On quantum character varieties of knots

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Introduction

Quantum character varieties of knots are considered to be constructed from the skein modules of the knot complements. Here, we start with the skein algebras of punctured disks, apply the theory of Haboro's bottom tangles to describe the braid group action, and then, to get the quantum character variety of the knot complement, pick up the invariant part of the action of the braid representing the knot . The actions of braids are given by matrices and the quantum character variety is given by relations that the determinants of certain matrices are equal to 0.

The first half is the reformulation of our previous work [4] presented in ILDT2020 [3]. Last time, the space of representation is constructed from a braided Hopf algebra, and this time, such space is constructed by using the bottom tangles. The second half is the construction of the quantum character variety of a knot by using the skein algebra of a punctured disk combined with the action of bottom tangles. This is a joint work with Roland van der Veen.

1. Algebra of free ribbons

1.1. **Free ribbons.** Let t be an indeterminate and K be the field $\mathbb{C}(t)$. Let D_k be a k-punctured disk, and $q_1, \ldots, q_k$ are its puncture points. Let p_1 be a point in ∂D_k, which is called a puncture on the boundary of D_k, and p_0 is another point in ∂D_k, which is called the base point. The thickened D_k is $D_k \times I$ where $I = [0, 1]$ is the unit interval. An *open ribbon* in the thickened $D_k \times I$ is non-intersecting framed arc in $I \times D_k$ whose boundary points are contained in $p_0 \times I$. A *closed free ribbon* is a closed framed loop in the thickened D_k. A ribbon in the thickened $D \times I$ is presented by a diagram on D_k as in Figure 1 where the base point p_0 is expressed by an arrow where the right hand point represents the higher points of $p_0 \times I$. Such diagram is called the *ribbon diagram.* Here the framing of a free ribbon is given by the black board framing, that is the framing determined by the normal vector perpendicular to D_k directed upward with respect to the orientation of D_k.

For $n = 0, 1, 2, \ldots$, let $\mathcal{F}_{k,n}$ be formal K-linear combinations of the set of isotopy classes of thickened k-puncture disks equipped with non-intersecting closed ribbons, and let $\mathcal{F}_{k,1}$ be the set of thickened k-puncture disks equipped with several or no non-intersecting closed ribbons and n ribbon with boundaries. Inside the thickened k-puncture disk, we require that there is no intersection of ribbons, and the labels $q_1, \ldots, q_k$ of puncture points are fixed. We call $\mathcal{F}_{k,n}$ *the space of free ribbons.*

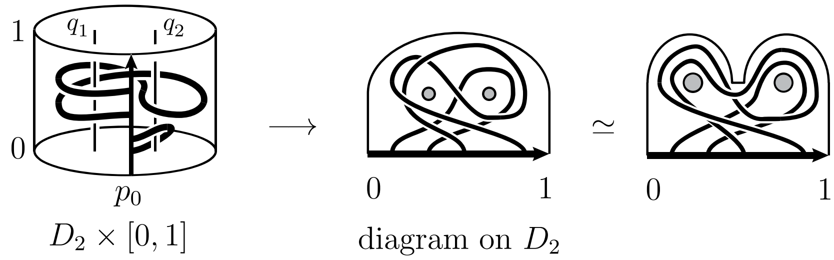


FIGURE 1. Free ribbons in thickened D_2.

The tensor product $\otimes$ from $\mathcal{F}_{k_1,n_1} \times \mathcal{F}_{k_2,n_2}$ to $\mathcal{F}_{k_1+k_2,n_1+n_2}$ is defined by concatenating two punctured disks as in Figure 2.

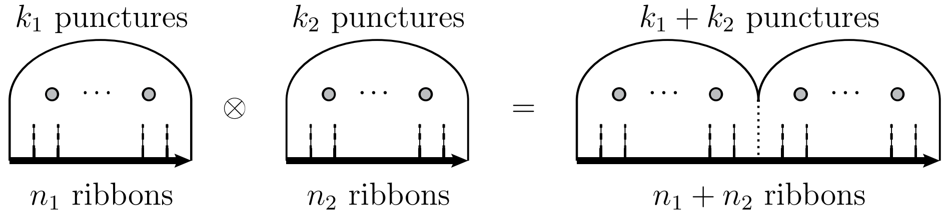


FIGURE 2. Tensor product $\otimes : \mathcal{F}_{k_1,n_1} \times \mathcal{F}_{k_2,n_2} \to \mathcal{F}_{k_1+k_2,n_1+n_2}$.

1.2. **Algebra of free ribbons.** We define a multiplication $\boldsymbol{\mu}$ from $\mathcal{F}_{k,n_1} \times \mathcal{F}_{k,n_2}$ to $\mathcal{F}_{k,n_1+n_2}$ by stacking two punctured disks as in FIgure 3. Let

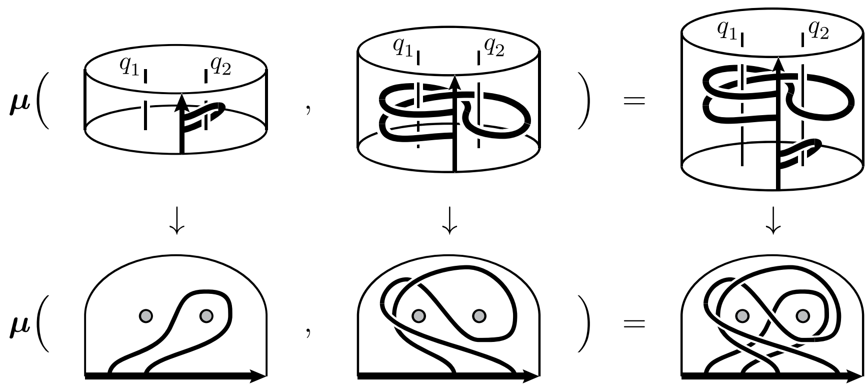


FIGURE 3. Multiplication $\boldsymbol{\mu} : \mathcal{F}_{k,n_1} \times \mathcal{F}_{k,n_2} \to \mathcal{F}_{k,n_1+n_2}$.

$$\mathcal{F}_k = \bigoplus_{n=0,1,\ldots} \mathcal{F}_{k,n},$$

then $\mathcal{F}_k$ is a graded algebra with the multiplication $\boldsymbol{\mu}$ whose grading is given by the number of open ribbons.

For $\mathcal{F}_{k,1}$, we define another product m from $\mathcal{F}_{k,1} \times \mathcal{F}_{k.1}$ to $\mathcal{F}_{k,1}$. Let F_1, F_2 be two ribbon diagrams in $\mathcal{F}_{k,1}$. Then $m(F_1, F_2)$ is obtained from $\boldsymbol{\mu}(F_1, F_2)$ by connecting the upper end point of F_1 to the lower end point of F_2.

2. Action of bottom tangles

2.1. Bottom tangles.

Definition 1. Let $\mathcal{T}_{k,n}$ be the subspace of $\mathcal{F}_{k,n}$, which consists of non-closed free arcs $\boldsymbol{\gamma} = (\gamma_1, \cdots, \gamma_n)$ such that the heights of their end points $h(\gamma_i(0))$ and $h(\gamma_i(1))$ satisfy

$$h(\gamma_1(1)) < h(\gamma_1(0)) < h(\gamma_2(1)) < \cdots < h(\gamma_n(1)) < h(\gamma_n(0)).$$

Then an element of $\mathcal{T}_{k,n}$ is called *a bottom tangle* of type (k, n).

For $T \in \mathcal{T}_{k,\ell}$ and $F \in \mathcal{F}_{\ell,n}$, the composition $T \circ F \in \mathcal{F}_{k,n}$ is defined by glueing the handles of F to the ribbons of T as in Figure 4. This composition gives an algebra structure in $\mathcal{T}_{k,k}$ and the action of $\mathcal{T}_{k,k}$ on $\mathcal{F}_{k,n}$ gives a $\mathcal{T}_{k,k}$ module structure on $\mathcal{F}_{k,n}$.

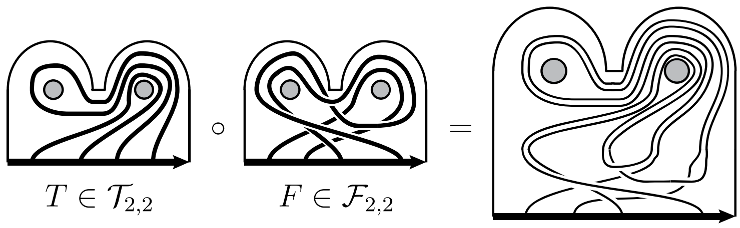


Figure 4. The composition of a bottom tangle $T \in \mathcal{T}_{k,\ell}$ and an element $F \in \mathcal{F}_{\ell,n}$ of the algebra of free ribbons in the case $k = n = \ell = 2$.

2.2. Braided Hopf algebra structure of bottom tangles.

A braided Hopf algebra structure is given to bottom tangles by Habiro in [2] as in Figure 5. The operations in the figure satisfies the axioms of the braided Hopf algebras. We define

$$\mu_i = id^{\otimes(i-1)} \otimes \mu \otimes id^{\otimes(n-i-1)}, \qquad \Delta_i = id^{\otimes(i-1)} \otimes \Delta \otimes id^{\otimes(n-i)},$$
$$\eta_i = id^{\otimes(i-1)} \otimes \eta \otimes id^{\otimes(n-i)}, \qquad \varepsilon_i = id^{\otimes(i-1)} \otimes \varepsilon \otimes id^{\otimes(n-i)},$$
$$S_i = id^{\otimes(i-1)} \otimes S_i \otimes id^{\otimes(n-i)}, \qquad \Psi_i = id^{\otimes(i-1)} \otimes \Psi \otimes id^{\otimes(n-i-1)}.$$

The multiplication $\boldsymbol{\mu}$ of free ribbons is expressed as follows.

$$\boldsymbol{\mu} = (\underbrace{\mu \otimes \cdots \otimes \mu}_{k}) \circ \Psi_{2k-2} \circ (\Psi_{2k-4} \circ \Psi_{2k-3}) \circ \cdots \circ (\Psi_4 \circ \Psi_5 \circ \cdots \circ \Psi_{k+1}) \circ (\Psi_2 \circ \Psi_3 \circ \cdots \circ \Psi_k).$$

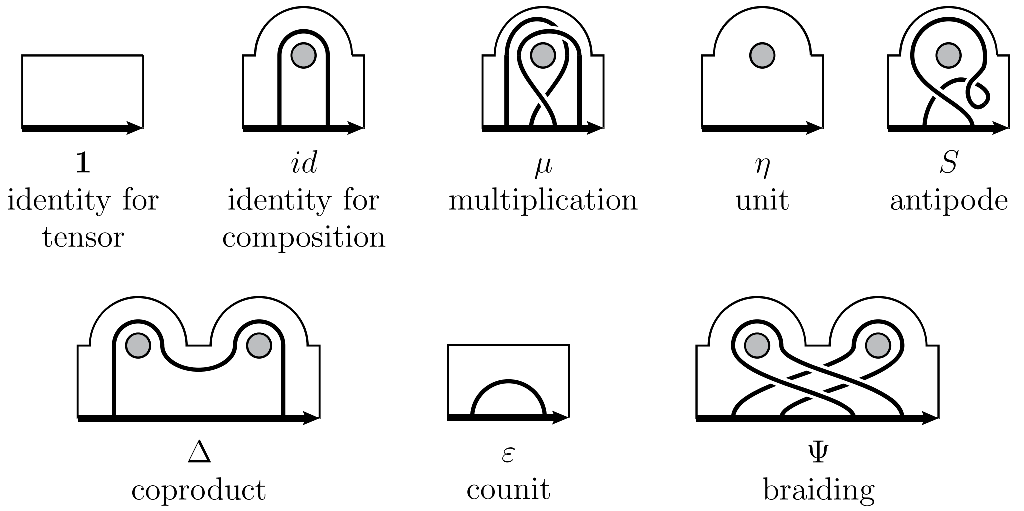


FIGURE 5. Braided Hopf algebra structure of bottom tangles.

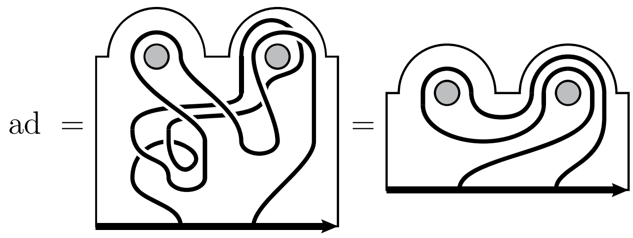


FIGURE 6. Bottom tangle expression of the adjoint action.

2.3. **Adjoint and braided commutativity.** We can define the adjoint ad as usual Hopf algebra by the following.

$$\mathrm{ad} = \mu_2 \circ \Psi_1 \circ (S \otimes \Delta) \circ \Delta.$$

Then ad is interpreted as a element of $\mathcal{T}_{2,1}$ as in Figure 6.

Proposition 1. *The adjoint satisfies the following relation.*

$$\mu_2 \circ (\mathrm{ad} \otimes id) = \mu_2 \circ \Psi_1 \circ (id \otimes \mathrm{ad}) \circ \Psi \in \mathcal{T}_{2,2}.$$

This relation is called the *braided commutativity*, which is crucial requirement for our previous work to construct a representation space of a knot from a braided Hopf algebra, which I presented this workshop of last year. In case of bottom tangles, the braided commutativity holds automatically.

Proof. It is proved by using diagrams. See Figure 7. □

2.4. **Flat bottom tangles.** Here we introduce the notion of a flat bottom tangle and see its properties.

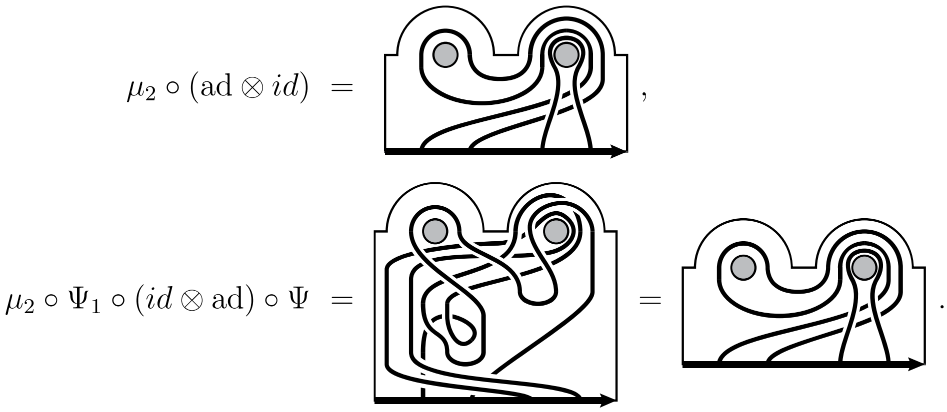


FIGURE 7. Proof of the braided commutativity.

Definition 2. A tangle $T \in \mathcal{T}_{k,n}$ is called a *flat bottom tangle* if T is presented a diagram without crossings. Let $\mathcal{T}^F_{k,n}$ be the subspace of $\mathcal{T}_{k,n}$ spanned by all the flat bottom tangles in $\mathcal{T}_{k,n}$.

Proposition 2. *The composition of two flat bottom tangles is a flat bottom tangle. So the flat bottom tangles form a subcategory $\mathcal{B}^F$ of $\mathcal{B}$.*

Proposition 3. *Any element T of $\mathcal{T}^F_{k,n}$ commutes with the multiplication*

$$\boldsymbol{\mu} : \mathcal{F}_{n,l_1} \otimes \mathcal{F}_{n,l_2} \to \mathcal{F}_{n,l_1+l_2}.$$

Therefore, T induces an algebra homomorphism from $\mathcal{F}_n$ to $\mathcal{F}_k$.

Proof. Let T be a flat bottom tangle. In the flat bottom tangle, the heights of ribbons can be arranged at any order, so the ribbons of $\boldsymbol{\mu}(F_1 \otimes F_2)$ can be separated so that the ribbons coming from F_1 are lower than the ribbons coming from F_2 as in Figure 8. In Figure 8, such separation is used at the equality with $*$. □

2.5. **Adjoint action.** Here we define the adjoint action to bottom tangles.

Definition 3. Let Ad be the element in $\mathcal{T}_{k+1,k}$ givern by

$$\mathrm{Ad} = (id^{\otimes k} \otimes \boldsymbol{\mu}) \circ (\Psi_k \Psi_{k-1} \cdots \Psi_2 \mathrm{ad}_1) \circ (\Psi_k \Psi_{k-1} \cdots \Psi_3 \mathrm{ad}_2) \circ \cdots \circ (\Psi_k \mathrm{ad}_{k-1}) \circ \mathrm{ad}_k$$

where $ad_i = id^{\otimes(i-1)} \otimes \mathrm{ad} \otimes id^{\otimes(k-i)}$.

The bottom tangle Ad is given by a flat bottom tangle as in Figure 9.

Proposition 4. *The bottom tangle* Ad *commutes with any bottom tangle $T \in \mathcal{T}_{k,n}$, i.e.*

$$\mathrm{Ad} \circ T = (T \otimes id) \circ \mathrm{Ad}.$$

Proof. Ad commutes with T as in Figure 10. □

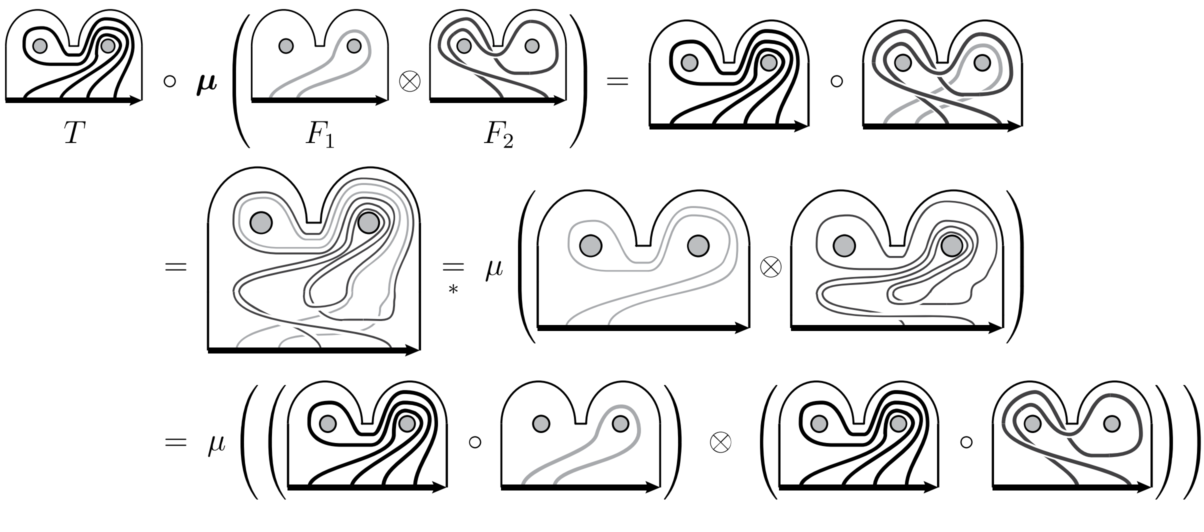


FIGURE 8. Commutativity of $\boldsymbol{\mu}$ and a flat bottom tangle.

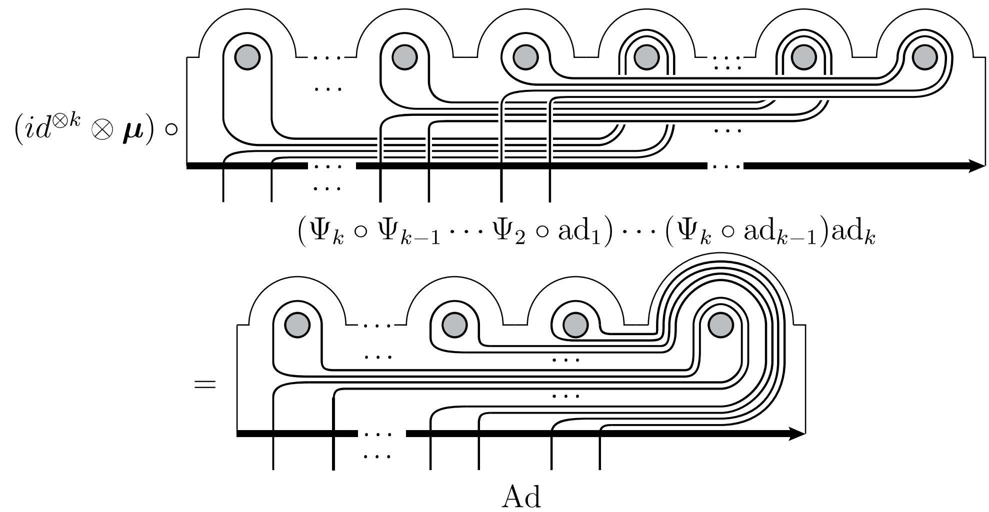


FIGURE 9. The adjoint action Ad.

3. Universal representation space

3.1. **Action of braids.** The braid group B_k acts on the punctured disk D_k. Let σ_1, ..., σ_{k-1} be the standard generators of B_k twisting the i-th and $(i+1)$-th strings. This action permutes the punctures and fixes the boundary of D_k. The generator σ_i swaps q_i and q_{i+1} by rotating counterclockwise a small disk containing q_i and q_{i+1}. This action induces an action of B_k to $\mathcal{F}_k$, and the actions of σ_i and σ_i^{-1} are given by the bottom tangles. For two strings case, the twist σ and σ^{-1} are given by T_σ and $T_{\sigma^{-1}}$ as ffollows.

$$T_\sigma = \mu_2 \circ \Psi_1 \circ (id \otimes \mathrm{ad}),$$
$$T_{\sigma^{-1}} = \mu_1 \circ \Psi_1^{-1} \circ \Psi_2^{-1} \circ \Psi_1^{-1} \circ S_2^{-1} \circ (\mathrm{ad} \otimes id).$$

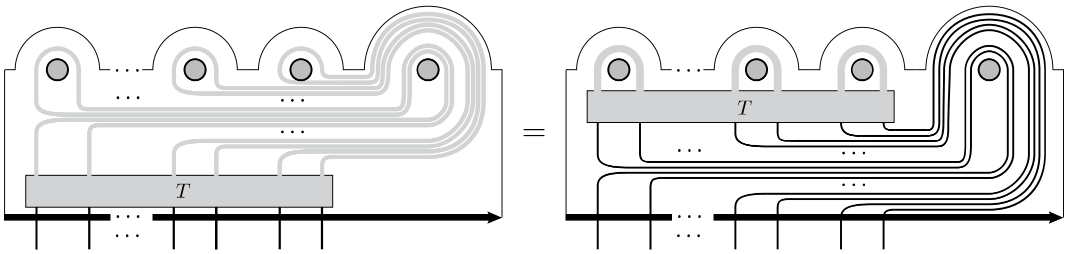


FIGURE 10. The adjoint action Ad commutes with $T \in \mathcal{T}_{k,n}$. The gray lines represent bunches of strings.

For general case, the action of $\sigma_i^{\pm 1}$ is given by $id^{\otimes(i-1)} \otimes T_{\sigma^{\pm 1}} \otimes id^{\otimes(n-i-1)}$. Since T_{σ_i} and

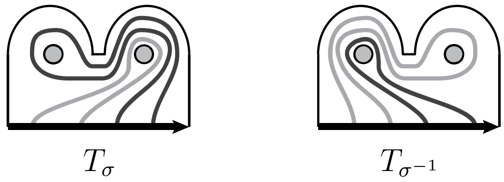


FIGURE 11. The bottom tangles corresponding σ and σ^{-1}.

$T_{\sigma_i^{-1}}$ are both flat bottom tangles and any element b in B_n is a composition of $\sigma_i^{\pm 1}$, so the bottom tangle corresponding to the action of b is a flat bottom tangle. Therefore, Proposition 3 implies the following.

Proposition 5. *The action of braids in B_k on $\mathcal{F}_k$ is an algebra automorphism.*

3.2. **Ideals of $\mathcal{F}_k$.** Here we introduce a notion of ideal for $\mathcal{F}_k$.

Definition 4. Let $b \in B_k$. The *left ideal of $\mathcal{F}_k$ associated with* b is a K-submodule $\mathrm{Image}\big(\boldsymbol{\mu} \circ (id^{\otimes k} \otimes (T_b - id^{\otimes k})\big)$, where $\boldsymbol{\mu} \circ (id^{\otimes k} \otimes (T_b - id^{\otimes k})$ is a K-module homomorphism from $\mathcal{F}_{2k}$ to $\mathcal{F}_k$. This submodule is denoted by I_b. The *right ideal of $\mathcal{F}_k$ associated with* b is a k-submodule $I_b^r = \mathrm{Image}\big(\boldsymbol{\mu} \circ ((T_b - id^{\otimes k}) \otimes id^{\otimes k})\big)$.

Proposition 6. *The left ideal I_b is equal to the right ideal I_b^r.*

Proof. It suffices to show that $\boldsymbol{\mu}\big(((T_b - id^{\otimes k}) \otimes id^{\otimes k})(x)\big) \in I_b$ for $x \in \mathcal{F}_{2k}$. Since T_b is a flat bottom tangle, it is K-algebra homomorphism. Hence we have $T_b \circ \boldsymbol{\mu} = \boldsymbol{\mu} \circ (T_b \otimes T_b)$ and

$$
\begin{aligned}
&\boldsymbol{\mu}\big(((T_b - id^{\otimes k}) \otimes id^{\otimes k})(x)\big) \\
&= \boldsymbol{\mu}\big((T_b \otimes id^{\otimes k})(x)\big) - \boldsymbol{\mu}(x) \\
&= \boldsymbol{\mu}\big((T_b \otimes id^{\otimes k})(x)\big) - \boldsymbol{\mu}\big((T_b \otimes T_b)(x)\big) + \boldsymbol{\mu}\big((T_b \otimes T_b)(x)\big) - \boldsymbol{\mu}(x) \\
&= -\boldsymbol{\mu}\big((T_b \otimes (T_b - id^{\otimes k}))(x)\big) + \big(T_b - id^{\otimes k}\big)\,\big(\boldsymbol{\mu}(x)\big).
\end{aligned}
$$

Since the terms $\boldsymbol{\mu}\big((T_b \otimes (T_b - id^{\otimes k}))(x)\big)$ and $\big(T_b - id^{\otimes k}\big)\,\big(\boldsymbol{\mu}(x)\big)$ are contained in I_b, $\boldsymbol{\mu}\big(((T_b - id^{\otimes k}) \otimes id^{\otimes k})(x)\big)$ is also contained in I_b. □

3.3. **Universal representation space.** Let L be a link, b be a braid in B_k whose closure is isotopic to L, and T_b be the bottom tangle corresponding to b. Let I_b be the ideal generated by the image of $T_b - id^{\otimes k}$, and $\mathcal{A}_b$ be the quotient space $\mathcal{F}_k/I_b$.

Theorem 1. *If the closures of two braids b_1 and b_2 are isotopic, then $\mathcal{A}_{b_1}$ and $\mathcal{A}_{b_2}$ are isomorphic as graded rings.*

Proof. The main idea is to show the isomorphism by using the Markov moves in Figure 12. The argument for the proof is similar to that for Theorem 2 in [4]. The detail of proof is omitted. □

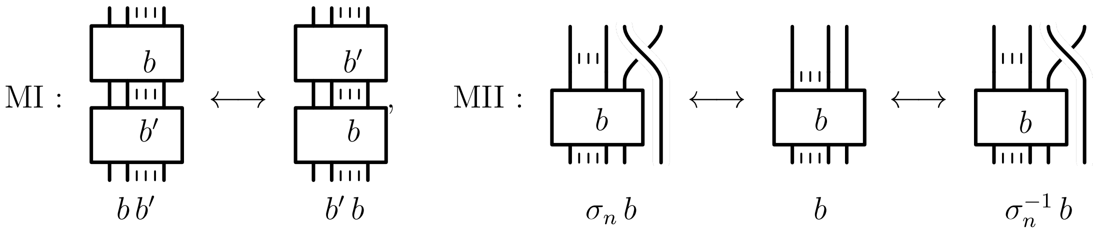


FIGURE 12. Markov moves.

4. SKEIN ALGEBRAS OF PUNCTURED DISKS

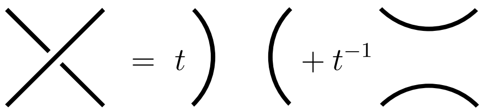
4.1. **Skein algebra $\mathcal{S}_k$.** For the punctured disk D_k, we define the corresponding skein algebra $\mathcal{S}_k$ as follows.

Definition 5. The skein module $\mathcal{S}_{k,n}$ is defined by the following.

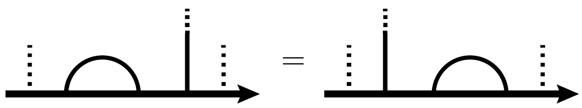
$$\mathcal{S}_{k,n} = \mathcal{F}_{k,n}/\!\sim$$

where $\sim$ is generated by the following two relations.

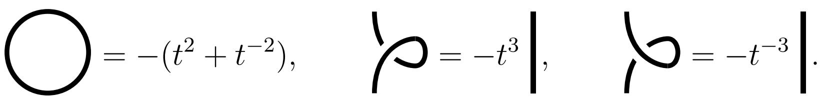
Kauffman bracket skein relation :



Boundary parallel relation :



The Kauffman bracket skein relation relations implies that



The skein algebra $\mathcal{S}_k$ is the direct sum of $\mathcal{S}_{k,n}$, i.e. $\mathcal{S}_k = \oplus_{n=0}^{\infty} \mathcal{S}_{k,n}$. Since the relations for $\sim$ are local and homogeneous relations, and the multiplication of $\mathcal{F}_k$ is just a stacking, the multiplication $\boldsymbol{\mu}$ induces a multiplication in $\mathcal{S}_k$. This multiplication gives the algebra structure of $\mathcal{S}_k$, and $\mathcal{S}_{k,0}$ is a subalgebra of $\mathcal{S}_k$. Moreover, $\mathcal{S}_k$ and $\mathcal{S}_{k,n}$ are both $\mathcal{S}_{k,0}$-module.

4.2. **Standart triangular decomposition of D_k.** We first introduce a standard triangular decomposition of D_k, which is given in Figure 13. D_k is decomposed into $2k-1$ triangles and the punctures p_1, q_1, …, q_k are vertices of triangles. Note that the base point p_0 is not a puncture and it is not a vertex. Bu cutting along the edges p_1q_j, we get

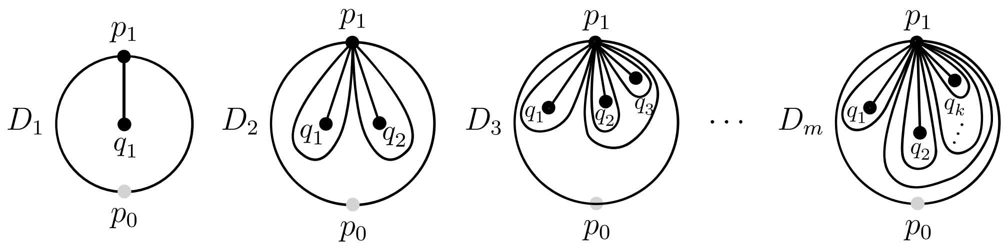


FIGURE 13. Standard decomposition of D_k.

a picture in Figure 14.

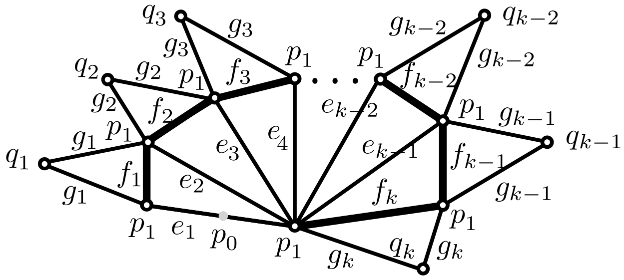


FIGURE 14. Developed standard triangulation of D_k.

4.3. **Flat basis of $\mathcal{S}_k$.** Any diagram in $\mathcal{F}_{k,n}$ can be represented as a K-linear combination of diagrams without crossings in $\mathcal{S}_{k,n}$ according to the Kauffman bracket skein relation. So, $\mathcal{S}_{k,n}$ is spanned by the elements of $\mathcal{T}^F_{k,n}$, which are flat bottom tangles.

Definition 6. A flat bottom tangle T is called *reduced* if there is no trivial loop and all the boundary parallel ribbons are located at the left of the bottom arrow of the diagram. Let $\mathcal{T}^{red}_{k,n}$ be the set of reduced flat bottom tangles in $\mathcal{T}_{k,n}$.

Proposition 7. *$\mathcal{T}^{red}_{k,n}$ is a basis of $\mathcal{S}_{k,n}$.*

To prove the proposition, we need the following.

Proposition 8. *Let T be a triangle and $\mathcal{S}(T)$ be the skein algebra on T with an extra relation that, if the diagram has an arc parallel to an edge, then this diagram is 0. Then the basis of $\mathcal{S}(T)$ is given by $T_{a,b,c}$ in Figure 15 where*

$$a, b, c \geq 0, \quad |a-b| \leq c \leq a+b, \quad a+b+c \text{ is even.} \tag{1}$$

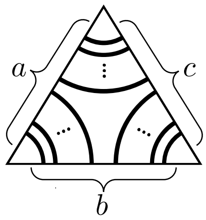


FIGURE 15. Basis $T_{a,b,c}$ of $\mathcal{S}(T)$.

Proof of Proposition 7. By the relations of the skein module, every element is expressed as a linear combination of elements in $\mathcal{T}_{k,n}^{red}$. For T in $\mathcal{T}_{k,n}^{red}$, let us consider the set of numbers of the intersection points with each edge, and the number of boundary parallel ribbons. Then These numbers gives a partial grading to $\mathcal{S}_{k,n}$, and for each grading, there is only one reduced flat bottom tangle having this grading by Proposition 8. This implies the linear independence of elements in $\mathcal{T}_{k,n}^{red}$. □

Corollary 1. *Let* $\boldsymbol{d}$ *be the grading of* $\mathcal{S}_{k,n}$ *given in the above proof and let* $\mathcal{S}_{k,n}^{\boldsymbol{d}}$ *is the span of the elements in* $\mathcal{S}_{k,n}$ *whose grading is equal to or less than* $\boldsymbol{d}$*, thenwe have the following.*

$$\dim \mathcal{S}_{k,n}^{\boldsymbol{d}} / \left(\bigoplus_{\boldsymbol{d}' < \boldsymbol{d}} \mathcal{S}_{k,n}^{\boldsymbol{d}'} \right) \leq 1.$$

The quotient space is spanned by at most one reduced flat bottom tangle.

4.4. **Basis of $\widetilde{\mathcal{S}}_{k,0}$.**

Definition 7. The skein module with $t = -1$ is called *classical skein module*, and the skein algebra with $t = -1$ is called *classical skein algebra*.

Proposition 9. $\mathcal{S}_{k,0}$ *is a* K *algebra generated by* $t_{j_1 \cdots j_m}$ $(j_1 < \cdots < j_m, m \leq 3)$ *given in Figure 16.*

Proof. For the classical case, it is proved by Bullok in [1]. The graded structure of $\mathcal{S}_{k,0}$ given by the number of intersection points of edges are the same for generic t and $t = -1$, so it is true for generic t. □

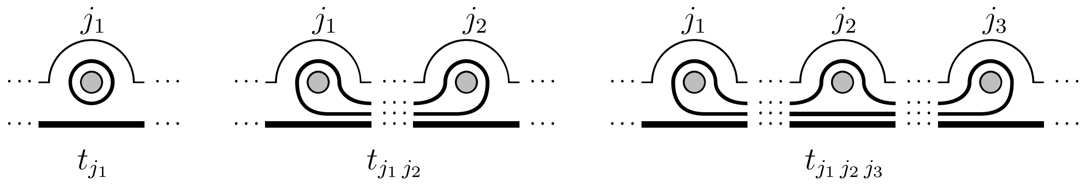


FIGURE 16. The generatros $t_{j_1 \cdots j_m}$ $(j_1 < \cdots < j_m, m \leq 3)$.

The graded structure of $\mathcal{S}_{k,0}$ also provides the following.

Proposition 10. $\mathcal{S}_{k,0}$ *is an integral domain.*

Let

$$\widetilde{\mathcal{S}}_{k,0} = \mathcal{S}_{k,0}[t_{j_1\cdots j_m}^{-1}] \quad (j_1 < \cdots < j_m, m \leq 3).$$

Then we have the following.

Proposition 11. *$\widetilde{\mathcal{S}}_{k,0}$ is generated by $t_{j\cdots j+m}$ $(m \leq 3)$. Moreover, the set of monomials of $t_{j\cdots j+m}$ is a basis of $\widetilde{\mathcal{S}}_{k,0}$.*

This proposition is proved by looking at the grading. The detail is omitted.

4.5. **Basis of $\widetilde{\mathcal{S}}_{k,1}$.** Let

$$\widetilde{\mathcal{S}}_{k,1} = \widetilde{\mathcal{S}}_{k,0} \otimes_{\mathcal{S}_{k,0}} \mathcal{S}_{k,1}$$

and

$$1 = \eta^{\otimes k} \circ \varepsilon, \quad \alpha_i = \eta^{\otimes(i-1)} \otimes id \otimes \eta^{\otimes(n-i)}, \quad \alpha_1\,\alpha_2 = m(\alpha_1 \otimes \alpha_2).$$

Proposition 12. *$\widetilde{\mathcal{S}}_{1,1}$ is a $\widetilde{\mathcal{S}}_{1,0}$ algebra spanned by 1, α_1, and $\widetilde{\mathcal{S}}_{k,1}$ is a $\widetilde{\mathcal{S}}_{k,0}$ algebra spanned by four elements 1, α_1, α_2, $\alpha_1\,\alpha_2 = m(\alpha_1 \otimes \alpha_2)$ if $k \geq 2$.*

This proposition is also proved by looking at the grading. The detail is omitted.

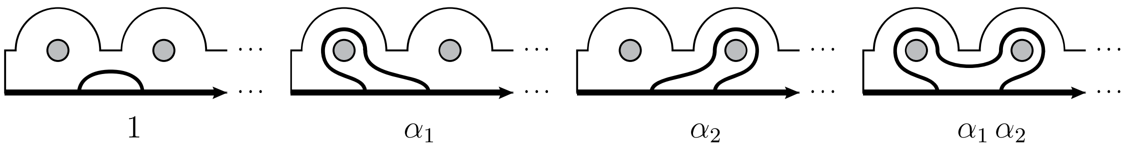


FIGURE 17. The generators 1, α_1, α_2, $\alpha_1\,\alpha_2$ of $\widetilde{\mathcal{S}}_{k,1}$.

4.6. **Action of braids.** Let L be a knot, $b \in B_k$ is a braid whose closure is isotopic to L, and $\widetilde{I}_b = \widetilde{\mathcal{S}}_{k,0} \otimes_{\mathcal{S}_{k,0}} I_b$.

Definition 8. Let $\widetilde{I}_b = \widetilde{\mathcal{S}}_{k,0} \otimes_{\mathcal{S}_{k,0}} (I_b/\sim)$, $\widetilde{I}_{b,n} = \widetilde{\mathcal{S}}_{k,0} \otimes_{\mathcal{S}_{k,0}} (I_b \cap \mathcal{F}_{k,n}/\sim)$, $\widetilde{\mathcal{A}}_b = \widetilde{\mathcal{S}}_k/\widetilde{I}_b$, and $\widetilde{\mathcal{A}}_{b,n} = \widetilde{\mathcal{S}}_{k,n}/\widetilde{I}_{b,n}$. We call $\widetilde{\mathcal{A}}_b$ the *space of quantum $SL(2)$ representations* of L.

Proposition 13. *The ideal $\widetilde{I}_{b,1}$ is generated by $T_b(\alpha_1) - \alpha_1, \cdots, T_b(\alpha_{k-1}) - \alpha_{k-1}$ as a left $\mathcal{S}_{k,0}$-module.*

Proof for this proposition is similar to that in [4].

Remark 1. By definition, $\widetilde{I}_{b,1}$ is generated by $T_b(x) - x$ for all $x \in \mathcal{F}_{k,1}$. But $x = \alpha_1$, $\cdots$, α_{k-1} are good enough.

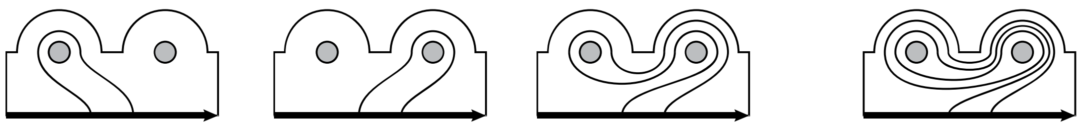
5. Quantum character variety

5.1. **The action of $T_b - id$ on $\widetilde{\mathcal{S}}_{2,0}$.** From now on, we consider the case that the number of punctures $k = 2$. Let b be a 2-braid and L be a link which is isotopic to the closure $\hat{b}$ of b. The ideal I_b is generated by the image of $T_b - id$. So, if L is a knot, $T_b(t_1) = t_2$, $T_b(t_2) = t_1$, $T_b(t_{12}) = t_{12}$, and the action of $T_b - id$ gives a relation $t_1 = t_2$ for $\widetilde{I}_{b,0}$. If L is a link, $T_b(t_1) = t_1$, $T_b(t_2) = t_2$ and $T_b(t_{12}) = t_{12}$, so the action of $T_b - id$ gives no relation for $\widetilde{\mathcal{S}}_{2,0}$. Let

$$\widetilde{\mathcal{S}}'_{2,0} = \begin{cases} \widetilde{\mathcal{S}}_{2,0}/(t_1 - t_2) & \text{if } \hat{b} \text{ is a knot,} \\ \widetilde{\mathcal{S}}_{2,0} & \text{if } \hat{b} \text{ is a two-component link.} \end{cases}$$

5.2. **The action of $T_b - id$ on $\widetilde{\mathcal{S}}_{2,1}$.** Recall that $\widetilde{\mathcal{S}}_{2,1}$ has a K-algebra structure with the product m. For the ideal $\widetilde{I}_b$, $\widetilde{I}_{b,1} = \widetilde{I}_b \cap \widetilde{\mathcal{S}}_{2,1}$ and $\widetilde{I}_{b,1}$ is also an ideal of $\widetilde{\mathcal{S}}_{2,1}$. $\widetilde{I}_{b,1}$ is generated by $(T_b - id)(\alpha_1)$ as a left ideal, and $\widetilde{\mathcal{S}}_{2,1}$ is generated by 1, α_1, α_2, $\alpha_1\,\alpha_2$ as $\widetilde{\mathcal{S}}_{2,0}$-module, the left ideal $\widetilde{\mathcal{S}}_{2,1}$ is spanned by $(T_b - id)(\alpha_1)$, $\alpha_1\,(T_b - id)(\alpha_1)$, $\alpha_2\,(T_b - id)(\alpha_1)$ and $\alpha_1\,\alpha_2\,(T_b - id)(\alpha_1)$ as an $\widetilde{\mathcal{S}}_{2,0}$-module.

The braid group B_2 is generated by a single element σ and the action of T_σ is given by $T_\sigma(\alpha_1) = \alpha_2$ and $T_\sigma(\alpha_2) = \alpha_2^{-1}\,\alpha_1\,\alpha_2$. The multiplications of α_1 and α_2 from the right in



α_1 $\qquad$ $T_\sigma(\alpha_1) = \alpha_2$ $\qquad$ $T_{\sigma^2}(\alpha_1) = \alpha_2^{-1}\alpha_1\alpha_2$ $\qquad$ $T_{\sigma^3}(\alpha_1) = \alpha_2^{-1}\alpha_1^{-1}\alpha_2\alpha_1\alpha_2$

Figure 18. The action of σ to α_1.

$\widetilde{\mathcal{S}}_{2,1}$ commute with the left multiplication of $\widetilde{\mathcal{S}}_{2,0}$ and are $\widetilde{\mathcal{S}}_{2,0}$-module maps, so they are given by the following matrices M_1, M_2 with coefficients in $\widetilde{\mathcal{S}}_{2,0}$.

$$(1, \alpha_1, \alpha_2, \alpha_1\,\alpha_2)\,\alpha_1 = (1, \alpha_1, \alpha_2, \alpha_1\,\alpha_2)\,M_1,$$
$$(1, \alpha_1, \alpha_2, \alpha_1\,\alpha_2)\,\alpha_2 = (1, \alpha_1, \alpha_2, \alpha_1\,\alpha_2)\,M_2,$$

where

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -t^4 & -t^2\,t_1 & 0 & 0 \\ -t^4\,t_1\,t_2 - t^6\,t_{12} & -t^2\,t_2 & -t^2\,t_1 & -t^4 \\ t^2\,t_2 & -t^2\,t_{12} & 1 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -t^4 & 0 & -t^2\,t_2 & 0 \\ 0 & -t^4 & 0 & -t^2\,t_2 \end{pmatrix}.$$

Let M_b the matrix corresponding to the right action of $(T_b - id)(\alpha_1)$. Then M_b is the relation matrix of the $\widetilde{\mathcal{S}}'_{2,0}$-module $\widetilde{\mathcal{S}}'_{2,0} \otimes_{\widetilde{\mathcal{S}}_{2,0}} \widetilde{\mathcal{S}}_{2,1}/\widetilde{I}_{b,1}$. Therefore, the elementary ideals of the matrix corresponding to $(T_b - id)(\alpha_1)$ is invariants of the module $\widetilde{\mathcal{S}}'_{2,0} \otimes_{\widetilde{\mathcal{S}}_{2,0}} \widetilde{\mathcal{S}}_{2,1}/\widetilde{I}_{b,1}$, and so invariants of the link L. Especially, the determinant of M_b is an invariant of L. Let $P_b = \det M_b$. Then P_b is also an invariant of L.

Definition 9. The *quantum character variety* of L is the algebraic variety determined by the radical of $P_b = 0$, where P_b is a polynomial in t_1 and t_{12} if L is a knot and is a polynomial in t_1, t_2, t_{12} if L is a two-component link.

Theorem 2. *By putting $A = -1$, the quantum character variety reduces to a multiple of the classical $SL(2,\mathbb{C})$ character variety of L.*

Proof. In the classical case,

$$(1, \alpha_1, \alpha_2, \alpha_1\,\alpha_2)\, M_b = (0,0,0,0)$$

gives the relations among representation matrices of 1, α_1, α_2 and $\alpha_1\,\alpha_2$. So, if the determinant $\det M_b \neq 0$, then only 0 matrices can be assigned to such element. So, to allow non-zero representation, $\det M_b$ must be 0, so it is a multiple of the polynomial for the classical $SL(2,\mathbb{C})$ character variety. □

Remark 2. In the examples given below, the radical of $\det M_b$ at $t = -1$ coincide the polyonomial for the classical $SL(2,\mathbb{C})$ character variety.

6. Examples

6.1. **Hopf link.** The Hopf link H is isotopic to the closure of σ^2. Since $T_{\sigma^2}(\alpha_1) = T_\sigma(\alpha_2) = \alpha_2\,\alpha_1\,\alpha_2^{-1}$, The matrix corresponding to $(T_{\sigma^2} - id)(\alpha_1)$ is $M_2^{-1}\,M_1\,M_2^{-1} - M_1$. So te quantum character variety of Hopf link is given by $\det(M_2^{-1}\,M_1\,M_2 - M_1) = 0$, which is the following.

$$\begin{aligned}\det(M_2^{-1}\,M_1\,M_2 - M_1) &= t^{16} + t^{14}t_1t_{12}t_2 \\ &+ t^{12}t_1^2t_{12}^2 + t^{12}t_1^2t_2^2 - 2t^{12}t_1^2 + t^{12}t_{12}^2t_2^2 - 2t^{12}t_{12}^2 - 2t^{12}t_2^2 + 4t^{12} \\ &+ t^{10}t_1^3t_{12}t_2 + t^{10}t_1t_{12}^3t_2 + t^{10}t_1t_{12}t_2^3 - 5t^{10}t_1t_{12}t_2 \\ &+ t^8t_1^4 + t^8t_1^2t_{12}^2t_2^2 - 4t^8t_1^2 + t^8t_{12}^4 - 4t^8t_{12}^2 + t^8t_2^4 - 4t^8t_2^2 + 6t^8 \\ &+ t^6t_1^3t_{12}t_2 + t^6t_1t_{12}^3t_2 + t^6t_1t_{12}t_2^3 - 5t^6t_1t_{12}t_2 \\ &+ t^4t_1^2t_{12}^2 + t^4t_1^2t_2^2 - 2t^4t_1^2 + t^4t_{12}^2t_2^2 - 2t^4t_{12}^2 - 2t^4t_2^2 + 4t^4 + t^2t_1t_{12}t_2 + 1.\end{aligned}$$

By substituting $t = -1$, we get a polynomial for the classical character variety with some multiplicity.

$$(-4 + t_1^2 + t_{12}^2 + t_1t_2t_{12} + t_2^2)^2.$$

Moreover, by substituting $t_1 = x + 1/x$, $t_2 = y + 1/y$, $t_{12} = z + 1/z$, we have

$$\det(M_2^{-1}\,M_1\,M_2 - M_1) =$$

$$\frac{1}{x^4y^4z^4}\,(t^2xy + z)(xy + t^2z)(t^2y + xz)(y + t^2xz)(t^2x + yz)(x + t^2yz)(t^2 + xyz)(1 + t^2xyz).$$

and its classical version is

$$\frac{1}{x^4y^4z^4}\,(xy+z)^2(y+xz)^2(x+yz)^2(1+xyz)^2.$$

6.2. **Trefoil.** The trefoil is isotopic to the closure of σ^3 and $T_{\sigma^3}(\alpha_1) = \alpha_2^{-1}\,\alpha_1^{-1}\,\alpha_2\,\alpha_1\,\alpha_2$. So the quantum character variety is given by $\det(M_2^{-1}M_1^{-1}M_2M_1M_2 - M_1)$, which is the following.

$$det(M_2^{-1}M_1^{-1}M_2M_1M_2 - M_1) = t^{-4}\,(1 - t^4 + t^8 + t^2t_{12} + t^6t_{12} + t^4t_{12}^2)^2$$
$$(1 + 2t^4 + t^8 - 4t^4t_1^2 + t^4t_1^4 - 2t^2t_{12} - 2t^6t_{12} + t^2t_1^2t - 12 + t^6t_1^2t_{12} + t^4t_{12}^2).$$

By substituting $t = -1$, we get the following classical one.

$$(1 + t_{12})^4(-2 + t_1^2 + t_{12})^2.$$

Moreover, by substituting $t_1 = x + 1/x$, $t_{12} = z + 1/z$, we have

$$\frac{1}{t^4x^4z^6}\,(t^2x^2 + z)(x^2 + t^2z)(t^2 + x^2z)(1 + t^2x^2z)(t^4 + t^2z + z^2)^2(1 + t^2z + t^4z^2)^2,$$

and, by putting $t = -1$, we have

$$\frac{1}{t^4x^4z^6}\,(x^2 + z)^2(1 + x^2z)^2(1 + z + z^2)^4.$$

6.3. **Figure-eight knot.** The figure-eight knot is given by a closure of 3-braid. But, by using the method to reduce the representation space given in [4], the relation for the ideal is given by $M_2M_1M_2^{-1}M_1^{-1}M_2M_1^{-1}M_2^{-1}M_1M_2 - M_1$. This is similar to a presentation of the fundamental group of the figure-eight knot complement. The determinant of this matrix is the following. Here t_1 and t_{12} is replaced by $x + 1/x$ and $z + 1/z$ respectively.

$$\det(M_2M_1M_2^{-1}M_1^{-1}M_2M_1^{-1}M_2^{-1}M_1M_2 - M_1) =$$
$$\frac{1}{t^{12}x^{12}z^{10}}\,(t^2x^2 + z)(x^2 + t^2z)(t^2 + x^2z)(1 + t^2x^2z)$$
$$(t^8x^2 + t^6z + 3t^6x^2z + t^6x^4z + 2t^4z^2 + 5t^4x^2z^2 + 2t^4x^4z^2 + t^2z^3 + 3t^2x^2z^3 + t^2x^4z^3 + x^2z^4)^2$$
$$(x^2 + t^2z + 3t^2x^2z + t^2x^4z + 2t^4z^2 + 5t^4x^2z^2 + 2t^4x^4z^2 + t^6z^3 + 3t^6x^2z^3 + t^6x^4z^3 + t^8x^2z^4)^2.$$

6.4. 5_2 **knot.** The 5_2 knot is given by a closure of 3-braid. But, the relation for the ideal is reduced as the figure-eight knot case, and is given by the matrix

$$M_2^{-1}M_1M_2^{-1}M_1^{-1}M_2M_1^{-1}M_2M_1M_2^{-1}M_1M_2M_1^{-1}M_2 - M_1.$$

This is similar to a presentation of the fundamental group of the 5_2 knot complement. By replacing t_1 and t_{12} by $x + 1/x$ and $z + 1/z$ respectively, the polynomial for the quantum character variety is the following.

$$\det(M_2^{-1}M_1M_2^{-1}M_1^{-1}M_2M_1^{-1}M_2M_1M_2^{-1}M_1M_2M_1^{-1}M_2 - M_1) =$$
$$\frac{1}{t^{20}x^{20}z^{14}}\,(t^2x^2 + z)(x^2 + t^2z)(t^2 + x^2z)(1 + t^2x^2z)$$
$$(t^{12}x^4 + 2t^{10}x^2z + 5t^{10}x^4z + 2t^{10}x^6z + t^8z^2 + 7t^8x^2z^2 + 13t^8x^4z^2 + 7t^8x^6z^2 + t^8x^8z^2$$

$$+ 2t^6z^3 + 10t^6x^2z^3 + 17t^6x^4z^3 + 10t^6x^6z^3 + 2t^6x^8z^3$$
$$+ t^4z^4 + 7t^4x^2z^4 + 13t^4x^4z^4 + 7t^4x^6z^4 + t^4x^8z^4 + 2t^2x^2z^5 + 5t^2x^4z^5 + 2t^2x^6z^5 + x^4z^6)^2$$
$$(x^4 + 2t^2x^2z + 5t^2x^4z + 2t^2x^6z + t^4z^2 + 7t^4x^2z^2 + 13t^4x^4z^2 + 7t^4x^6z^2 + t^4x^8z^2$$
$$+ 2t^6z^3 + 10t^6x^2z^3 + 17t^6x^4z^3 + 10t^6x^6z^3 + 2t^6x^8z^3$$
$$+ t^8z^4 + 7t^8x^2z^4 + 13t^8x^4z^4 + 7t^8x^6z^4 + t^8x^8z^4 + 2t^{10}x^2z^5 + 5t^{10}x^4z^5 + 2t^{10}x^6z^5 + t^{12}x^4z^6)^2.$$

6.5. **Observation.** The examples of knots in the above computation satisfy the following. Let $Q(t_1, t_{12})$ be the polynomial to determine the classical character variety. Then the polynomial to determine the quantum character variety is given by

$$Q(t_1, t\,z + t^{-1}\,z^{-1})\,Q(t_1, t^{-1}\,z + t\,z^{-1}).$$

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