases}$$

$$f_2((x, a), (y, b)) = \begin{cases} 0 & \text{if } a = 0, \\ -1 - xy^{-1} & \text{if } a = 1 \text{ and } b = 0, \\ 1 + yx^{-1} & \text{if } a = 1 \text{ and } b = 1. \end{cases}$$



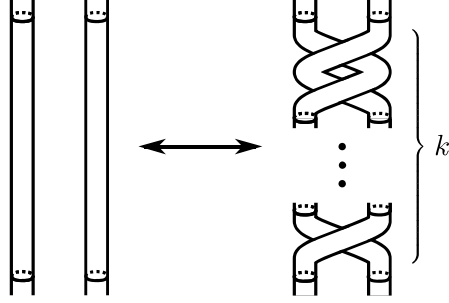


Figure 3: A  $k$ -move for a handlebody-link.

Then for any handlebody-link  $H$ , the multiset

$$\{\Delta_d(H, \rho; f_1, f_2) \mid \rho \in \text{Hom}(MCQ(H), X)\}$$

is an invariant under 4-moves for  $H$  for each  $d \in \mathbb{Z}_{\geq 0}$ .

**Example 4.3.** Let  $H$  be the three component handlebody-link represented by the Y-oriented diagram  $D$  depicted in Figure 4. Let  $X$  and  $f = (f_1, f_2)$  be the MCQ and MCQ Alexander pair that are the same as Proposition 4.2, respectively. Let  $\rho : MCQ(H) \rightarrow X$  be the MCQ representation depicted in Figure 4. The Wirtinger presentation of  $MCQ(H)$  with respect to  $D$  is given by

$$\left\langle \begin{array}{l} x_1, x_2, x_3; x_4; \\ x_5, x_6, x_7; x_8; x_9 \end{array} \middle| \begin{array}{l} x_6 \triangleleft x_1 = x_7, x_1 \triangleleft x_7 = x_2, x_8 \triangleleft x_3 = x_8, x_4 \triangleleft x_8 = x_3, \\ x_9 \triangleleft x_4 = x_9, x_5 \triangleleft x_9 = x_4, x_3 x_1 = x_2, x_7 x_5 = x_6 \end{array} \right\rangle.$$

Hence we have

$$A(H, \rho; f_1, f_2) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 - t^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 - t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -t^{-1} & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \end{pmatrix} \\ \sim (2 + 2t \ 0 \ 0 \ 0)$$

and  $\Delta_3(H, \rho; f_1, f_2) = 2 + 2t$ . On the other hand, let  $H_0$  be the three component trivial handlebody-link consisting of one genus 2 component and two genus 1 components. Then for any MCQ representation  $\rho_0 : MCQ(H_0) \rightarrow X$ , we have

$$A(H_0, \rho_0; f_1, f_2) \sim (0 \ 0 \ 0 \ 0)$$

and  $\Delta_3(H_0, \rho_0; f_1, f_2) = 0$ . Consequently,  $H$  is not 4-move equivalent to the trivial handlebody-link by Proposition 4.2.

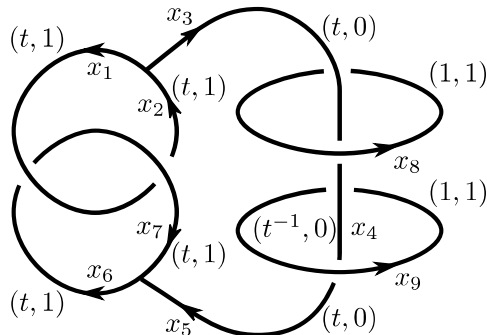


Figure 4: A Y-oriented diagram  $D$  of the three component handlebody-link  $H$ .

## Part II

# Spatial surfaces

A *spatial surface* [13] is a compact surface embedded in the 3-sphere  $S^3 = \mathbb{R}^3 \cup \{\infty\}$ . Two (oriented) spatial surfaces  $F$  and  $F'$  are *equivalent*, denoted by  $F \cong F'$ , if there is an orientation-preserving self-homeomorphism of  $S^3$  which sends one to the other (with its orientation). In this part, unless otherwise stated, we suppose the following conditions:

- Each component of a spatial surface has non-empty boundary.
- A spatial surface has no disk components.

In this part, we introduce a coloring invariant for oriented spatial surfaces by using a multiple group rack, which is an algebra whose axioms are motivated from Reidemeister moves for oriented spatial surfaces. Further, we distinguish some oriented spatial surfaces by using the coloring invariants. This is a joint work with Atsushi Ishii and Shosaku Matsuzaki.

## 5 Oriented spatial surfaces

In figures of this part, the front side and the back side of an oriented spatial surface are colored by light gray and dark gray as illustrated in Figure 5, respectively.



Figure 5: The front side and the back side.

Let  $D$  be a diagram of a spatial trivalent graph. We obtain a spatial surface  $F$  from  $D$  by taking a regular neighborhood of  $D$  in  $\mathbb{R}^2$  and perturbing it around all crossings of  $D$ ,

according to its over/under information. Then we give  $F$  an orientation so that the front side of  $F$  faces into the positive direction of the  $z$ -axis of  $\mathbb{R}^3$  as illustrated in Figure 6. We then call  $D$  a *diagram* of the oriented spatial surface  $F$ . We remark that any oriented spatial surface is equivalent to an oriented spatial surface obtained by this process [13].

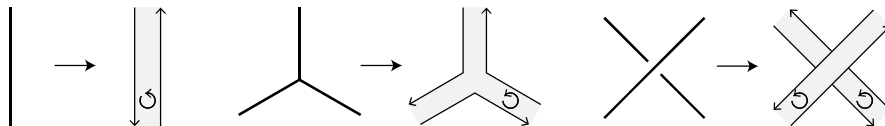


Figure 6: The process for obtaining an oriented spatial surface.

We introduce elementary methods to distinguish oriented spatial surfaces and show some examples. Let  $F$  be an oriented spatial surface. The regular neighborhood  $N(F)$  of  $F$  in  $S^3$  is a handlebody-link. We denote by  $\partial F$  the boundary of  $F$ , where we assume that  $\partial F$  is oriented so that the orientation is coherent with that of  $F$ .

**Remark 5.1.** If oriented spatial surfaces  $F_1$  and  $F_2$  are equivalent, then we have the following:

1. Two oriented links  $\partial F_1$  and  $\partial F_2$  are equivalent.
2. Two handlebody-links  $N(F_1)$  and  $N(F_2)$  are equivalent.

**Example 5.2.** For  $i, j, k \in \mathbb{Z}$ , let  $F(i, j, k)$  be the oriented spatial surface as illustrated in Figure 7, where the integers  $i, j$  and  $k$  indicate  $i, j$  and  $k$  full-twists, respectively. We note that a negative integer indicates the reverse twists. Then, the following conditions are equivalent:

1. The multiset  $\{i, j, k\}$  coincides with the multiset  $\{i', j', k'\}$ .
2.  $F(i, j, k)$  and  $F(i', j', k')$  are equivalent.
3.  $\partial F(i, j, k)$  and  $\partial F(i', j', k')$  are equivalent as oriented links.

**Example 5.3.** Let  $F_1$  and  $F_2$  be the spatial surfaces illustrated in Figure 8. Both  $\partial F_1$  and  $\partial F_2$  are trivial knots. It is easy to see that  $N(F_1)$  is the genus 2 trivial handlebody-knot, but  $N(F_2)$  is not. Hence  $F_1$  and  $F_2$  are not equivalent.

**Remark 5.4.** Let  $\widetilde{F}_1$  and  $\widetilde{F}_2$  be the spatial closed surfaces illustrated in Figure 9, from which  $F_1$  and  $F_2$  in Figure 8 are obtained by removing a disk, respectively. Then  $\widetilde{F}_1$  and  $\widetilde{F}_2$  are not equivalent, since  $\widetilde{F}_1$  splits  $S^3$  into two solid tori, but  $\widetilde{F}_2$  does not. This also implies that  $F_1$  and  $F_2$  are not equivalent by [13].

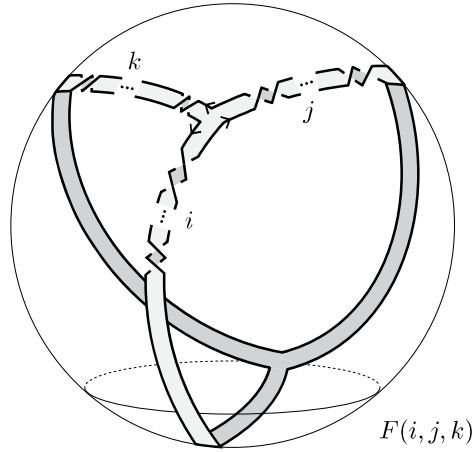


Figure 7: An oriented spatial surface  $F(i, j, k)$  whose boundary has three components.

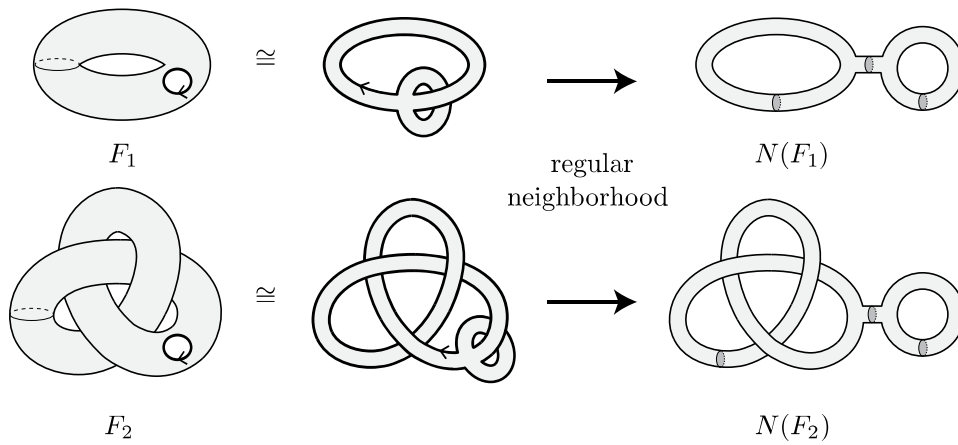


Figure 8: Inequivalent spatial surfaces.

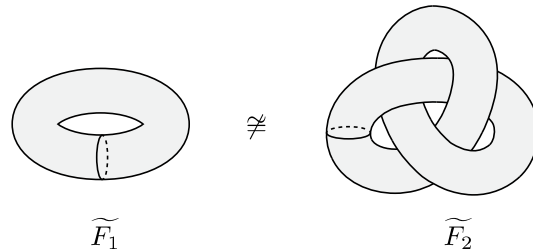


Figure 9: Inequivalent spatial closed surfaces.

The oriented spatial surfaces  $F_1$  and  $F_2$  depicted in Figure 11 can not be distinguished by using these elementary methods. In Section 7, we demonstrate that they are not equivalent by using coloring invariants.

## 6 Multiple group racks and a coloring invariant for oriented spatial surfaces

A *rack* [5] is a non-empty set  $X$  with a binary operation  $\triangleleft : X \times X \rightarrow X$  satisfying the following axioms:

- For any  $a \in X$ , the map  $\triangleleft a : X \rightarrow X$  defined by  $\triangleleft a(x) = x \triangleleft a$  is bijective.
- For any  $a, b, c \in X$ ,  $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ .

We denote  $(\triangleleft a)^n : X \rightarrow X$  by  $\triangleleft^n a$  for  $n \in \mathbb{Z}$ .

We remark that a quandle is a rack. For a positive integer  $n$ , we define a binary operation  $\triangleleft$  on  $\mathbb{Z}_n$  by  $a \triangleleft b = a + 1$ . Then,  $C_n := (\mathbb{Z}_n, \triangleleft)$  is a rack, called the *cyclic rack* of order  $n$ . Let  $R$  be a ring and  $M$  a left  $R[t^{\pm 1}, s]/(s(t+s-1))$ -module. We define a binary operation  $\triangleleft$  on  $M$  by  $x \triangleleft y = tx + sy$ . Then,  $(M, \triangleleft)$  is a rack, called the  $(t, s)$ -*rack*.

**Proposition 6.1.** *Let  $X$  be a rack. Fix  $e_1, \dots, e_m \in \mathbb{Z}$  and  $i_1, \dots, i_m \in \{1, \dots, n\}$ . Then  $X^n$  is a rack with the binary operation defined by*

$$\begin{aligned} (x_1, \dots, x_n) \triangleleft (y_1, \dots, y_n) \\ = ((\dots (x_1 \triangleleft^{e_1} y_{i_1}) \triangleleft^{e_2} \dots) \triangleleft^{e_m} y_{i_m}, \dots, (\dots (x_n \triangleleft^{e_1} y_{i_1}) \triangleleft^{e_2} \dots) \triangleleft^{e_m} y_{i_m}). \end{aligned}$$

We define the *type* of a rack  $X$  by

$$\text{type } X = \min\{n \in \mathbb{Z}_{>0} \mid x \triangleleft^n y = x \text{ (for any } x, y \in X)\},$$

where we set  $\min \emptyset := \infty$  for the empty set  $\emptyset$ . We note that  $(X, \triangleleft^i)$  is also a rack for any  $i \in \mathbb{Z}$ , and any finite rack is of finite type.

**Definition 6.2.** A *multiple group rack (MGR)*  $X$  is a disjoint union of groups  $G_\lambda$  ( $\lambda \in \Lambda$ ) with a binary operation  $\triangleleft : X \times X \rightarrow X$  satisfying the following axioms:

- For any  $x \in X$  and  $a, b \in G_\lambda$ ,  $x \triangleleft e_\lambda = x$  and  $x \triangleleft (ab) = (x \triangleleft a) \triangleleft b$ , where  $e_\lambda$  is the identity of  $G_\lambda$ .
- For any  $x, y, z \in X$ ,  $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$ .
- For any  $x \in X$  and  $a, b \in G_\lambda$ ,  $(ab) \triangleleft x = (a \triangleleft x)(b \triangleleft x)$ , where  $a \triangleleft x, b \triangleleft x \in G_\mu$  for some  $\mu \in \Lambda$ .

We remark that an MGR is a rack, and an MCQ is an MGR. Let  $X$  be a rack. Then  $X \times \mathbb{Z}_{\text{type } X} = \bigsqcup_{x \in X} (\{x\} \times \mathbb{Z}_{\text{type } X})$  is an MGR, called the *associated MGR* of  $X$ , with

$$(x, a) \triangleleft (y, b) := (x \triangleleft^b y, a), \quad (x, a)(x, b) := (x, a + b)$$

for any  $x, y \in X$  and  $a, b \in \mathbb{Z}_{\text{type } X}$ , where we put  $\mathbb{Z}_\infty := \mathbb{Z}$ .

Let  $D$  be a  $Y$ -oriented diagram of an oriented spatial surface and let  $X$  be an MGR. We denote by  $\mathcal{A}(D)$  the set of arcs of  $D$ . We denote by  $u_c, v_c, w_c, u_\tau, v_\tau, w_\tau$  the arcs around each crossing  $c$  and vertex  $\tau$  as illustrated in Figure 2, respectively. An  $X$ -*coloring* of  $D$  is a map  $C : \mathcal{A}(D) \rightarrow X$  satisfying the conditions

$$C(u_c) \triangleleft C(v_c) = C(w_c) \quad \text{and} \quad C(u_\tau)C(v_\tau) = C(w_\tau)$$

for each crossing  $c$  and vertex  $\tau$ . We denote by  $\text{Col}_X(D)$  the set of  $X$ -colorings of  $D$ . Then we have the following theorem.

**Theorem 6.3.** *Let  $X$  be an MGR. Let  $D$  and  $D'$  be  $Y$ -oriented diagrams of an oriented spatial surface  $F$ . Then there is a one-to-one correspondence between  $\text{Col}_X(D)$  and  $\text{Col}_X(D')$ . In particular, the cardinality  $\#\text{Col}_X(D)$  is an invariant of  $F$ .*

## 7 Coloring examples

In this section, we give examples to distinguish oriented spatial surfaces by using MGR coloring invariants.

Let  $F$  be an oriented spatial surface. The *reverse* of  $F$ , denoted by  $-F$ , is the oriented spatial surface obtained by reversing the orientation of  $F$ . An oriented spatial surface  $F$  is *reversible* if  $F$  and its reverse are equivalent. Let  $h$  be an orientation-reversing self-homeomorphism of  $S^3$ . The oriented spatial surface  $h(F)$  is called the *mirror image* of  $F$  and denoted by  $F^*$  (see Figure 10).

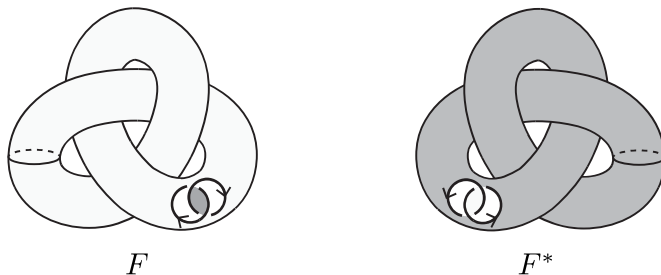


Figure 10: An oriented spatial surface  $F$  and its mirror image  $F^*$ .

**Example 7.1.** Let  $F_1$  and  $F_2$  be the oriented spatial surfaces illustrated in Figure 11, where  $F_2$  is the reverse of  $F_1$ . Let  $D_1$  and  $D_2$  be the  $Y$ -oriented diagrams of  $F_1$  and  $F_2$  illustrated in the figure, respectively. Let  $R_3$  be the dihedral quandle. Then  $R_3^3$  is a rack with

$$(x_1, x_2, x_3) \triangleleft (y_1, y_2, y_3) = (((x_1 \triangleleft y_1) \triangleleft y_2) \triangleleft y_3, ((x_2 \triangleleft y_1) \triangleleft y_2) \triangleleft y_3, ((x_3 \triangleleft y_1) \triangleleft y_2) \triangleleft y_3).$$

Let  $R_3^3 \times \mathbb{Z}_2$  be the associated MGR of  $R_3^3$ . Then we have  $\#\text{Col}_{R_3^3 \times \mathbb{Z}_2}(D_1) = 144$  and  $\#\text{Col}_{R_3^3 \times \mathbb{Z}_2}(D_2) = 72$ . Therefore,  $F_1$  and  $F_2$  are not equivalent, that is, they are not reversible.

**Example 7.2.** Let  $F_1$  and  $F_2$  be the oriented spatial surfaces illustrated in Figure 12. We note that  $F_1$  and  $F_2$  are Seifert surfaces of the same oriented knot. Let  $D_1$  and  $D_2$  be the  $Y$ -oriented diagrams of  $F_1$  and  $F_2$  illustrated in the figure, respectively. Let  $X := \mathbb{Z}_3[t^{\pm 1}]/(t^2 + t + 1)$  be the  $(t, s)$ -rack ( $s = t + 2$ ), and let  $X \times \mathbb{Z}_3$  be the associated MGR of  $X$ . Then we have  $\#\text{Col}_{X \times \mathbb{Z}_3}(D_1) = 45$  and  $\#\text{Col}_{X \times \mathbb{Z}_3}(D_2) = 33$ . Therefore,  $F_1$  and  $F_2$  are not equivalent.

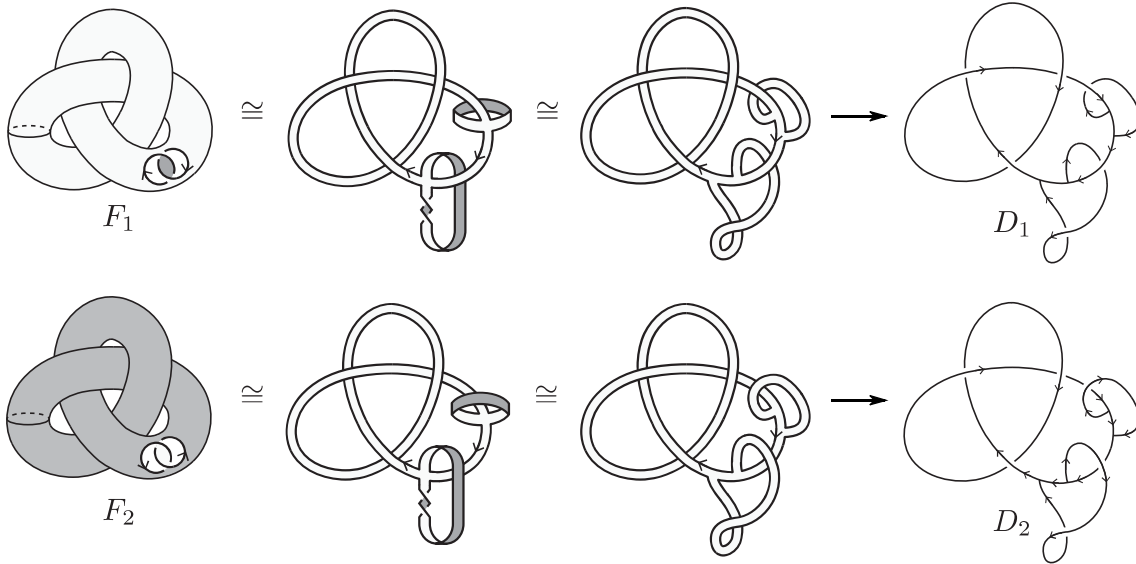


Figure 11: Oriented spatial surfaces  $F_1, F_2$  and their diagrams  $D_1, D_2$ .

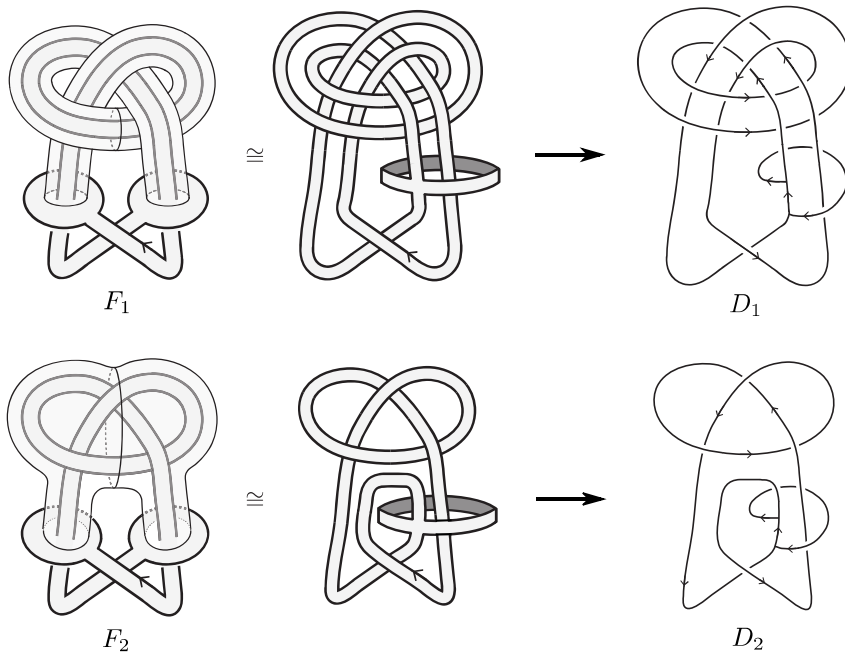


Figure 12: Two Seifert surfaces  $F_1, F_2$  of the same oriented knot and their diagrams  $D_1, D_2$ .

**Example 7.3.** Let  $\widetilde{F}_1$  and  $\widetilde{F}_2$  be the oriented spatial closed surfaces illustrated in Figure 13, where we note that they do not bound handlebodies in  $S^3$ . Let  $F_1$  and  $F_2$  be the oriented spatial surfaces obtained by removing a disk from  $\widetilde{F}_1$  and  $\widetilde{F}_2$  as shown in

the figure, respectively. By [13],  $\widetilde{F}_1$  and  $\widetilde{F}_2$  are equivalent if and only if  $F_1$  and  $F_2$  are equivalent. Let  $D_1$  and  $D_2$  be Y-oriented diagrams of  $F_1$  and  $F_2$ , respectively. Let  $R_3$  be the dihedral quandle, and let  $R_3 \times \mathbb{Z}_2$  be the associated MGR of  $R_3$ . Then we have  $\#\text{Col}_{R_3 \times \mathbb{Z}_2}(D_1) = 96$  and  $\#\text{Col}_{R_3 \times \mathbb{Z}_2}(D_2) = 144$ . Therefore,  $F_1$  and  $F_2$  are not equivalent, that is,  $\widetilde{F}_1$  and  $\widetilde{F}_2$  are not equivalent.

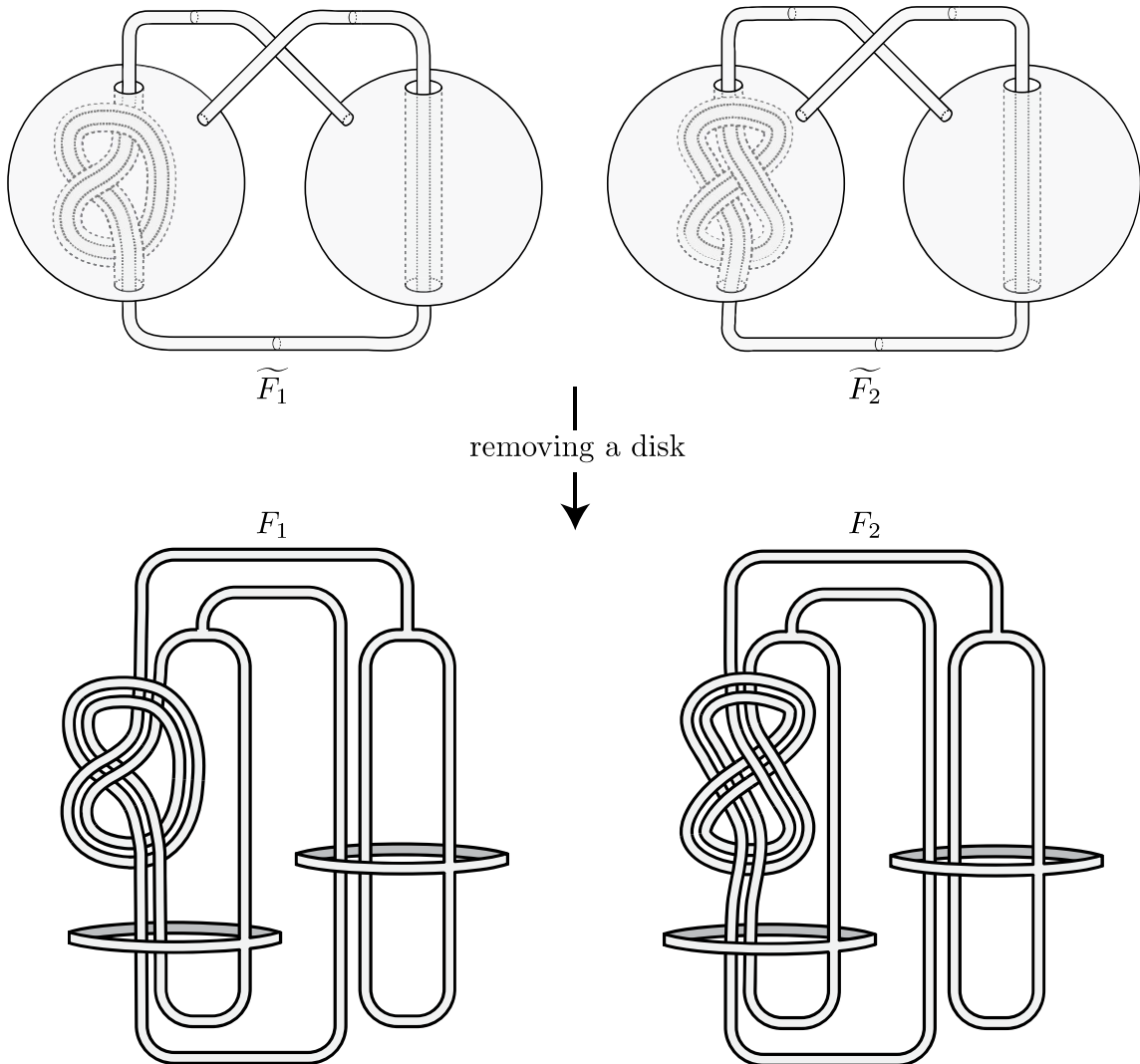


Figure 13: Two oriented spatial closed surfaces  $\widetilde{F}_1$  and  $\widetilde{F}_2$ .

**Proposition 7.4.** *Let  $X$  be an MGR. Let  $F$  be an oriented spatial surface and let  $D$  and  $-D^*$  be Y-oriented diagrams of  $F$  and  $-F^*$ , respectively. Then there is a one-to-one correspondence between  $\text{Col}_X(D)$  and  $\text{Col}_X(-D^*)$ .*

By the above proposition, We can not distinguish an oriented spatial surface and its mirror image of its reverse by using MGR coloring numbers.



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