

Seiberg-Witten Floer homotopy and contact structures

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1 Backgrounds

We develop homotopy theoretical aspects of Seiberg-Witten theory and give two kinds of applications to low dimensional topology. This paper is a survey of [IMT21].

1.1 Several open questions to low dimensional topology

In [IMT21], we consider the following two problems:

- (i) sliceness problem of knots in general 4-manifolds, and
- (ii) giving constraints on Betti numbers of symplectic caps.

We mainly review background of (i) in this subsection. For (ii), see Subsection 1.4.

Definition 1.1. A knot K in S^3 is *smoothly (resp. topologically) slice* if K bounds a smooth (resp. locally flat) properly embedded 2-disk in D^4 .

It is known that a topologically slice but not smoothly slice knot provides an exotic \mathbb{R}^4 . Thus sliceness is closely related to smooth structures of 4-manifolds. Also, the concordance relation $K_1 \sim K_2$ is defined as the sliceness of $K_1 \# (-\overline{K}_2)$, where \overline{K} means the mirror image and $-K$ is K with the opposite sign. The quotient set

$$\mathcal{C} := \{ \text{all oriented knots} \} / \sim \text{ :concordance}$$

is called the *knot concordance group* and it admits an abelian group structure via the connected sum. The group \mathcal{C} has been studied via various techniques. There are several effective tools to study the subgroup

$$\mathcal{T} := \{ \text{topologically slice knots} \} \subset \mathcal{C}$$

in various theories including Heegaard Floer theory [OS03, MO07, OS08, OS11, Hom14, HW16, OSS17, HM17, DHST19, AKS20], Khovanov homology [Ras10] and gauge theory [KM13, DS19, KM19]. For example, there are several studies finding \mathbb{Z}^∞ -subgroups or summands in \mathcal{T} in Heegaard Floer theory [OSS17, KP18, Hom19, AKS20].

Instead of D^4 , we consider sliceness for general 4-manifolds with S^3 -boundary. Let X be a closed, oriented, connected, smooth 4-manifold and K a knot in S^3

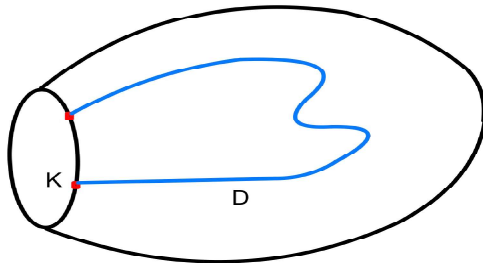


Figure 1: Slice disk in $X \setminus \text{int } D^4$

Definition 1.2. [MMSW19, Definition 6.2] K is *smoothly (resp. topologically) H-slice* in X if K bounds a properly embedded smooth (resp. locally flat) null-homologous disc D in $X^\circ = X - \text{Int } B^4$.

In 4-dimensional topology, the following quantity is important:

$$b^+(X) := \dim(\text{a maximal positive definite subspace of the intersection form of } X).$$

The following are known typical obstructions to H-sliceness:

- (i) obstructions to topological H-sliceness : signature and Rochlin type theorem [Gil81, KR20, MMP20a, Rob65]
- (ii) obstructions to smooth H-sliceness in definite 4-manifolds ($b^+ = 0$) : Heegaard Floer τ -invariant [OS03], Rasmussen invariant [Ras10], Thurston-Bennequin number [Pla04], other Heegaard Floer and gauge theoretic obstructions [KM13, MMSW19]
- (iii) obstructions to smooth H-sliceness in indefinite 4-manifolds ($b^+ > 0$): generalized Thurston-Bennequin inequality [MR06], 10/8-type constraints [MMP20a], an obstruction from Bauer-Furuta invariant [MMP20a]

It is proved in [Schneiderman'10] that, for a knot K in S^3 , there exists $N > 0$ such that K is smoothly H-slice in $\#_N S^2 \times S^2$. Thus H-sliceness depends on 4-manifolds. Also, recently, in [MMP20b], the existence of an exotic pair (X, X') of closed 4-manifolds admitting smoothly H-slice knot K in X but not in X' is proved. As above, H-sliceness is also related to smooth structures of 4-manifolds. As a main result, we give an obstruction to H-sliceness in a certain class of 4-manifolds including a certain class of symplectic 4-manifolds. Our main tool is the author's invariant [Iid19] defined via Seiberg-Witten theory.

1.2 Spin^c structures and Dirac operators

In this subsection, we introduce Spin^c structures and Dirac operators as preliminaries of Seiberg-Witten theory. For more details, see [KM07]. Spin^c structures give a nice 1st order elliptic operator called the Dirac operator. Dirac operators originate in Dirac's approach to give a physical description of electrons based on the theory of special relativity and quantum mechanics.

Definition 1.3. A *Spin^c structure* on a Riemannian 4-manifold (X, g) is a pair

$$\mathfrak{s} = (S = S^+ \oplus S^-, \rho : \Lambda^* T^* X \otimes \mathbb{C} \rightarrow \text{Hom}(S, S)),$$

where S^+ and S^- are hermite vector bundles with rank 2 called positive and negative spinor bundles and ρ is a bundle map called the *Clifford multiplication*. Relations

$$\rho(e^i)\rho(e^j) + \rho(e^j)\rho(e^i) = -2g(e^i, e^j)$$

and

$$\rho(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \rho(e^{i_1}) \cdots \rho(e^{i_k}), \quad i_1 < \cdots < i_k$$

are imposed for any local orthonormal frame e^0, e^1, e^2, e^3 . The bundles S^+ and S^- are -1 and $+1$ eigenspaces for

$$\rho(e^0 \wedge e^1 \wedge e^2 \wedge e^3) : S \rightarrow S.$$

respectively.

Two *Spin^c* structures (S, ρ) and (S', ρ') are *isomorphic* if there exists unitary isomorphism $S \rightarrow S'$ which intertwines with Clifford multiplications.

Definition 1.4. A unitary connection $\nabla^A = A$ on S is called a *Spin^c structure* if it satisfies

$$\nabla_V^A(\rho(W)\Phi) = \rho(\nabla_V^A W)\Phi + \rho(W)\nabla_V^A \Phi$$

for any vector fields V, W and any section Φ of S .

Let us denote by $\mathcal{A}(\mathfrak{s})$ the space of *Spin^c* connections. This is an affine space modeled on the space of imaginary valued 1-forms $i\Omega_X^1$.

Definition 1.5. The *Dirac operator* is given by

$$D_A = \rho \circ \nabla_A = \sum_{i=0}^3 \rho(e_i) \nabla_{e_i}^A : \Gamma(S) \rightarrow \Gamma(S).$$

The operator D_A is decomposed as

$$D_A = \begin{bmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{bmatrix}$$

with respect to the decomposition $S = S^+ \oplus S^-$, where

$$D_A^+ : \Gamma(S^+) \rightarrow \Gamma(S^-) \text{ and } D_A^- : \Gamma(S^-) \rightarrow \Gamma(S^+).$$

The operators D_A^\pm are first order elliptic operators. Although a spin structure does not always exist on 4-manifolds, a *spin^c* structure exists on any 4-manifold. Let us write

$$\text{Spin}_X^c := \{\text{all spin}^c \text{ structures}\} / \text{isomorphism}.$$

There is a free transitive action of $H^2(X; \mathbb{Z})$ on Spin_X^c for any closed 4-manifolds. Thus there a one to one correspondence (non-canonical) between $H^2(X; \mathbb{Z})$ and Spin_X^c .

1.3 Seiberg-Witten theory

In this subsection, we review Seiberg-Witten theory. For more details, see [KM07]. Seiberg-Witten theory has been an effective tool in the studies of 3 and 4-manifolds. A typical example of such studies is finding an exotic pair of 4-manifolds. This theory is based on a non-linear PDE called the Seiberg-Witten equation.

Let X be a closed oriented 4-manifold. For simplicity we assume $b_1(X) = 0$. Fix a Riemannian metric and a Spin^c structure \mathfrak{s} on X .

Definition 1.6. The *Seiberg-Witten equation* is a non-linear 1-st order PDE for a pair $(A, \Phi) \in \mathcal{A}(\mathfrak{s}) \times \Gamma(S^+)$ defined by

$$\begin{cases} \frac{1}{2}\rho(F_{A^t}^+) = (\Phi\Phi^*)_0 \\ D_A^+\Phi = 0, \end{cases} \quad (1)$$

where F_{A^t} is the curvature 2-form of the $U(1)$ connection induced from A on the $U(1)$ bundle $\det S^+$, $+$ in $F_{A^t}^+$ means the self dual component with respect to the Hodge star operator $*$: $\Omega_X^2 \rightarrow \Omega_X^2$, and $(\Phi\Phi^*)_0$ is the traceless part of $\Phi\Phi^* : S^+ \rightarrow S^+$.

Roughly speaking, the Seiberg-Witten invariant of (X, \mathfrak{s}) is obtained by counting the number of points in the quotient space

$$\mathcal{M}(X, \mathfrak{s}) := \{ \text{all solutions } (A, \Phi) \text{ to (1)} \} / \mathcal{G}_X,$$

where

$$\mathcal{G}_X := \text{Map}(X, U(1))$$

and the action of $u \in \mathcal{G}_X$ on (A, Φ) is given by

$$u \cdot (A, \Phi) := (A - u^{-1}du, u\Phi).$$

However, in general, the moduli space $\mathcal{M}(X, \mathfrak{s})$ may have quotient singularities. The quantity $b^+(X)$ control the existence of singularities. When $b^+ > 0$, for a generic perturbation, $\mathcal{M}(X, \mathfrak{s})$ is an orientable compact smooth manifold of dimension

$$d(\mathfrak{s}) := \frac{1}{4}(c_1^2(S^+)[X] - 2\chi(X) - 3\sigma(X)) = c_2(S^+)[X].$$

Fix an orientation of a line

$$\Lambda(\mathfrak{s}) := \Lambda^{\max}(H^0(X; \mathbb{R}) \oplus H^1(X; \mathbb{R}) \oplus H^+(X; \mathbb{R})).$$

Then an orientation of $\mathcal{M}(X, \mathfrak{s})$ is induced from it. Then the Seiberg-Witten invariant is defined to be

$$SW(X, \mathfrak{s}) := \int_{\mathcal{M}(X, \mathfrak{s})} U^{\frac{1}{2}d(\mathfrak{s})},$$

where U is a cohomology class coming from an isomorphism

$$H^*((\mathcal{A}(\mathfrak{s}) \times (\Gamma(S^+) \setminus \{0\})) / \mathcal{G}_X : \mathbb{Z}) \cong H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[U].$$

When $b^+(X) > 1$, $SW(X, \mathfrak{s})$ is independent on the choices of perturbations. Thus $SW(X, \mathfrak{s})$ is an invariant of (X, \mathfrak{s}) . This construction gives a map

$$SW_X : \text{Spin}_X^c \rightarrow \mathbb{Z}.$$

The map SW gives a strong tool to study smooth structures of 4-manifolds.

Example 1.7. When $X = 3\mathbb{C}P^2 \# 20\overline{\mathbb{C}P^2}$, $SW_X = 0$. But for $X' = K3 \# \overline{\mathbb{C}P^2}$, $SW_{X'} \neq 0$. Since X and X' have the same intersection form, by Freedman theory [Fre82], X and X' are homeomorphic. Thus we can conclude that X and X' give an exotic pair.

In 2004, Bauer-Furuta [BF04] introduced a refinement of the Seiberg-Witten invariant:

$$BF_X : \text{Spin}_X^c \rightarrow \bigoplus_{(m,n) \in \mathbb{Z} \times \mathbb{Z}} \lim_{M,N \rightarrow \infty} [(\mathbb{R}^{m+M} \oplus \mathbb{C}^{n+N})^+, (\mathbb{R}^M \oplus \mathbb{C}^N)^+]_{S^1}.$$

The invariant BF_X is defined by using a method called *finite dimensional approximation* of the Seiberg-Witten equation.

It is proved in [BF04] that BF_X recovers SW_X when we fix an orientation of $\Lambda_{\mathfrak{s}}$ for any \mathfrak{s} .

The following example gives a pair of 4-manifolds detected by the Bauer-Furuta invariant but not by the Seiberg-Witten invariant.

Example 1.8. When $X = 6\mathbb{C}P^2 \# 39\overline{\mathbb{C}P^2}$, $BF_X = \text{constant}$. But, when $X' = K3 \# K3 \# \overline{\mathbb{C}P^2}$, then we still have nontrivial result $BF_{X'} \neq \text{constant}$. Since X and X' have the same intersection form, by Freedman theory [Fre82], X and X' are homeomorphic. Thus X and X' give an exotic pair.

1.4 Symplectic and contact structures

In order to obtain obstruction to H-sliceness of knots, we use symplectic fillings and find a solution to Seiberg-Witten equation. For more details on contact structures in 3-dimension and symplectic structures in 4-dimension, see [OS04, GS99].

Symplectic structures are structure on even dimensional manifolds and originate from the classical mechanics. Let X be an even dimensional oriented manifold. Denote its dimension by $2n$.

Definition 1.9. A 2-form ω on X is called a *symplectic form* if it is a closed form and $\omega^n > 0$.

Liouville vector fields and contact type hypersurfaces serve as convenient tools to conduct cut-and-paste operation respecting symplectic structures. Contact type hypersurfaces are also typical examples of contact manifolds. First we give the definition of the contact structure.

Definition 1.10. Let Y be an oriented odd dimensional manifold. Denote its dimension by $2n + 1$. A codimension 1 distribution $\xi \subset TY$ is called a contact structure if there exists a 1-form θ on Y such that

$$\text{Ker } \theta = \xi \text{ and } \theta \wedge (d\theta)^n > 0.$$

θ is called a contact 1-form. The following is a natural situation in which contact structures appear.

Definition 1.11. A vector field v defined on a symplectic manifold (X, ω) is called a *Liouville vector field* if

$$\mathcal{L}_v \omega = \omega$$

holds. A codimension 1 submanifold $Y \subset X$ is called a contact type hypersurface if there exists a Liouville vector field defined near Y and transverse to Y . Then $\theta := \iota_v \omega$ is a contact form on Y .

Let $(Y_0, \xi_0), (Y_1, \xi_1)$ be contact manifolds with the same dimensions.

Definition 1.12. A cobordism equipped with a symplectic structure (W, ω) from Y_0 to Y_1 is called a *strong symplectic cobordism* if there exists a Liouville vector field v defined near collar neighborhoods on ∂W such that

1. v is transverse to ∂W
2. v is inward on Y_0 and outward on Y_1 , and
3. for $i = 0, 1$, $\xi_i = \text{Ker}(\iota_v \omega)$ on Y_i .

The contact manifold (Y_0, ξ_0) is called the *concave boundary* of (W, ω) and (Y_1, ξ_1) called the *convex boundary* of (W, ω) .

Definition 1.13. A symplectic cobordism (W, ω) is called a *symplectic filling* of (Y_1, ξ_1) when Y_0 is empty and called a *symplectic cap* of (Y_0, ξ_0) when Y_1 is empty.

If $(W_0, \omega_0) : (Y_0, \xi_0) \rightarrow (Y_1, \xi_1)$ and $(W_1, \omega_1) : (Y_1, \xi_1) \rightarrow (Y_2, \xi_2)$ are strong symplectic cobordisms, we can glue two symplectic structures and construct another strong symplectic cobordism $(W_0 \cup W_1, \omega_0 \cup \omega_1) : (Y_0, \xi_0) \rightarrow (Y_2, \xi_2)$. An important remark is that there is asymmetry between convex and concave boundaries.

Definition 1.14. A contact structure on a closed manifold is called *symplectically fillable* if it has a symplectic filling.

In this paper, we focus on dimension 3 and 4. It is shown by Etnyre-Honda [EH02] independently that any closed contact 3-manifold has a symplectic cap. On the other hand, a closed contact manifold does not have a symplectic filling in general. For example, it is proved in [Lis98] that $-\Sigma(2, 3, 5)$ does not admit any symplectic fillings. There are several known constraints on the topology of symplectic fillings of a contact 3-manifold (Y, ξ) , but not much is known about the topology of symplectic caps.

We give a topological constraint on symplectic caps for certain spherical 3-manifolds.

We will also use a knot invariant coming from contact geometry called *maximal Thurston-Bennequin invariant*. Let Y be an oriented homology 3-sphere and ξ a contact structure.

Definition 1.15. A *Legendrian knot* K is a knot in (Y, ξ) such that $T_p K \subset \xi_p$ for any $p \in K$.

A Legendrian knot has a diagrammatic representation called a *front projection*. A Legendrian knot K admits a framing coming from $\xi|_K$, which is called the *contact framing*.

Definition 1.16. For any knot $K \subset S^3$, *maximal Thurston–Bennequin invariant* for K is defined to be

$$TB(K) := \max_{\forall \text{Legendrian rep of } K \in (S^3, \xi_{std})} \{ (\text{contact framing}) - (\text{Seifert framing}) \},$$

where ξ_{std} is the standard contact structure on S^3 .

One can check that $TB(K)$ is a \mathbb{Z} -valued isotopy invariant of knots. The important property of $TB(K)$ is following Thurston–Bennequin type inequality proved by Plamenevskaya [Pla04]:

$$TB(K) \leq 2\tau(K) - 1,$$

where $\tau(K)$ is a concordance invariant called the *tau-invariant* introduced in [OS03]. Since the tau invariant gives a lower bound for the smooth 4-ball genus, $TB(K)$ obstructs H-sliceness in D^4 and, also in negative definite 4-manifold with S^3 -boundary satisfying $b_1 = 0$.

1.5 Seiberg-Witten theory and symplectic/contact structures

Taubes’s non-vanishing result of the Seiberg-Witten invariant for symplectic manifolds is one of the most fundamental results in the relation between Seiberg-Witten theory and symplectic/contact structures. Before stating this result, note that a symplectic structure determines a canonical Spin^c structure. We explain it in the case of dimension 4. Let (X, ω) be a symplectic 4-manifold. There exists a compatible almost complex structure J unique up to homotopy. A Riemannian metric on X is determined by

$$g_J = \omega(\cdot, J\cdot).$$

The triple (ω, J, g_J) consist so-called almost Kähler structure.

Then

$$\begin{cases} S^+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \\ S^- = \Lambda^{0,1} \\ \rho = \sqrt{2} \text{Symb}(\bar{\partial} + \bar{\partial}^*) : T^*X \rightarrow \text{Hom}(S^+, S^-) \end{cases}$$

gives a Spin^c structure on X . Here $\text{Symb}(\bar{\partial} + \bar{\partial}^*)$ is the principal symbol of

$$\bar{\partial} + \bar{\partial}^* : \Omega_X^{0,0} \oplus \Omega_X^{0,2} \rightarrow \Omega_X^{0,1}.$$

This construction gives a well-defined isomorphism class of Spin^c structure. We denote it by \mathfrak{s}_ω .

Theorem 1.17 ([Tau94]). *Let (X, ω) be a closed symplectic 4-manifold with $b^+(X) \geq 2$. Then*

$$SW_X(\mathfrak{s}_\omega) = \pm 1.$$

Actually, we can give a solution under a certain perturbation: Fix (ω, J, g_J) as before. Define $(A_0, \Phi_0) \in \mathcal{A}(\mathfrak{s}_\omega) \times \Gamma(S^+)$ as follows.

$$\Phi_0 = (1, 0) \in \Omega_X^{0,0} \oplus \Omega_X^{0,2} = \Gamma(S^+).$$

Note that the virtual dimension is

$$d(\mathfrak{s}_\omega) = c_2(S^+) = 0$$

since Φ_0 is a nowhere-vanishing section of S^+ . The map

$$\begin{aligned} \mathcal{A}(\mathfrak{s}_\omega) &\rightarrow \Omega^1(i\mathbb{R} \oplus \Lambda^{0,2}) \\ A &\mapsto \nabla_A \Phi_0 \end{aligned}$$

is well defined since a Spin^c connection is unitary. Since this is a $i\Omega_X^1$ -affine map, there exists unique $A_0 \in \mathcal{A}(\mathfrak{s}_\omega)$ such that $\Omega^1(i\mathbb{R})$ part of $\nabla_{A_0} \Phi_0$ is zero. Then we can check that

$$D_{A_0}^+ \Phi_0 = 0.$$

See for example [HT99]. Obviously, (A_0, Φ_0) is a solution to

$$\frac{1}{2}\rho(F_A^+) - (\Phi\Phi^*)_0 = \frac{1}{2}\rho(F_{A_0}^+) - (\Phi_0\Phi_0^*)_0.$$

Thus (A_0, Φ_0) is a solution to the Seiberg-Witten equation perturbed by $\frac{1}{2}\rho^{-1}(F_{A_0}^+) - (\Phi_0\Phi_0^*)_0$. Furthermore, we can show that for a large constant r , $(A_0, r\Phi_0)$ is the unique solution to the perturbed equation

$$\begin{cases} \frac{1}{2}\rho(F_A^+) - (\Phi\Phi^*)_0 &= \frac{1}{2}\rho(F_{A_0}^+) - r(\Phi_0\Phi_0^*)_0 \\ D_A^+ \Phi &= 0. \end{cases}$$

and actually contributes to ± 1 to the Seiberg-Witten invariant (i.e. transversality is automatically satisfied. See Lemma 3.11 of [KM97]).

Inspired by Taubes's work on Seiberg-Witten theory on symplectic manifolds, Kronheimer-Mrowka ([KM97]) constructed an integer valued invariant (defined up to sign) for 4-manifold with contact boundary. Note that an oriented 2-plane field ξ on a oriented 3-manifold Y defines a Spin^c structure \mathfrak{s}_ξ as follows: Let W be a compact oriented 4-manifold with non-empty boundary and $\xi = \text{Ker } \theta$ be a contact structure on the boundary. Fix a complex structure J on ξ . Then

$$g_1 = \theta \otimes \theta + \frac{1}{2}d\theta(\cdot, J\cdot)$$

is a Riemannian metric on ∂W . Consider a manifold

$$C = \mathbb{R}_t^{\geq 1} \times \partial W$$

equipped with a conical metric

$$g_0 = dt^2 + t^2 g_1$$

and symplectic structure

$$\omega_0 = \frac{1}{2}d(t^2\theta).$$

These define an almost Kähler structure on C and in turn determines a Spin^c structure \mathfrak{s}_0 . Let

$$W^+ = W \cup_{\partial W} C$$

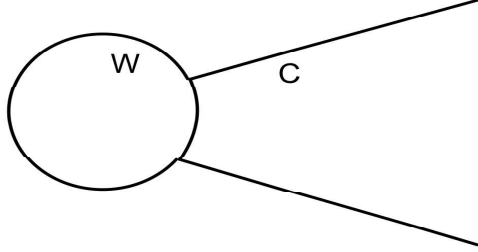


Figure 2: Conical-end 4-manifold

be a 4-manifold with a conical end obtained by gluing W and C along boundaries and $\text{Spin}^c(W, \xi)$ be the set of isomorphis class of pairs (\mathfrak{s}, h) where \mathfrak{s} is a Spin^c structure on W^+ and $h : \mathfrak{s}|_W \rightarrow \mathfrak{s}_0$ is an isomorphism. This is a $H^2(W, \partial W; \mathbb{Z})$ torsor. Kronheimer-Mrowka defnied an invariant

$$KM_{X, \xi} : \text{Spin}^c(W, \xi) \rightarrow \mathbb{Z}$$

up to overall sign. This is constructed from the perturbed Seiberg-Witten equation on W^+ .

Using Bauer-Furuta's method of finite dimensional approximation, the first author [Iid19] defined a refinement of Kronheimer-Mrowka's invariant

$$BF(W, \xi, \mathfrak{s}) \in \pi_{d(\mathfrak{s})}^{st} / \pm 1$$

for $\mathfrak{s} \in \text{Spin}^c(W, \xi)$ when $b_3(W) = 0$. Here

$$d(\mathfrak{s}) = \langle e(S^+, \Phi_0), [W, \partial W] \rangle$$

is the relative Euler number relative to the section Φ_0 on C defined from the almost Kähler structure. The first author proved that the mapping degree of $BF(W, \xi, \mathfrak{s})$ is equal to Kronheimer-Mrowka's invariant when $d(\mathfrak{s}) = 0$.

Using the connected sum formula below, we can prove a certain non-vanishing result.

Theorem 1.18 ([Iid19]). *Let (W, \mathfrak{s}_W) be an oriented Spin^c compact 4-manifold whose boundary is a contact 3-manifold (Y, ξ) with $b_3(W) = 0$, $\mathfrak{s}_W|_Y = \mathfrak{s}_\xi$ and let (X, \mathfrak{s}_X) be a closed Spin^c 4-manifold with $b_1(X) = 0$. Then, we have*

$$\Psi(W \# X, \xi, \mathfrak{s}_W \# \mathfrak{s}_X) = \Psi(W, \xi, \mathfrak{s}_W) \wedge BF(X, \mathfrak{s}_X) \quad (2)$$

in the stable homotopy group up to sign. Here we forget the S^1 action of $BF(X, \mathfrak{s}_X)$.

Since Iida's invariant is $\pm \text{Id}$ up to sign and stable homotopy for any weak symplectic filling with $b_3 = 0$ [Iid19, Corollary 4.3], thus by Theorem 1.18, we obtain the following non-vanishing results.

Theorem 1.19 ([Iid19]). *Let (W, ω) be a weak symplectic filling of a contact 3-manifold (Y, ξ) with $b_3(W) = 0$. We consider a closed symplectic 4-manifold (X_1, ω_1) with $b_1(X_i) = 0$ and $b_2^+(X_i) \equiv 3 \pmod{4}$. Then, one has*

$$\Psi(W \# X_1, \mathfrak{s}_\omega \# \mathfrak{s} \# \mathfrak{s}_{\omega_1}) \neq 0.$$

2 Statements of results

2.1 Obstructions to H-sliceness

Using techniques in contact/ symplectic topology and a certain adjunction type inequality for Lida's invariant, we prove the following non-H-sliceness results for symplectic 4-manifolds.

Theorem 2.1 ([IMT21]). *For a knot $K \subset S^3$ satisfying $TB(K) > 0$, K is not smoothly H-slice in any closed symplectic 4-manifold X with $b_1 = 0$ and $b^+ \equiv 3 \pmod{4}$.*

A sequence of closed symplectic 4-manifolds with $b_1 = 0$ and $b^+ \equiv 3 \pmod{4}$ is given by elliptic surfaces $\{E(2n)\}_{n \in \mathbb{Z}_{>0}}$. Let $T(p, q)$ be the (p, q) -torus knot. As a consequence, we can prove the following result:

Corollary 2.2. *The Whitehead doubles $\{Wh_0^+(T_{2,2^n-1})\}$ of torus knots are not smoothly H-slice in $K3$.*

Proof. It is proven in [EH01] that

$$TB(T(p, q)) = pq - p - q \text{ if } p, q > 0.$$

It is not difficult to see the following fact using front projection of Legendrian knots:

Lemma 2.3. *For any knot K in S^3 satisfying $TB(K) > 0$, one has $TB(Wh_0^+(K)) > 0$.*

Thus $\{Wh_0^+(T_{2,2^n-1})\}$ and $K3$ satisfy assumptions of Theorem 2.1. \square

Moreover, our sequence $\{Wh_0^+(T_{2,2^n-1})\}$ is interesting in the following sense:

- (i) $\{Wh_0^+(T_{2,2^n-1})\}$ bounds topologically disk in D^4 , hence topologically disk in $K3$ ([Fre82]),
- (ii) $Wh_0^+(T_{2,2^n-1})$ is smoothly slice in $K3$ ([MMP20b]), and
- (iii) $\{Wh_0^+(T_{2,2^n-1})\}$ are linearly independent in the knot concordance group \mathcal{C} ([HK12]).

Let us give a sketch of proof. The main gauge theoretic ingredient is the following vanishing result:

Theorem 2.4 ([IMT21]). *Let (W, \mathfrak{s}) be an oriented $Spin^c$ compact 4-manifold whose boundary is a contact 3-manifold (Y, ξ) with $b_3(W) = 0$ and $\mathfrak{s}|_Y = \mathfrak{s}_\xi$.*

If a non-torsion homology class in $H_2(W, \partial W; \mathbb{Z})$ is realized by a smoothly embedded 2-sphere whose self-intersection number is non-negative, then

$$\Psi(W, \xi, \mathfrak{s}_W) = 0.$$

This is a special case of the adjunction inequality given in [IMT21]. The proof of Theorem 2.4 is a standard neck stretching argument as in the original proof of Kronheimer-Mrowka's adjunction inequality [KM94].

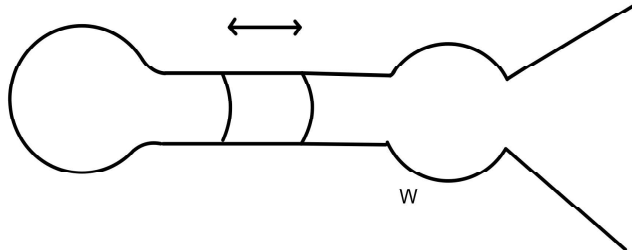


Figure 3: Neck stretching argument in the proof of Theorem 2.4

Sketch of proof of Theorem 2.1. A tubular neighborhood of a H-slice disk in $X \setminus \text{int } D^4$ is diffeomorphic to the trace $W_0(\overline{K})$ of 0-surgery along the mirror image of K . We consider the trace $W_0(K)$ of 0-surgery along K , which admits a symplectic filling structure by the assumption $TB(K) > 0$. We take a symplectic cobordism W from $\partial(W_0(K))$ to a some contact rational homology 3-sphere (Y, ξ) . Set

$$Z := (W_0(K) \cup_{\partial W_0(K)} W) \# X.$$

Then by the non-vanishing result Theorem 1.19, one has

$$\Psi(Z, \xi, \mathfrak{s}_Z) \neq 0.$$

where Z is the Spin^c structure induced by the symplectic structures on X , W , and $W_0(K)$. On the other hand,

$$S := (\text{core of } W_0(K)) \cup K \times I \cup (\text{core of } W_0(\overline{K}))$$

gives a smoothly embedded 2-sphere in Z such that

- $S \cdot S = 0$ and
- S is not a torsion in $H_2(Z, \partial Z)$.

Thus, by Theorem 2.4, we see

$$\Psi(Z, \xi, \mathfrak{s}_Z) = 0.$$

This gives a contradiction. □

2.2 Topology of symplectic caps

We give a constraint of topology of symplectic caps. In particular, we give a constraint on b^+ .

Let X be a compact 4-manifold with connected boundary Y . The proof uses vanishing results of Ψ , which follows from the following geometric setting.

Definition 2.5 ([IMT21]). We say X has a *geometrically isolated 2-handle* if there is a 2-handle h in a handle decomposition of X such that h does not intersect any 1-handles in that handle decomposition, and the core of h is not a torsion in $H_2(X; \mathbb{Z})$.

Any 4-manifolds obtained as 2-handle surgery have geometrically isolated 2-handle.

Our result on symplectic caps is as follows.

Theorem 2.6 ([IMT21]). *The following results hold:*

- (i) *The contact 3-manifold (S^3, ξ_{std}) does not admit any positive definite symplectic cap having a geometrically isolated 2-handle such that $b_1 = 0$ and $b_2^+ \geq 2$.*
- (ii) *The contact 3-manifold $\Sigma(2, 3, 5)$ with the unique tight contact structure does not admit any positive definite symplectic cap having a geometrically isolated 2-handle such that $b_1 = 0$ and $b_2^+ \geq 2$, and there is no 2-torsion on its homology.*

Remark 2.7. We have three remarks:

1. Since $\mathbb{C}P^2 \setminus \text{int } D^4$ gives a symplectic cap of (S^3, ξ_{std}) , $b_2^+(X) > 1$ is necessary.
2. We can recover the result by using Theorem 2.6(i) that any positive definite geometrically simply connected closed 4-manifold with $b_2^+ > 1$ does not admit a symplectic structure proved in [HL19, Theorem 1.1], [Yas19, Corollary 1.6].
3. The proof uses classification results for intersection forms of negative definite 4-manifolds bounded by S^3 and $\Sigma(2, 3, 5)$ proven by Donaldson [Don83] and Scaduto [Sca18].

Since there are few studies of topology of symplectic caps, Theorem 2.6 gives an interesting constraint.

Let us give a sketch of proof of Theorem 2.6. First, we prove the following constraint coming from Bauer-Furuta type invariants:

Theorem 2.8 ([IMT21]). *Let (Y, ξ) be a contact 3-manifold with a symplectic filling that has $b_1 = 0$. If $b_1(Y) = 0$ and Y admits a positive scalar curvature metric, then (Y, ξ) does not have positive definite symplectic cap X with $b_1 = 0$ and $b_2^+ \geq 2$ having a geometrically isolated 2-handle and a Spin^c structure \mathfrak{s}_X such that*

$$\frac{-c_1^2(\mathfrak{s}_X) + b_2(X)}{8} = \delta(Y, \mathfrak{s}_\xi),$$

where $\delta(Y, \mathfrak{s})$ is Frøyshov invariant of (Y, \mathfrak{s}) with the convention $\delta(\Sigma(2, 3, 5)) = 1$.

Sketch of proof of Theorem 2.8. Let W be a symplectic filling of (Y, ξ) . We consider

$$Z := (X \cup W) \# (-X).$$

We use the relative Bauer-Furuta invariant ([Man03, Kha15]) BF for the 4-manifold Z with boundary Y . Note that $X \cup W$ admits a symplectic structure. The assumptions for X imply

$$BF_{-X}(\mathfrak{s}_X) = id.$$

Since $X \cup W$ has a symplectic structure, the Spin^c structure \mathfrak{s}_Z obtained by the connected sum of the Spin^c structure coming from the symplectic structure on $X \cup W$ and \mathfrak{s}_X satisfies

$$BF_Z(\mathfrak{s}_Z) \neq 0.$$

By sliding a geometrically isolated 2-handle on X to the corresponding geometrically isolated 2-handle of $-X$, we again find an embedded 2-sphere with self intersection number 0. The standard neck stretching argument shows

$$BF_Z(\mathfrak{s}) = 0.$$

for all Spin^c structure \mathfrak{s} . This gives a contradiction. \square

Sketch of proof of Theorem 2.6. Using Theorem 2.8 and Donaldson's theorem A [Don83] or a classification result of intersection forms of negative definite 4-manifolds bounded by $\Sigma(2, 3, 5)$ [Sca18], we can obtain a contradiction. \square

3 Open problems

Theorem 2.8 suggests the following conjecture:

Conjecture 3.1 ([IMT21]). A contact manifold (Y, ξ) , where Y is an L -space and ξ is symplectically fillable, does not have a simply connected positive definite symplectic cap with $b_2^+ > 1$.

As a related question, for lens spaces, we have:

Question 3.2. Can we prove an analogue of Theorem 2.6 for lens spaces?

In the proof of Theorem 2.6, the classification of negative definite lattices of 4-manifolds bounding a given 3-manifold is important. Thus we relate Question 3.2 with the following problem:

Question 3.3. Classify all isomorphism classes of negative definite lattices of 4-manifolds bounding the lens space $L(p, q)$.

This problem is related to $\mathcal{I}(L(p, q))$ defined in [CP18].

We did not consider general 3-manifolds in Theorem 2.1. However, the generalized Thurston-Bennequin inequality is still true for general 3-manifolds [MR06].

Question 3.4. Can we prove Theorem 2.1 for general contact 3-manifolds?

Also, we have another direction to prove the generalized Thurston-Bennequin inequality given in [MR06] for Baur-Furuta type invariant. Originally, the generalized Thurston-Bennequin inequality is proved when Kronheimer-Mrowka's invariant [KM97] is non-trivial. Since Iida's invariant refines Kronheimer-Mrowka's invariant, it is natural to ask:

Question 3.5. Can we prove the generalized Thurston-Bennequin inequality for Iida's invariant?

In [IT20], the relative version of Iida's invariant [Iid19]

$$\Psi(Y, \xi) : S^0 \rightarrow \Sigma^* SWF(-Y, \mathfrak{s}_\xi)$$

is defined for any contact rational homology 3-sphere (Y, ξ) , where $SWF(Y, \mathfrak{s}_\xi)$ is Manolescu's Seiberg-Witten-Floer stable homotopy type given in [Man03] and $*$ is a rational number determined by (Y, ξ) . The invariant $\Psi(Y, \xi)$ is called *Seiberg-Witten Floer homotopy contact invariant*. On this invariant, we have the following conjecture:

Conjecture 3.6 ([IT20]). The invariant $\Psi(Y, \xi) : S^0 \rightarrow \Sigma^* SWF(-Y, \mathfrak{s})$ recovers Kronheimer-Mrowka-Ozváth-Szabó ([KMOS07])'s contact invariant

$$\psi(Y, \xi) \in \check{H}M_\bullet(-Y), \quad (3)$$

where $\check{H}M_\bullet(-Y)$ is the monopole Floer homology of Y .

See table 1 for relations between several "Seiberg-Witten type" invariants.

Also, there are not many calculations of $\Psi(Y, \xi)$. For example, the following is open:

Conjecture 3.7. For any overtwisted contact structure, the invariant Ψ given in [IT20] vanishes.

We list related invariants as follows:

	Counting	Finite dimensional approximation
closed 4-manifolds	SW-invariant $\in \mathbb{Z}$	BF-invariant $BF(X) : (\mathbb{R}^m \oplus \mathbb{C}^n)^+ \rightarrow (\mathbb{R}^{m'} \oplus \mathbb{C}^{n'})^+$
4-manifolds with contact boundary	KM-invariant $\in \mathbb{Z}/\{\pm 1\}$	Iida's invariant $\Psi(W, \xi) : (\mathbb{R}^M)^+ \rightarrow (\mathbb{R}^{M'})^+$
closed 3-manifolds	monopole Floer homology group " $HM_\bullet(Y)$ "	SW Floer homotopy type $SWF(Y)$
4-manifolds with boundary	relative SW invariant " $\psi(X) \in HM_\bullet(\partial X)$ "	relative BF invariant $BF(X) : (\mathbb{R}^m \oplus \mathbb{C}^n)^+ \rightarrow SWF(\partial X)$
contact 3-manifolds	contact invariant $\psi(Y, \xi) \in \check{H}M_\bullet(-Y)$	homotopy contact invariant $\Psi(Y, \xi) : (\mathbb{R}^M)^+ \rightarrow SWF(-Y)$

Table 1: Invariants

Also, it is natural to ask:

Question 3.8. Is there a contact rational homology 3-sphere (Y, ξ) such that

- (1) Floer homotopy contact invariant $\Psi(Y, \xi) : S^0 \rightarrow \Sigma^* SWF(-Y, \mathfrak{s}_\xi)$ ([IT20]) does not vanish, but
- (2) Kronheimer-Mrowka-Ozváth-Szabó ([KMOS07])'s contact invariant $\psi(Y, \xi) \in \check{H}M_\bullet(-Y)$ vanishes?

Since the double branched cover of transverse links admits a natural contact structure, it is interesting to ask:

Question 3.9. Can we define an invariant of transverse links from $\Psi(Y, \xi) : S^0 \rightarrow \Sigma^* SWF(-Y, \mathfrak{s}_\xi)$?

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