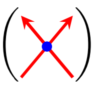
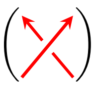
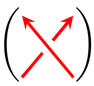
On Khovanov homology and Vassiliev theory

Jun Yoshida
RIKEN Center for Advanced Intelligence Project

1 Introduction

A knot invariant v is called a *Vassiliev invariant* if it vanishes on singular knots with sufficiently large number of double points. More precisely, we extend v to an invariant $v^{(r)}$ on singular knots with exactly r (transverse) double points with $v^{(0)} = v$ and

$v^{(r+1)}$  $= v^{(r)}$  $- v^{(r)}$  . (1.1)

We say v is *of order* r if $v^{(r)} \not\equiv 0$ and $v^{(r+1)} \equiv 0$, and the singular knot invariant $v^{(r)}$ is sometimes called the *Vassiliev derivative of order* r. The recurrence formula (1.1) is called *Vassiliev skein relation* while it was first explicitly introduced in [Birman, 1993, Birman and Lin, 1993]. As an important result in that paper, it was pointed out that Vassiliev invariants are deeply related to quantum invariants. Namely, consider the following Taylor series of the Jones polynomial V:

$$V_t(K)|_{t=e^x} = \sum_n a_n(K)x^n \quad .$$

It was shown that each a_n is of order n and hence a Vassiliev invariant. Actually, it was even shown that analogues hold for other quantum invariants; i.e. ones coming from representations of quantum groups.

On the other hand, in light of recent developments of knot theory, it has been revealed that quantum invariants are kinds of "de-categorifications" of categorical invariants. The first concrete study was carried out by Khovanov; in the seminal work [Khovanov, 2000]; he showed that the Jones polynomial $V(K)$ is obtained as the (graded) Euler characteristic of a homology invariant $Kh(K)$, which is nowadays called *Khovanov homology*. A number of researches follow it, and quantum invariants of type A are all categorified, e.g. [Khovanov and Rozansky, 2008a, Khovanov and Rozansky, 2008b, Cautis and Kamnitzer, 2008a, Cautis and Kamnitzer, 2008b, Rasmussen, 2015].

It seems a natural question to ask whether there is a direct relationship between categorified knot invariants and Vassiliev invariants. The goal of our project is to provide a higher-categorical background for Vassiliev theory to answer the question positively. As a first step, we defined Vassiliev derivatives of Khovanov homology in [Ito and Yoshida, 2020, Ito and Yoshida, 2021] and proved that they are singular knot invariants. This is done by constructing an explicit chain map, which we usually write $\widehat{\Phi}$, that categorifies the subtraction in Vassiliev skein relation (1.1). However, although we showed homotopy invariance of $\widehat{\Phi}$ under certain moves, the geometric meanings of the higher homotopies are unclear; this is because there is no such thing as "equality of equality" in the combinatorial context where (1.1) lies in.

In this article, we present our result in view of the cohomology of the knot space, which may reveal the meaning of higher categorical structures. In fact, in contrast to the combinatorial characterization above, Vassiliev invariants were originally discovered in the study of the space of knots [Vassiliev, 1990]. Indeed, if $\mathcal{K}$ denotes the space of knots, then every knot invariant is uniquely associated with a 0-cocycle in the cohomology group $H^0(\mathcal{K})$ or the reduced on $\widetilde{H}^0(\mathcal{K})$ if it is appropriately normalized on the unknot:

$$\{\text{knot invariants}\}/\{\text{constants}\} \cong [\pi_0(\mathcal{K}), \mathbb{Z}]/\mathbb{Z} \cong \widetilde{H}^0(\mathcal{K}) \quad .$$

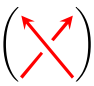
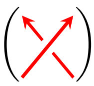
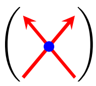
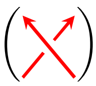
Vassiliev invariants arise from a spectral sequence converging to $\widetilde{H}^*(\mathcal{K})$. Hence, if his construction is described categorically, then it may automatically explain the geometric meaning of categorified knot invariants. Specifically, we investigate the *fundamental groupoid* $\Pi_1\mathcal{K}$ and show that it has a combinatorial description in terms of knot diagrams and Reidemeister moves. In addition, to take discriminant Σ into account, we embed $\Pi_1\mathcal{K}$ into a category $\mathcal{C}$ obtained by attaching crossing-changes to $\Pi_1\mathcal{K}$. In this setting, our main result can be presented in the following form.

Main Theorem. *Khovanov homology extends to a functor*

$$Kh : \mathcal{C} \to D^b(\mathbf{grAb}^{\mathsf{f}}) \quad ,$$

where the target category is the bounded derived category of graded abelian groups of finite total dimension.

We note that Section 1 implies that each morphism in $\mathcal{C}$ representing a crossing-change induces a morphism between Khovanov homologies of knots that is itself invariant of a singular knot. Therefore, it extends Khovanov homology to singular knots so that the following is an exact triangle in $D^b(\mathbf{grAb}^{\mathsf{f}})$:

Kh  $\to Kh$  $\to Kh$  $\to \Sigma Kh$  .

Furthermore, we discuss categorical analogues of relations coming from the Vassiliev's spectral sequence; namely, the FI-relation and the 4T-relation.

2 Categorification of 0-cocycles

We first quickly sketch the basic idea of the categorification of 0-cocycles in terms of fundamental groupoid. For a topological space X and an abelian group A, an A-valued 0-cocycle on X is an element of either of the following equivalent sets:

$$[X, A] \cong [\pi_0 X, A] \cong H^0(X; A) \quad ,$$

where $[-,-]$ is the set of homotopy classes of continuous maps, and $\pi_0 X$ is the set of path-connected components of X.

Definition. Let X be a topological space. The *fundamental groupoid* of X is the category $\Pi_1 X$ described as follows:

- objects are points on X;
- morphisms are boundary-fixed homotopy classes of continuous paths $c : [0,1] \to X$, with $\operatorname{dom} c = c(0)$ and $\operatorname{cod} c = c(1)$;
- the composition is concatenation of paths.

It is easily seen that every morphism of $\Pi_1 X$ is invertible; in other words, it $\Pi_1 X$ is in fact a groupoid. Also, an isomorphism class of $\Pi_1 X$ is nothing but a path-connected component of X. It follows that there is a bijection $\Pi_1 X/\mathrm{isom} \cong \pi_0 X$, so one may say $\Pi_1 X$ is a categorification of $\pi_0 X$.

On the other hand, we consider categorification of coefficient abelian group in the following form. Recall that, for a triangulated category $\mathcal{T}$, the *Grothendieck group* $K(\mathcal{T})$ is the abelian group generated by the isomorphism classes of $\mathcal{T}$ subject to the relation $[X] - [Y] + [Z] = 0$ for every exact triangle in $\mathcal{T}$ of the following form:

$$X \to Y \to Z \to \Sigma X \quad .$$

Definition. Let A be an abelian group. A triangulated category $\mathcal{T}$ is called a *categorification* of A if it is equipped with an isomorphism $K(\mathcal{T}) \cong A$.

Example 2.1. Let $A = \mathbb{Z}$ be the ring of integers. For a field k, we denote by $D^b(\mathbf{Vect}_k^{\mathsf{f}})$ the bounded derived category of chain complexes of finite dimension k-modules. Then, the *Euler characteristic*

$$\chi(C^*) := \sum_i (-1)^i \dim_k C^i$$

induces an isomorphism $K(D^b(\mathbf{Vect}_k^{\mathsf{f}})) \cong \mathbb{Z}$. Hence, $D^b(\mathbf{Vect}_k^{\mathsf{f}})$ is a categorification of $\mathbb{Z}$.

Example 2.2. Let $A = \mathbb{Z}[q, q^{-1}]$ be the Laurent polynomial ring. For a field k, we denote by $D^b(\mathbf{grVect}_k^{\mathsf{f}})$ the bounded derived category of graded k-modules of totally finite dimension. More precisely, the objects are bounded bigraded k-modules $C^{*,\star}$ together with k-homomorphisms $d : C^{i,j} \to C^{i+1,j}$ such that $d \circ d = 0$. Then, the *graded Euler characteristic*

$$\chi(C^{*,\star}) := \sum_{i,j} (-1)^i q^j \dim_k C^{i,j}$$

induces an isomorphism $K(D^b(\mathbf{grVect}_k^{\mathsf{f}})) \cong \mathbb{Z}[q, q^{-1}]$. In other words, $D^b(\mathbf{grVect}_k^{\mathsf{f}})$ is a categorification of the Laurent polynomial ring.

If a categorification $\mathcal{T}$ of an abelian group A, then an A-valued 0-cocycle on a topological space X will be categorified as a functor

$$F : \Pi_1 X \to \mathcal{T} \quad .$$

Indeed, given such an F, we obtain a map $\pi_0(X) \to K(\mathcal{T})$ as the one induced by F on the isomorphism classes. As mentioned in the introduction, one of the goals of the article is to exhibit Khovanov homology in this form in the case $X = \mathcal{K}$ and $\mathcal{T} = D^b(\mathbf{grVect}_k^{\mathsf{f}})$.

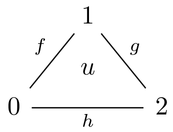


Figure 3.1: A 2-simplex defining a composition

3 The fundamental groupoid of the knot space

To consider categorification of knot invariants in the sense of the previous section, we need more convenient description of the fundamental groupoid of the knot space. For this, recall that the *singular simplicial set* of a topological space X is the simplicial set $S_\bullet X$ such that

$$S_n X \coloneqq \{\text{continuous maps } \Delta^n \to X\} \quad ,$$

where Δ^n is the standard simplex. The face and degeneracy operators are defined in the obvious way. The simplicial set $S_\bullet X$ has all the weak-homotopy information on X: specifically, the fundamental groupoid $\Pi_1 X$ is equivalently given as follows:

- objects are elements of $S_0 X$;
- morphisms are equivalent classes of $S_1 X$ under the relation $d_1 u \sim d_2 u$ for each 2-simplex $u \in S_2 X$ with $d_0 u$ degenerate;
- the composition is defined so that $[g] \circ [f] = [h]$ holds for $f, g, h \in S_1 X$ if and only if there is a 2-simplex $u \in S_2 X$ with $d_0 u = g$, $d_1 u = h$, and $d_2 u = f$ (see Figure 3.1).

Generally, the above construction works for every *Kan complex* K; we write $\Pi_1 K$ the resulting category and call it the *fundamental groupoid* of K.

Definition. For a compact manifold X and a smooth fiber bundle $q : Y \to B$, we define a simplicial set $\mathcal{E}^B_\bullet(X, Y)$ with $\mathcal{E}^B_i(X, Y)$ consisting of smooth maps $f : X \times \Delta^i \to Y$ such that

- for each $t \in \Delta^i$, the restriction $f|_{X \times \{t\}} : X \to Y$ is an embedding;
- for each face $\sigma \subset \Delta^i$, the map

$$\widehat{qf}_\sigma : X \times \sigma \to B \times \sigma \; ; \quad (x, t) \mapsto (qf(x, t), t)$$

  is "generic" in the sense that its (multi-)jets are transverse to all the discriminants.

Proposition 3.1 (cf. [Lurie, 2009, Proposition 1] and [Lee, 2003, Theorem 10.16])**.** *Let X be a compact manifold and $q : Y \to B$ a smooth fiber bundle. Then, $\mathcal{E}^B_\bullet(X, Y)$ is a Kan complex equipped with a canonical homotopy equivalence*

$$\mathcal{E}^B_\bullet(X, Y) \to S_\bullet \operatorname{Emb}(X, Y) \quad ,$$

where $\operatorname{Emb}(X, Y)$ is the space of smooth embeddings $X \to Y$ with the Whitney C^∞-topology. Specifically, there is an equivalence of categories $\Pi_1 \mathcal{E}^B_\bullet(X, Y) \simeq \Pi_1 \operatorname{Emb}(X, Y)$.

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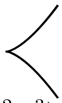
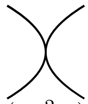
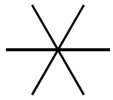
		
(t^2, t^3)	$(\pm t^2, t)$	triple point

Table 3.1: Singularities associated with the Reidemeister moves

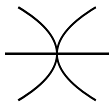
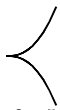
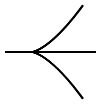
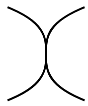
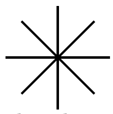
MM6	MM7	MM8	MM9	MM10
				
$(\pm t^2, t)$ and $(t, 0)$	(t^2, t^5)	(t^2, t^3) and $(0, t)$	$(\pm t^3, t)$	quadruple point

Table 3.2: Singularities associated with the movie moves

We set $X = S^1$, $Y = \mathbb{R}^3$, and $B = \mathbb{R}^2$ in Proposition 3.1. Then, we obtain an equivalence of categories

$$\Pi_1 \mathcal{E}_\bullet^{\mathbb{R}^2}(S^1, \mathbb{R}^3) \simeq \Pi_1 \mathcal{K} \quad ,$$

where $\mathcal{K} = \mathrm{Emb}(S^1, \mathbb{R}^3)$ is the space of knots in $\mathbb{R}^3$. We write $\mathcal{D} \coloneqq \Pi_1 \mathcal{E}_\bullet^{\mathbb{R}^2}(S^1, \mathbb{R}^3)$. Now, the classification of the singularities yields the following combinatorial description of the category $\mathcal{D}$.

Theorem 3.2 (cf. [Carter and Saito, 1998, Roseman, 2000, Roseman, 2004])**.** *The category $\mathcal{D}$ is equivalent to the one described as follows:*

- *objects are knots associated with a knot diagram;*
- *morphisms are sequences of isotopies the moves associated with the singularities in Table 3.1, which are usually called the* Reidemeister moves*;*
- *morphisms are subject to the relation generated by*
  - (i) *invertibility of Reidemeister moves,*
  - (ii) *commutativity of Reidemeister moves applied to remote parts, and*
  - (iii) *the moves associated with the singularities in Table 3.2.*

Remark 3.3. The relations coming from the singularities in Table 3.2 form a part of *movie moves* in [Carter and Saito, 1998].

4 Khovanov homology as a categorified 0-cocycle

In the previous section, we obtained a category of knot diagrams $\mathcal{D}$ together with an equivalence $\mathcal{D} \simeq \Pi_1 \mathcal{K}$. Note that each object in $\mathcal{D}$ is canonically associated with a knot diagram D. Hence, in view of the combinatorial description in Theorem 3.2, one obtains a categorified 0-cocycle in the following steps:

(i) construct a bounded chain complex $C(D)$ of graded modules for each knot diagram D;

(ii) specify a chain homotopy along each Reidemeister moves;

(iii) verify the homotopy commutativity of diagrams associated with movie moves in Table 3.2.

In this section, We carry out the process in the case of Khovanov homology.

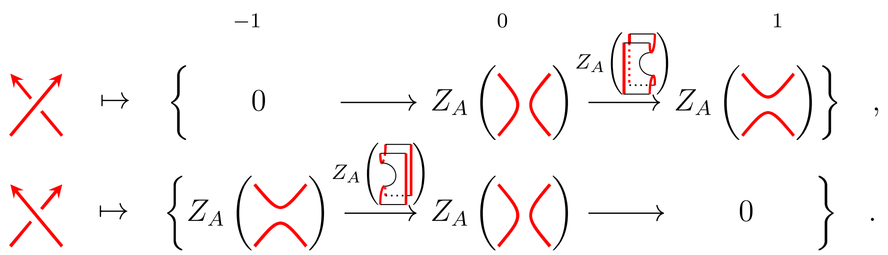
For this, we quickly review the construction of Khovanov homology. For a fixed base ring k, we write $A = k[x]/(x^2)$ the Frobenius algebra with the following comultiplication and counit:

$$\Delta(1) = 1 \otimes x + x \otimes 1 \ , \quad \Delta(x) = x \otimes x \ , \quad \varepsilon(1) = 0 \ , \quad \varepsilon(x) = 1 \quad .$$

As Frobenius algebras are equivalent to 2-dimensional topological quantum theory (aka. TQFT), we denote by Z_A the associated TQFT. In addition, by setting $\deg 1 = 1$ and $\deg x = -1$ in $A \cong Z_A(S^1)$, one sees that Z_A lifts to a functor

$$Z_A : \mathbf{Cob}_2 \to \mathbf{grMod}_k^{\mathsf{f}} \quad ,$$

where $\mathbf{Cob}_2$ is the category of 2-dimensional cobordisms. Using this functor, we define a complex $C(D)$ in $\mathbf{grMod}_k^{\mathsf{f}}$ for each diagram D as follows: for each crossing c of D, consider either of the following complexes according to the sign of c:



(4.1)

Then, construct $C(D)$ by stacking up all such elementary parts.

Theorem 4.1 (cf. [Clark et al., 2009])**.** *For every field k, Khovanov homology with coefficients in k extends to a functor*

$$Kh : \Pi_1\mathcal{K} \to D^b(\mathbf{grMod}_k^{\mathsf{f}})$$

whose image in $[\pi_0\mathcal{K}, \mathbb{Z}[q, q^{-1}]]$ agrees with the (unnormalized) Jones polynomial.

Proof. Khovanov [Khovanov, 2000] constructed a concrete chain map between complexes $C(D)$ associated with Reidemeister moves. Furthermore, in [Clark et al., 2009], it was shown that Khovanov homology satisfies the relation associated with all movie moves containing Table 3.2. Therefore, in view of Example 2.2 and the recipe in the last paragraph in the previous section, we obtain the required functor. □

5 Crossing-change

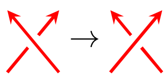
We next consider crossing-changes from the viewpoint of categorified cocycles. Let us denote by $\boldsymbol{\mathcal{X}} \subset C^\infty(S^1, \mathbb{R}^3)$ the subset of generic smooth immersions. According to the singularity theory of smooth maps, $\mathcal{K}$ is actually the 0-th stratum of a stratification $\boldsymbol{\mathcal{X}} = \mathcal{K} \cup \bigcup_{i>0} \Sigma_i'$; the set $\Sigma' := \bigcup_i \Sigma_i'$ is hence the *discriminant*. For example, the lower strata are described as follows:

- Σ_1' is the space of immersions with *unique transverse double points*, which are exactly singular knots with single double points;
- Σ_2' is the space of immersions with exactly two transverse double points.

We denote by $\Pi_1^{\Sigma'}\boldsymbol{\mathcal{X}} \subset \Pi_1\boldsymbol{\mathcal{X}}$ the subcategory whose morphisms are those paths which are transverse to the strata Σ_i'. Since each stratum $\Sigma_i' \subset \boldsymbol{\mathcal{X}}$ is of codimension i, morphisms in $\Pi_1^{\Sigma'}\boldsymbol{\mathcal{X}}$ are continuous families $\{K_t\}_{0\le t\le 1}$ of knots which may be singular with single double points only at finitely many parameters. Specifically, Σ_1' has a normal bundle which is canonically oriented in the direction from a negative resolution to a positive one on each double point. We further define $\vec{\Pi}_1^{\Sigma'}\boldsymbol{\mathcal{X}} \subset \Pi_1^{\Sigma'}\boldsymbol{\mathcal{X}}$ the subcategory consisting of paths which are along the orientation of the normal bundle of Σ_1' on each intersection.

By an analogous argument to Section 3, we again obtain a combinatorial model for the category $\vec{\Pi}_1^{\Sigma'}\boldsymbol{\mathcal{X}}$.

Proposition 5.1. *The category $\vec{\Pi}_1^{\Sigma'}\boldsymbol{\mathcal{X}}$ is equivalent to the category generated by $\mathcal{D}$ given in Theorem 3.2 and morphisms of the form*

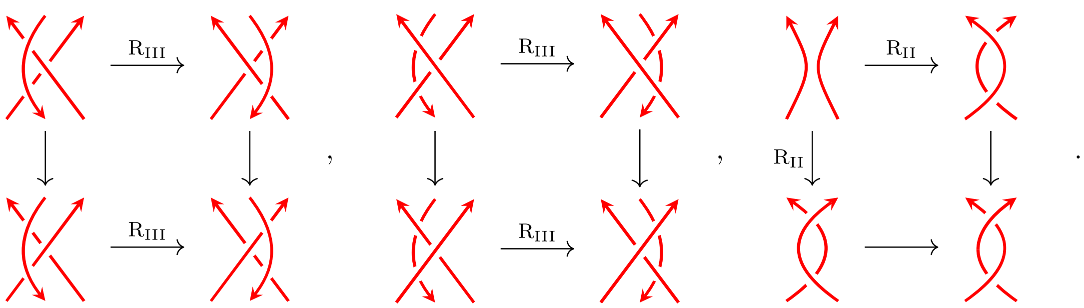


(5.1)

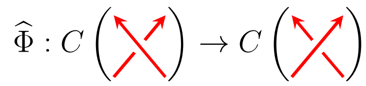
subject to the following relations:

(i) *the "crossing-change"* (5.1) *commutes with Reidemeister moves applied to separated parts;*

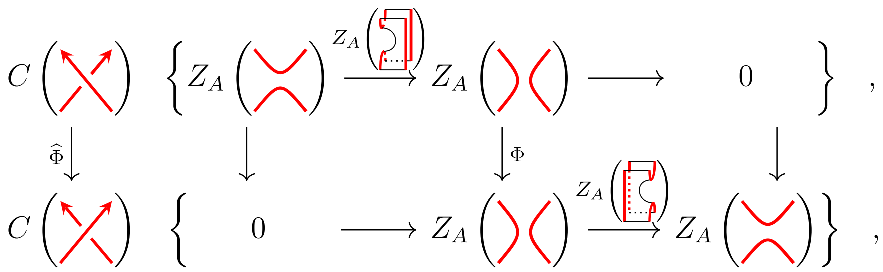
(ii) *the following are commutative:*



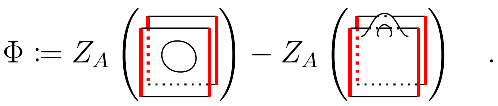
We use Proposition 5.1 to exhibit our extension of Khovanov homology [Ito and Yoshida, 2021] as a functor out of the category $\vec{\Pi}_1^{\Sigma'}\boldsymbol{\mathcal{X}}$. The construction is similar to that in the previous section. All we have to do is to assign a chain map to the crossing-change (5.1) and to verify the conditions in Proposition 5.1. As for the first part, we define a morphism



by the following morphism:



where the morphism Φ on degree 0 is given as the following sum of two cobordisms:



Theorem 5.2 ([Ito and Yoshida, 2020, Ito and Yoshida, 2021])**.** *With the above crossing change $\widehat{\Phi}$, Khovanov homology extends to a functor*

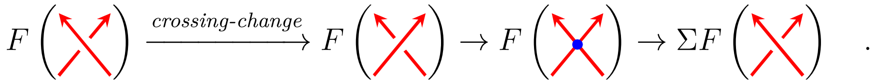
$$Kh : \vec{\Pi}_1^{\Sigma'} \boldsymbol{\mathcal{X}} \to D^b(\mathbf{grMod}_k^{\mathsf{f}}) \quad .$$

In fact, Theorem 5.2 defines Vassiliev derivatives of Khovanov homology thanks to the next result.

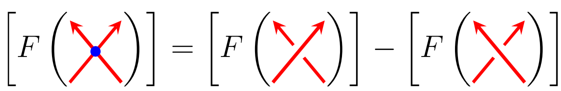
Proposition 5.3. *Let $\mathcal{T}$ be a triangulated category, and suppose we are given a functor*

$$F : \vec{\Pi}_1^{\Sigma'} \boldsymbol{\mathcal{X}} \to \mathcal{T} \quad .$$

Then, it defines a singular knot invariant with values in $\mathcal{T}$ so that, for each double point in a singular knot diagram, it comes equipped with the following exact triangle:

 (5.2)

Notice that, by definition of the Grothendieck group $K(\mathcal{T})$, the exact triangle (5.2) gives rise to the equation



in $K(\mathcal{T})$. This implies that the extension of F to singular knots is a categorification of Vassiliev derivatives.

6 Relations from Vassiliev's spectral sequence

In this final section, we discuss further property of our extension of Khovanov homology in Theorem 5.2. To begin with, we review Vassiliev's idea, which is roughly as follows: let $\mathcal{M} \subset C^\infty(S^1, \mathbb{R}^3)$ be the subspace of "generic" smooth maps $S^1 \to \mathbb{R}^3$. As in the case of the space of immersions in Section 5, the space $\mathcal{M}$ also has a stratification $\mathcal{M} = \mathcal{K} \cup \bigcup_{i>0} \Sigma_i$ according to the singularity theory with the discriminant the set $\Sigma := \bigcup_i \Sigma_i$. Each stratum Σ_i consists of smooth maps with singularities of degree i in the sense of [Boardman, 1967]. For example, the lower strata are described as follows:

- Σ_1 is the space of immersions with *unique transverse double points*, which are exactly singular knots with single double points;
- Σ_2 is the space consists of
  (a) immersions with *exactly two transverse double points* and
  (b) injective maps with *unique singular values*

Since $\mathcal{M}$ is a contractible spaces, Alexander duality[1] yields the following isomorphism:

$$\widetilde{H}^*(\mathcal{K}) \cong \overline{H}_{\infty-*-1}(\Sigma) \quad ,$$

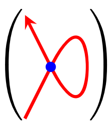
where the right hand side is (a certain colimit of) Borel-Moore homology. With respect to the filtration $F^p\Sigma = \bigcup_{i>p} \Sigma_i$, we therefore obtain a spectral sequence of the following form:

$$E_1^{p,q} \cong \overline{H}_{\infty-p-q-1}(F^p\Sigma) \Rightarrow \widetilde{H}^{p+q}(\mathcal{K}) \quad .$$

Vassiliev invariants of order i are exactly the invariants that lie in the i-th filtration of $\widetilde{H}^0(\mathcal{K})$ with respect to the spectral sequence. Note that the elements of the groups of the form $E_r^{i,-i}$ may be related to an order i invariants. Combinatorial descriptions of such groups are given by [Vassiliev, 1990] and [Kontsevich, 1993] in terms of *chord diagrams*. It however turns out that not all functions on chord diagrams are related to knot invariants. A criterion was given by [Kontsevich, 1993]: functions should satisfy two relations called the *FI-relation* and the *4T-relation*. In the rest, we discuss their categorical analogue on Khovanov homology.

6.1 The FI-relation

For a function v on singular knot diagrams, the *FI relation* is represented as vanishing at the following type of double points:

v  $= 0 \quad .$ (6.1)

In terms of the category $\vec{\Pi}_1^{\Sigma'} \boldsymbol{\mathcal{X}}$ or $\Pi_1\mathcal{M}$, the relation arises from comparison of the two paths in Figure 6.1. Hence, in the categorical context, the FI relation can be seen as a condition for the crossing-change morphisms to commute with Reidemeister moves of type I.

[1]We cheat here; $\mathcal{M}$ is not finite dimensional. For details, we refer the reader to [Vassiliev, 1990]

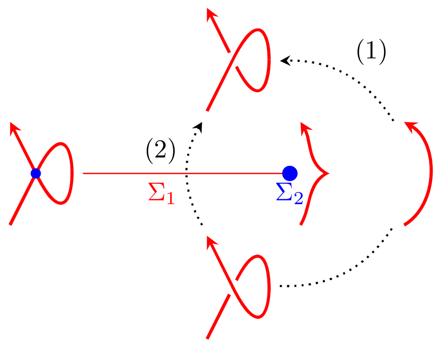
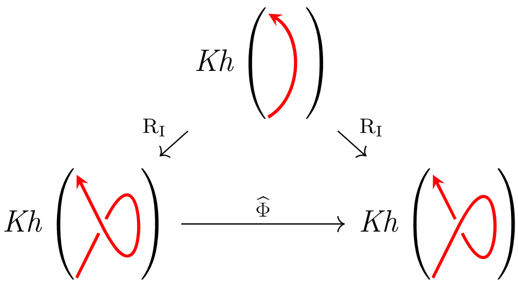


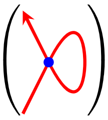
Figure 6.1: Two paths for the FI relation

Theorem 6.1. *The morphism $\widehat{\Phi}$ defined in Section 5 commutes with the morphisms associated with Reidemeister moves of type I. More precisely, the following diagram commutes in $D^b(\mathbf{grMod}_k^{\mathsf{f}})$:*



. (6.2)

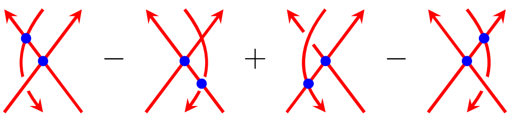
We note that since the top-to-bottom morphisms in the diagram (6.2) are quasi-isomorphisms, so is the bottom. As Khovanov homology on singular knots are defined using exact triangles induced by the morphism $\widehat{\Phi}$, we obtain the following vanishing of Vassiliev derivative:

Kh () $\cong 0$.

In this point of view, Theorem 6.1 is a categorification of the FI-relation (6.1).

6.2 The 4T-relation

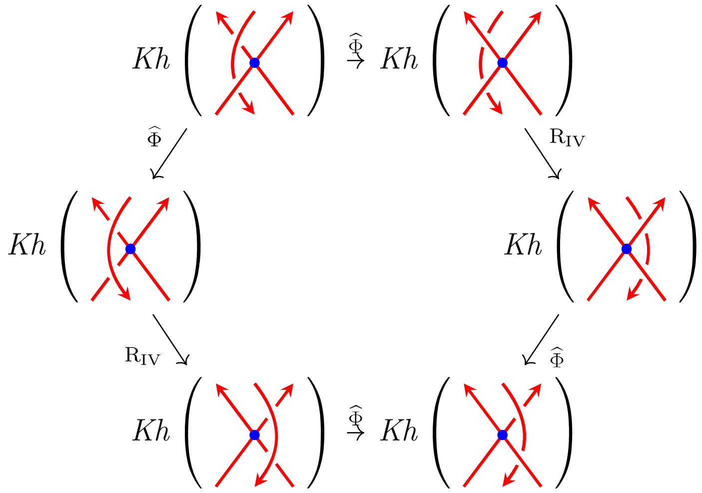
As for the 4T-relation (aka. the *four-term relation*), its knot diagrammatic representation is as follows:



$= 0$.

In contrast to the FI-relation, the categorical situation is a little bit subtle. Indeed, associated categorical condition is not commutativity but the "higher-commutativity" of 3-dimensional diagrams. Since triangulated categories do not carry sufficient higher-dimensional information in general, the categorical analogue of 4T-relation cannot be stated as conditions of morphisms in triangulated categories. Thus, we instead state it as a condition on categorical Vassiliev derivatives.

Theorem 6.2 (in preparation)**.** *The Vassiliev derivatives of Khovanov homology defined in Section 5 satisfies the* $4T$*-relation in the sense that the diagram below commutes up to chain homotopies:*



,

where the arrows with label R_{IV} *are the quasi-isomorphisms induced by the Reidemeister moves of type III (see Theorem 5.2).*

Remark 6.3. The difficulty of categorical 4T-relation mentioned above suggests that we should need higher categorical setting in order to categorify Vassiliev theory. One possible candidate is the notion of *pretriangulated dg-categories*. In fact, using this framework, 4T-relation is represented as exactness of a 2-cycle spanned by crossing changes and Reidemeister III-moves.

Obtained the fundamental relations in Kontsevich invariants, we are now sure that the crossing-change morphism $\widehat{\Phi}$ is the "right" one in view of Vassiliev theory. In addition, we highly expect that there should be a categorified version of *weight systems* so that Khovanov homology is expanded into them, which we are searching for.

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RIKEN Center for Advanced Intelligence Project, Tokyo 103-0027
JAPAN
E-mail address: `jun.yoshida@riken.jp`

理化学研究所革新知能統合研究センター　吉田純