Annulus presentation and dualizable pattern

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1 Introduction

The 0-trace $X_K(0)$ of a knot is the 4-manifold obtained from $\mathbf{B}^4$ by attaching a 2-handle along the 0-framing of the knot. The following theorem gives one of the important properties of 0-traces.

Theorem 1.1 (folklore, [5, Theorem 1.8])**.** *A knot is slice if and only if its* 0*-trace smoothly embeds in* $\mathbf{S}^4$.

In particular, the 0-trace of a knot has the complete information to determine the sliceness of the knot. On techniques to construct knots with the same 0-trace, the following are known.

- Osoinach [6] introduced an "annular twisting" technique to construct infinitely many knots admitting the same 0-surgery. Abe, Jong, Omae and Takeuchi [1] extended the technique to 4-dimensional cases under some additional conditions (see Section 2).
- Gompf and Miyazaki [3] introduced a technique to construct a pair of knots admitting the same 0-surgery by utilizing a pattern with duality, which is called a *dualizable pattern* in this manuscript. Miller and Piccirillo [5] explained that Gompf and Miyazaki's pairs of knots always have the diffeomorphic 0-trace (for dualizable patterns, see Section 3).

Miller and Piccirillo [5] proved that Abe, Jong, Omae and Takeuchi's extension of Osoinach's technique can be explained in terms of dualizable patterns. In particular, they constructed a dualizable pattern from Abe, Jong, Omae and Takeuchi's "(special) annulus presentation" (see Section 3.2).

In this manuscript, we overview these works on 0-trace and introduce the author's resent result [8]. We also introduce an analogy of dualizable pattern, which is called "r-dualizable pattern". An r-dualizable pattern can be used to construct a pair of knots with a homeomorphism between their 0-surgeries which may not extend to a diffeomorphism on 0-traces. Moreover, we explain that r-dualizable patterns can be regarded as "RBG links" given by Manolescu and Piccirillo [4].

This manuscript is organized as follows. In Section 2, we recall Osoinach's technique and related works. In Section 3, we survey Miller and Piccirillo's work. In Section 4, we concretely draw the dual of Miller and Piccirillo's dualizable pattern which is obtained from Abe, Jong, Omae and Takeuchi's special annulus presentation. This is the main result of [8]. In Section 5, we introduce a notion of r-dualizable pattern.

Throughout this manuscript,

- unless specifically mentioned, all knots and links are smooth and unoriented, and all other manifolds are smooth and oriented,
- for an n-component link $L_1 \cup \cdots \cup L_n$, we denote the 3-manifold obtained by m_i-surgery on a knot L_i for any i by $M_{L_1 \cup \cdots \cup L_n}(m_1, \ldots, m_n)$,
- we denote a tubular neighborhood of a knot K in a 3-manifold by $\nu(K)$,
- we denote the unknot in $\mathbf{S}^3$ by U.

2 Annulus twist and annulus presentation

Let $A \subset \mathbf{S}^3$ be an embedded annulus with ordered boundaries $\partial A = c_1 \cup c_2$. An *$n$-fold annulus twist along* A is to apply $(\mathrm{lk}(c_1, c_2) + 1/n)$-surgery on c_1 and $(\mathrm{lk}(c_1, c_2) - 1/n)$-surgery on c_2, where $\mathrm{lk}(c_1, c_2)$ is the linking number of c_1 and c_2 and we give c_1 and c_2 parallel orientations. We see that the resulting manifold obtained by an annulus twist is also $\mathbf{S}^3$.

Let $A \subset \mathbf{S}^3$ be an embedded annulus with $\partial A = c_1 \cup c_2$. Take an embedding of a band $b\colon I \times I \to \mathbf{S}^3$ such that

- $b(I \times I) \cap \partial A = b(\partial I \times I)$,
- $b(I \times I) \cap \operatorname{Int} A$ consists of ribbon singularities, and
- $A \cup b(I \times I)$ is an immersion of an orientable surface,

where $I = [0, 1]$. If a knot $K \subset \mathbf{S}^3$ is isotopic to the knot $(\partial A \setminus b(\partial I \times I)) \cup b(I \times \partial I)$, then we call (A, b) an *annulus presentation* of K. An annulus presentation (A, b) is *special* if A is a Hopf band (that is, A is unknotted and $\mathrm{lk}(c_1, c_2) = \pm 1$) (see Figure 1).

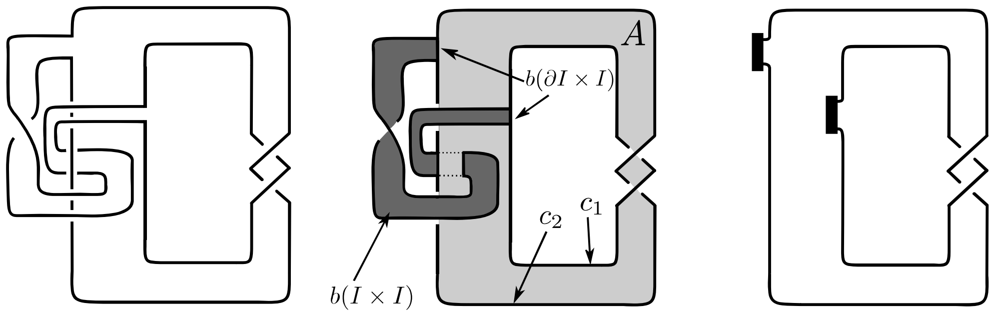

Figure 1: Special annulus presentation

Remark 2.1. In this manuscript, for an annulus presentation (A, b), we often draw the attaching regions $A \cap b$ by bold arcs and we omit the band b (see the right picture in Figure 1).

Let K be a knot with an annulus presentation (A, b). Let $A' \subset A$ be a shrunken annulus with $\partial A' = c'_1 \cup c'_2$ which satisfies the following:

- $\overline{A \setminus A'}$ is a disjoint union of two annuli,
- each c'_i is isotopic to c_i in $\overline{A \setminus A'}$ for $i = 1, 2$ and
- $A \setminus (\partial A \cup A')$ does not intersect $b(I \times I)$.

Then, by $A^n(K)$, we denote the knot obtained from K by the n-fold annulus twist along A' (for example, see Figure 2). More precisely, $A^n(K)$ is defined as follows. Put

$$M_n = M_{c'_1 \cup c'_2}(\mathrm{lk}(c_1, c_2) + 1/n, \mathrm{lk}(c_1, c_2) - 1/n).$$

Since

$$K \subset \mathbf{S}^3 \setminus \nu(c'_1 \cup c'_2) \subset M_n,$$

we can regard K as a knot in M_n. Let $\mathcal{A}^n\colon M_n \to \mathbf{S}^3$ be an orientation-preserving homeomorphism. Then, $A^n(K)$ is given by $\mathcal{A}^n(K) \subset \mathbf{S}^3$. For simplicity, we denote $A^1(K)$ by $A(K)$ and $A^0(K)$ by K.

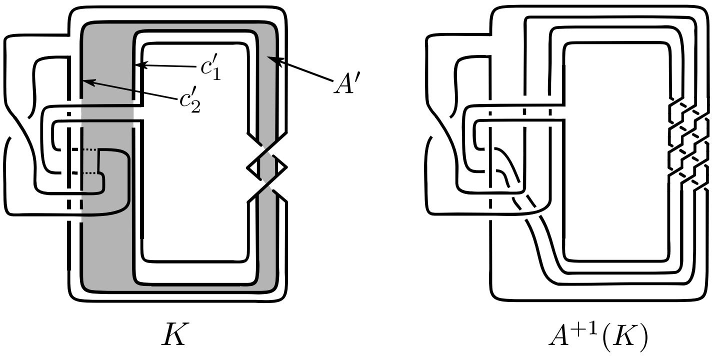

Figure 2: The knot $A^{+1}(K)$ obtained from K by an annulus twist. It seems that $A^n(K)$ is obtained from K by twisting along A.

By utilizing Osoinach's work [6, Theorem 2.3], for a knot K with an annulus presentation (A, b), we see that there is a homeomorphism $\phi_n\colon M_K(0) \to M_{A^n(K)}(0)$. Teragaito [10] explained the homeomorphism by using surgery descriptions (see Figure 3). We call ϕ_n the *n-th Osoinach-Teragaito's homeomorphism*. Moreover, if (A, b) is special, by applying Abe, Jong, Omae and Takeuchi's result [1, Theorem 2.8] to the knot, we see that the homeomorphism ϕ_n extends to an orientation-preserving diffeomorphism $\Phi_n\colon X_K(0) \to X_{A^n(K)}(0)$ for any $n \in \mathbf{Z}$. As a consequence, we obtain the following.

Theorem 2.2. *Let $K \subset \mathbf{S}^3$ be a knot with an annulus presentation (A, b). Then, there is an orientation-preservingly homeomorphism $\phi_n\colon M_K(0) \to M_{A^n(K)}(0)$ for any $n \in \mathbf{Z}$. In particular, ϕ_n is given as in Figure 3. Moreover, if (A, b) is special, ϕ_n extends to an orientation-preserving diffeomorphism $\Phi_n\colon X_K(0) \to X_{A^n(K)}(0)$.*

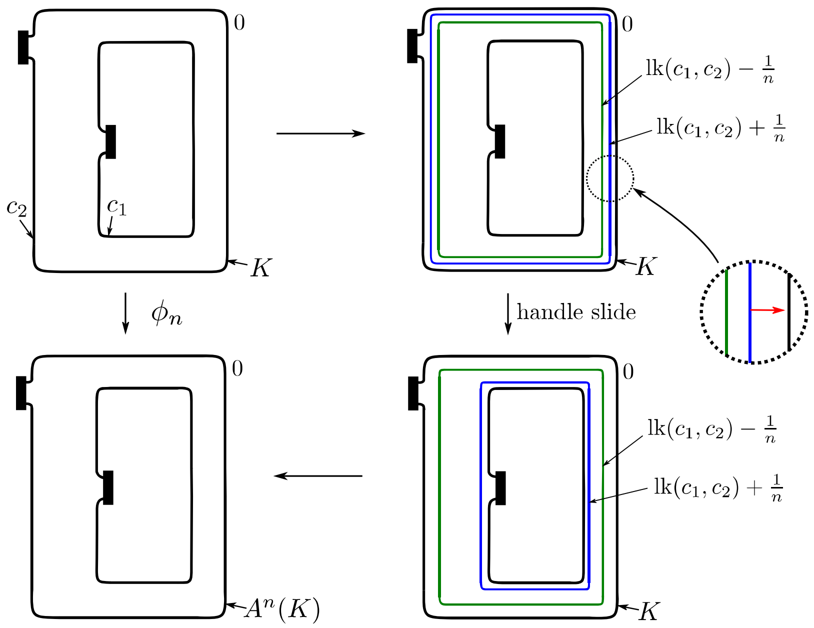

Figure 3: (color online) Osoinach-Teragaito's homeomorphism ϕ_n. For simplicity we draw A as a flat annulus although A may be knotted and twisted.

3 Relation between annulus presentation and dualizable pattern

3.1 Dualizable pattern

Here, we recall the definition of dualizable patterns [3].

Let $P\colon \mathbf{S}^1 \to V$ be an oriented knot in a solid torus $V = \mathbf{S}^1 \times D^2$. Suppose that the image $P(\mathbf{S}^1)$ is not null-homologous in V. Such a P is called a *pattern*. By an abuse of notation, we use the notation P for both a map and its image. Define λ_V, μ_P, μ_V and λ_P as follows:

- put $\lambda_V = \mathbf{S}^1 \times \{x_0\} \subset \partial V \subset V$ for some $x_0 \in \partial D^2$ and orient λ_V so that P is homologous to $r\lambda_V$ in V for some positive $r \in \mathbf{Z}_{>0}$,
- define $\mu_P \subset V$ by a meridian of P and orient μ_P so that the linking number of P and μ_P is 1,
- put $\mu_V = \{x_1\} \times \partial D^2 \subset \partial V \subset V$ for some $x_1 \in \mathbf{S}^1$ and orient μ_V so that μ_V is homologous to $s\mu_P$ in $V \setminus \nu(P)$ for some positive $s \in \mathbf{Z}_{>0}$,
- define λ_P by a longitude of P which is homologous to $t\lambda_V$ in $V \setminus \nu(P)$ for some positive $t \in \mathbf{Z}_{>0}$.

For an oriented knot $K \subset \mathbf{S}^3$, let $\iota_K\colon V \to \mathbf{S}^3$ be an embedding which identifies V with $\overline{\nu(K)}$ and sends λ_V to the prefered longitude of K. Then $\iota_K \circ P\colon \mathbf{S}^1 \to \mathbf{S}^3$ represents an oriented knot. The knot is called the *satellite* of K with pattern P and denoted by $P(K)$.

A pattern $P\colon \mathbf{S}^1 \to V$ is *dualizable* if there is a pattern $P^*\colon \mathbf{S}^1 \to V^*$ and an orientation-preserving homeomorphism $f\colon V \setminus \nu(P) \to V^* \setminus \nu(P^*)$ such that $f(\lambda_V) = \lambda_{P^*}$, $f(\lambda_P) = \lambda_{V^*}$, $f(\mu_V) = -\mu_{P^*}$ and $f(\mu_P) = -\mu_{V^*}$. We call the pattern P^* the *dual* of P. It is easy to see that P^* is uniquely determined and $(P^*)^* = P$.

There is a convenient technique to determine whether a given pattern is dualizable as follows. Define $\Gamma\colon \mathbf{S}^1 \times D^2 \to \mathbf{S}^1 \times \mathbf{S}^2$ by $\Gamma(t,d) = (t, \gamma(d))$, where $\gamma\colon D^2 \to \mathbf{S}^2$ is an arbitrary orientation preserving embedding. For any curve $c\colon \mathbf{S}^1 \to \mathbf{S}^1 \times D^2$, define $\widehat{c} = \Gamma \circ c\colon \mathbf{S}^1 \to \mathbf{S}^1 \times \mathbf{S}^2$. Then, we obtain the following proposition.

Proposition 3.1 ([5, Proposition 2.5])**.** *A pattern P in a solid torus V is dualizable if and only if $\widehat{P}$ is isotopic to $\widehat{\lambda_V}$ in $\mathbf{S}^1 \times \mathbf{S}^2$.*

Moreover, in the proof of [5, Proposition 2.5], we can find a way how to draw the dual P^* (see Figure 4). In Figure 4, the isotopy on $\mathbf{S}^1 \times \mathbf{S}^2$ sends $\widehat{\lambda_V}$ to $\widehat{P^*}$ and $\widehat{P}$ to $\widehat{\lambda_{V^*}}$, respectively, where the curves are 0-framed. For example, by Proposition 3.1, we see that a patten of geometric winding number one is dualizable and its dual is itself (see Figure 5).

By Figure 4, we obtain the following theorem obviously.

Theorem 3.2 (e.g. [5, Theorem 3.1])**.** *Let P be a dualizable pattern. Then there is a homeomorphism $\phi\colon M_{P(U)}(0) \to M_{P^*(U)}(0)$ which extends to an orientation-preserving diffeomorphism $\Phi\colon X_{P(U)}(0) \to X_{P^*(U)}(0)$.*

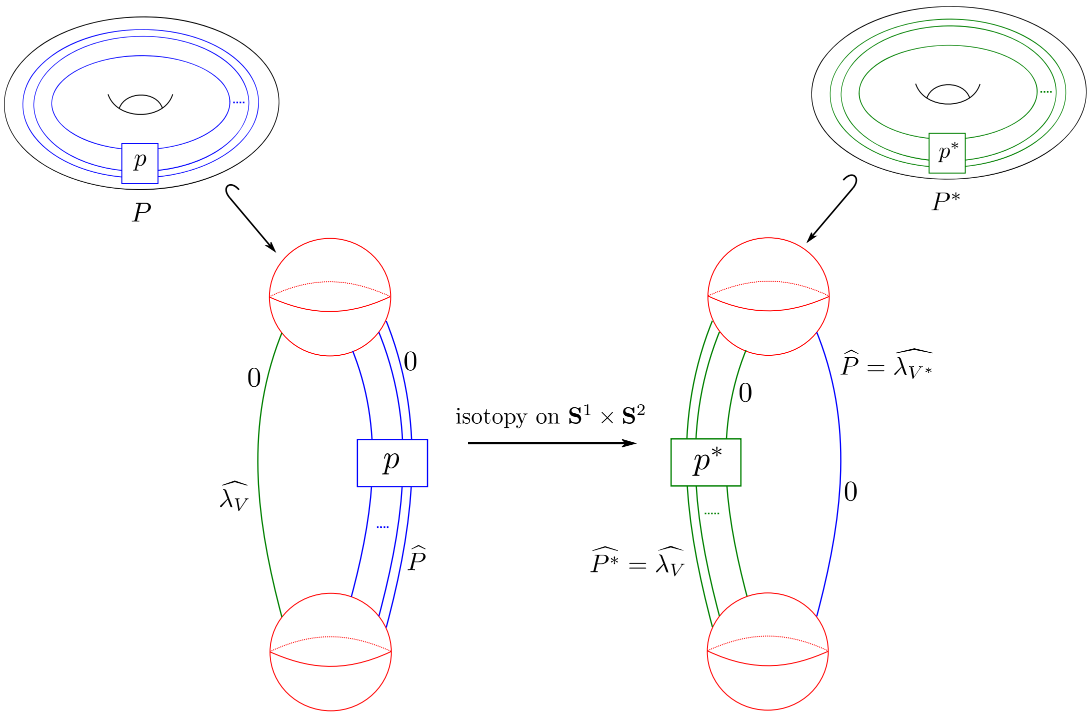

Figure 4: (color online) A dualizable pattern P and its dual P^*. The pairs of two balls represent $\mathbf{S}^1 \times \mathbf{S}^2$. The boxes labeled by p and p^* are the tangles corresponding to P and P^*, respectively.

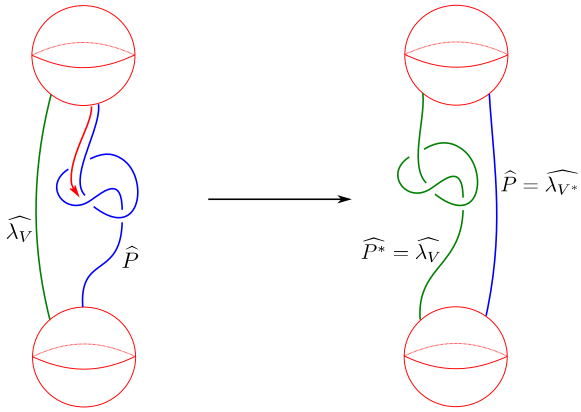

Figure 5: (color online) A pattern of geometric winding number one. Such a pattern is dualizable and its dual is itself.

3.2 From special annulus presentations to dualizable patterns

In this section, we recall Miller and Piccirillo's construction [5, Section 5] of dualizable patterns from a special annulus presentation (see also [9]).

Let $K \subset \mathbf{S}^3$ be a knot with a special annulus presentation (A, b). In Figure 6, the left knots represent K, and each right knot represents $A^{\pm 1}(K)$ for the corresponding left K. Then, for each case, take curves $\beta_K^{\pm} \subset \mathbf{S}^3 \setminus \nu(K)$ as in Figure 6.

For example, in the top left of Figure 6, β_K^+ runs near c_1 and there is a sufficiently thin annulus bounded by β_K^+ and c_1 such that the thin annulus does not intersect the band b except the attaching regions. We remark that β_K^+ is determined by a tubular neighborhood of c_1 on A.

Let P_+ (resp. P_-) be the pattern given by $K \subset V_+ = \mathbf{S}^3 \setminus \nu(\beta_K^+)$ (resp. $K \subset V_- = \mathbf{S}^3 \setminus \nu(\beta_K^-)$), where we give a parameter of $V_\pm$ so that $P_\pm(U) = K$. Moreover, we give an orientation of $P_\pm$ arbitrarily. Then, we can check that $P_\pm$ are dualizable patterns (for example, slide K along the 0-framing of $\beta_K^\pm$ and apply Proposition 3.1). These dualizable patterns satisfy the following.

Proposition 3.3 (e.g. [5, Proposition 5.3] and [9, Proposition 3.9])**.** *Let K be a knot with a special annulus presentation (A, b). Let P_+ and P_- be the dualizable patterns as above. Then we have $P_\pm(U) = K$ and $P_\pm^*(U) = A^{\pm 1}(K)$.*

Remark 3.4. The homeomorphisms given in Figure 3 induces homeomorphisms

$$\phi_{\pm 1} \colon (M_K(0), \beta_K^\pm) \to (M_{A^{\pm 1}(K)}(0), \alpha_{A^{\pm 1}(K)}),$$

where $\alpha_{A^{\pm 1}(K)} \subset \mathbf{S}^3 \setminus \nu(A^{\pm 1}(K))$ is a meridian of $A^{\pm 1}(K)$. Here we regard $\beta_K^\pm$ and $\alpha_{A^{\pm 1}(K)}$ as curves in $M_K(0)$ and $M_{A^{\pm 1}(K)}(0)$ under the identifications

$$\mathbf{S}^3 \setminus \nu(K) \cong M_K(0) \setminus \nu(L_K),$$
$$\mathbf{S}^3 \setminus \nu(A^{\pm 1}(K)) \cong M_{A^{\pm 1}(K)}(0) \setminus \nu(L_{A^{\pm 1}(K)}),$$

respectively, where L_K and $L_{A^{\pm 1}(K)}$ are the corresponding surgery duals.

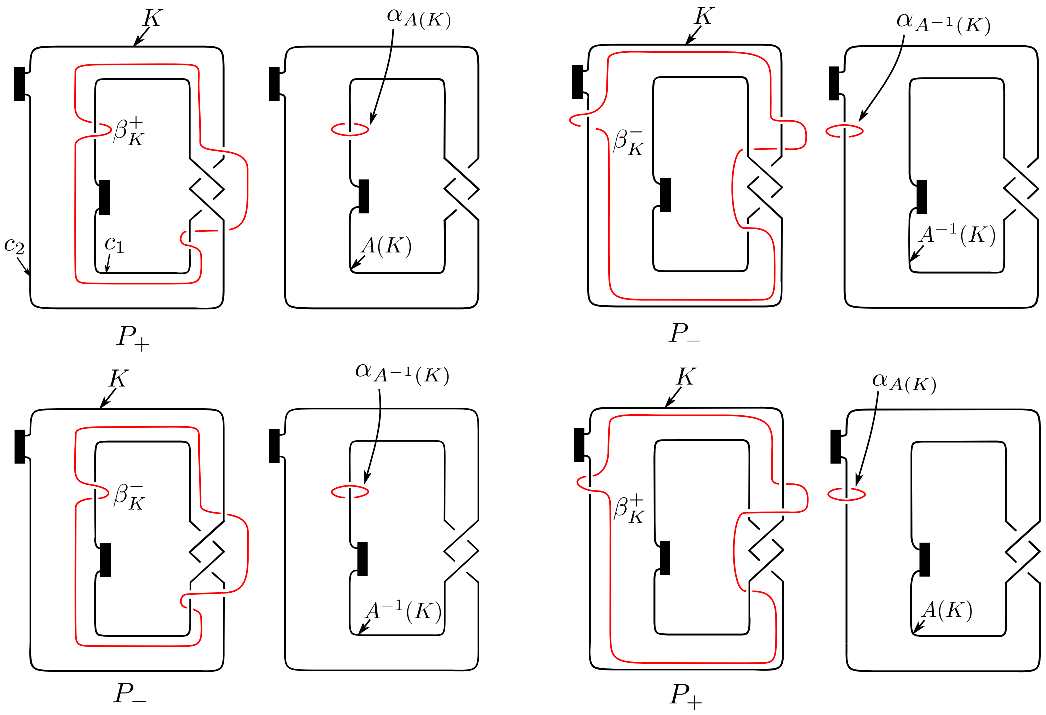

Figure 6: (color online) From a special annulus presentation (A, b) of a knot K to dualizable patterns P_+ and P_- given by $K \subset \mathbf{S}^3 \setminus \nu(\beta_K^{\pm}) = V_{\pm}$

Moreover, we can check that $\phi_{\pm 1}$ induce the following homeomorphisms

$$\phi_{\pm 1} \colon M_{K \cup \beta_K^{\pm}}(0,0) \to M_{A^{\pm 1}(K) \cup \alpha_{A^{\pm 1}(K)}}(0,0) \cong \mathbf{S}^3.$$

4 Naturality of the correspondence between annulus presentation and dualizable pattern

4.1 The dual $P_{\pm}^*$

The dual of the dualizable pattern obtained from an annulus presentation in Section 3.2 is given as follows.

Theorem 4.1 ([8, Theorem 4.3])**.** *Let K be a knot with a special annulus presentation (A, b). Then the dual $P_{\pm}^* \subset V_{\pm}^* = \mathbf{S}^3 \setminus \nu(A^{\pm 1}(K))$ of the dualizable pattern $P_{\pm}$ in Section 3.2 is given as in Figure 7.*

Example 4.2 (Self duality)**.** A pattern of geometric winding number one is a self-dual dualizable pattern (see Figure 5). However, the converse is not true. For example, for any dualizable pattern P, the composition $P \circ P^*$ of P and its dual P^* is a self-dual dualizable pattern (see [5, Proposition 3.3]). We also find a self-dual dualizable pattern by utilizing Theorem 4.1. See Figure 8. The left picture in Figure 8 represents a dualizable pattern $P = 8_3 \subset \mathbf{S}^3 \setminus \nu(\beta)$. The center is obtained from the left by rotating around the horizontal axis. The right is obtained from the center by flipping over the annulus of the annulus presentation. By Theorem 4.1, the right is the dual P^* of P. Hence, $P = P^*$. On the other hand the geometric winding number of the dualizable pattern is not one since the red curves in Figure 8 are not meridians of 8_3 (see [2, Proof of Lemma 5.4]). Since the algebraic winding number of a dualizable pattern is always one, the geometric winding number of a dualizable pattern is odd. Hence, the geometric winding number of the dualizable pattern in Figure 8 is three.

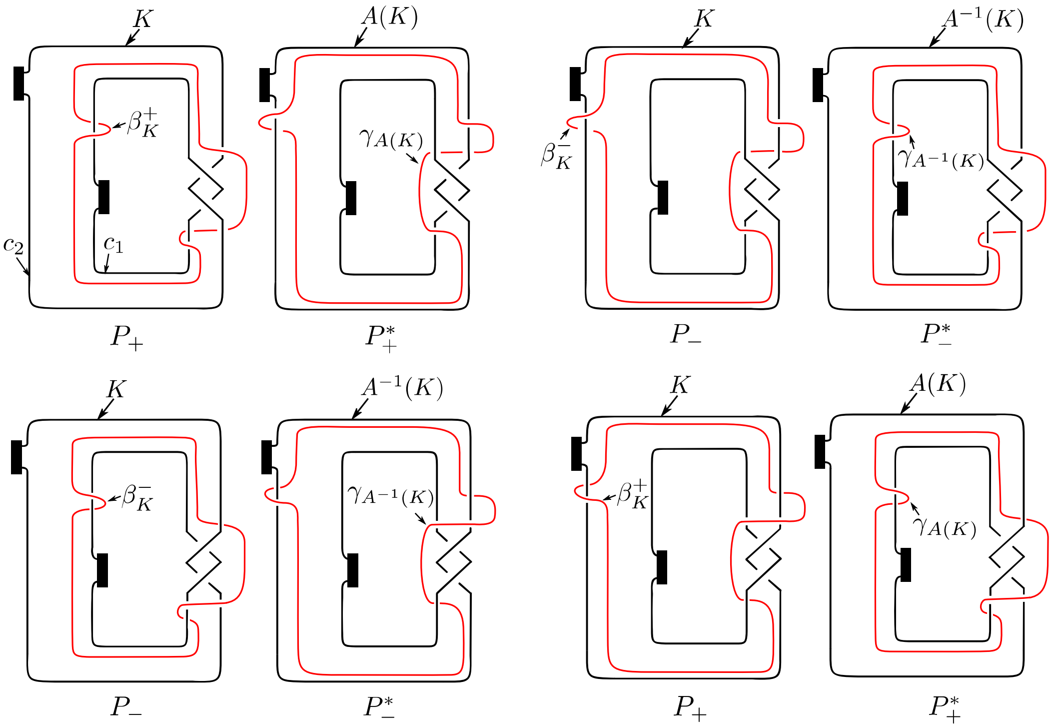

Figure 7: (color online) The dualizable patterns $P_\pm \subset V_\pm = \mathbf{S}^3 \setminus \nu(\beta_K^\pm)$ and $P_\pm^* \subset V_\pm^* = \mathbf{S}^3 \setminus \nu(\gamma_{A^{\pm 1}(K)})$

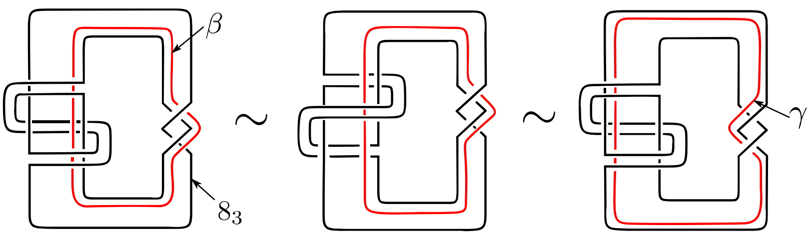

Figure 8: (color online) The left dualizable pattern $P = 8_3 \subset \mathbf{S}^3 \setminus \nu(\beta)$ is isotopic to its dual $P^* = 8_3 \subset \mathbf{S}^3 \setminus \nu(\gamma)$.

4.2 Naturality

Let K be a knot with a special annulus presentation (A, b). Then, we obtain a dualizable pattern P_+ as in Section 3.2. Put $\check{K} = A(K)$ and give the natural annulus presentation $(\check{A}, \check{b})$ of $\check{K}$ from (A, b). Then we obtain another dualizable pattern $\check{P}_-$ from $\check{K}$ as in Section 3.2. We see that these patterns satisfy $P_+(U) = K$, $P_+^*(U) = A(K)$, $\check{P}_-(U) = A(K)$ and $\check{P}_-^*(U) = K$. More strongly, Theorem 4.1 and Figure 7 imply that $P_+ = \check{P}_-^*$ and $P_+^* = \check{P}_-$.

Let SAP be the set of special annulus presentations and DP be the set of unoriented patterns which are dualizable after giving some orientation. Then, by the above discussion, we obtain the following commutative diagram:

$$\begin{array}{ccccc} SAP & \xrightarrow{A^{\pm 1}} & SAP & \longrightarrow & \mathcal{K} \\ \pm\big\downarrow & & \big\downarrow\mp & & \big\| \\ DP & \xrightarrow{\ *\ } & DP & \longrightarrow & \mathcal{K} \end{array}$$

where

- $A^{\pm 1}\colon SAP \to SAP$ is the map induced by ± 1-fold annulus twist,
- $\pm\colon SAP \to DP$ is given by $(A, b) \mapsto P_\pm$ as in Section 3.2,
- $*\colon DP \to DP$ is given by $P \mapsto P^*$,
- $SAP \to \mathcal{K} : (A, b) \mapsto K$ and
- $DP \to \mathcal{K} : P \mapsto P(U)$.

4.2.1 Equivalence relations on DP and SAP

Two dualizable patterns are equivalent if they are isotopic in the solid torus after forgetting the orientations. Denote the set of equivalence classes of dualizable patterns by $DP/\sim$.

In [2], we give an equivalence relation of annulus presentations. Here, we consider another equivalence relation.

Let (A, b) be an annulus presentation of a knot K. Let $A' \subset A$ be a shrunken annulus given in Section 2. Define a surface $F_{(A,b)}$ by

$$F_{(A,b)} = (\overline{A \setminus A'}) \cup b(I \times I),$$

(see Figure 9). By definition of A', we see that $F_{(A,b)}$ is an embeded surface in $\mathbf{S}^3$ whose boundary $\partial F_{(A,b)}$ is the ordered 3-component link $K \cup c_1' \cup c_2'$.

Let (A_1, b_1) and (A_2, b_2) be two annulus presentations. Then (A_1, b_1) and (A_2, b_2) are *strongly equivalent* if $F_{(A_1,b_1)}$ and $F_{(A_2,b_2)}$ are isotopic in $\mathbf{S}^3$ and the isotopy preserves the orders of the boundary components. Denote the set of strong equivalence classes of special annulus presentations by $SAP/\sim_{st}$. Since the curves $\beta_K^\pm$ are determined by the tubular neighborhood of c_1 and c_2 (see Section 3.2), the maps

$$\pm\colon SAP/\sim_{st} \to DP/\sim$$

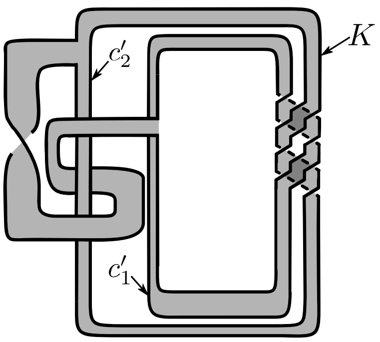

Figure 9: $F_{(A,b)}$ for the annulus presentation in Figure 1

are well-defined. Moreover, the following diagram is commutative:

$$\begin{array}{ccccc} SAP/{\sim_{st}} & \xrightarrow{A^{\pm 1}} & SAP/{\sim_{st}} & \longrightarrow & \mathcal{K} \\ \pm\downarrow & & \downarrow\mp & & \| \\ DP/{\sim} & \xrightarrow{*} & DP/{\sim} & \longrightarrow & \mathcal{K} \end{array}$$

4.2.2 Is the correspondence from SAP to DP injective?

It has explained that a patten of geometric winding number one is dualizable. In particular, for any knot K, there is a dualizable pattern P such that $P(U) = K$. On the other hand, the four-ball genus $g_4(K)$ of a knot K with an annulus presentation is smaller than 2. Hence, the map $\pm\colon SAP/{\sim_{st}} \to DP/{\sim}$ in Section 4.2 is not surjective. How about the injectivity?

Question 4.3. *Are the maps $\pm\colon SAP/{\sim_{st}} \to DP/{\sim}$ injective? If the answer is "No", find a new equivalence relation on SAP so that the maps $\pm$ induce injective maps.*

5 0-surgery homeomorphism and its extension to 0-trace

Dualizable patterns are used to construct a pair of knots with a diffeomorphism between their 0-traces. In this section, we consider a technique to construct a pair of knots with a homeomorphism between their 0-surgeries which may not extend to a diffeomorphism between their 0-traces. The technique introduced in this section is essentially due to Manolescu and Piccirillo [4].

5.1 r-dualizable pattern

Define $\Gamma_r\colon \mathbf{S}^1 \times D^2 \to L(r,1)$ by Figure 10, where $r \in \mathbf{Z}$ and $L(r,1)$ is the Lens space of type $(r,1)$ represented by the r-framed unknot R. For convenience, define $L(0,1) = \mathbf{S}^1 \times \mathbf{S}^2$ and $L(\pm 1, 1) = \mathbf{S}^3$.

Fix an integer $r \in \mathbf{Z}$. For any curve $c\colon \mathbf{S}^1 \to \mathbf{S}^1 \times D^2$, define $\widetilde{c} = \Gamma_r \circ c\colon \mathbf{S}^1 \to L(r,1)$. Then, a pattern Q in a solid torus W is *r-dualizable* if $\widetilde{Q}$ is isotopic to $\widetilde{\lambda_W}$ in $L(r,1)$

and the isotopy preserves their 0-framings (the 0-framings are the images under Γ_r of the 0-framings of Q and λ_W in W). Obviously, a 0-dualizable pattern is a dualizable pattern.

Let Q be an r-dualizable pattern in W. Define a pattern $Q^\dagger$ by

$$Q^\dagger = \widetilde{\lambda_W} \subset L(r,1) \setminus \nu(\widetilde{Q}) \cong L(r,1) \setminus \nu(\widetilde{\lambda_W}) \cong \mathbf{S}^1 \times D^2,$$

where the parameter of $L(r,1) \setminus \nu(\widetilde{Q})$ is given so that $\widetilde{Q}$ is a longitude of the solid torus. Denote this solid torus by $W^\dagger$. We call $Q^\dagger$ the *dual* of Q (see Figure 10). We see that

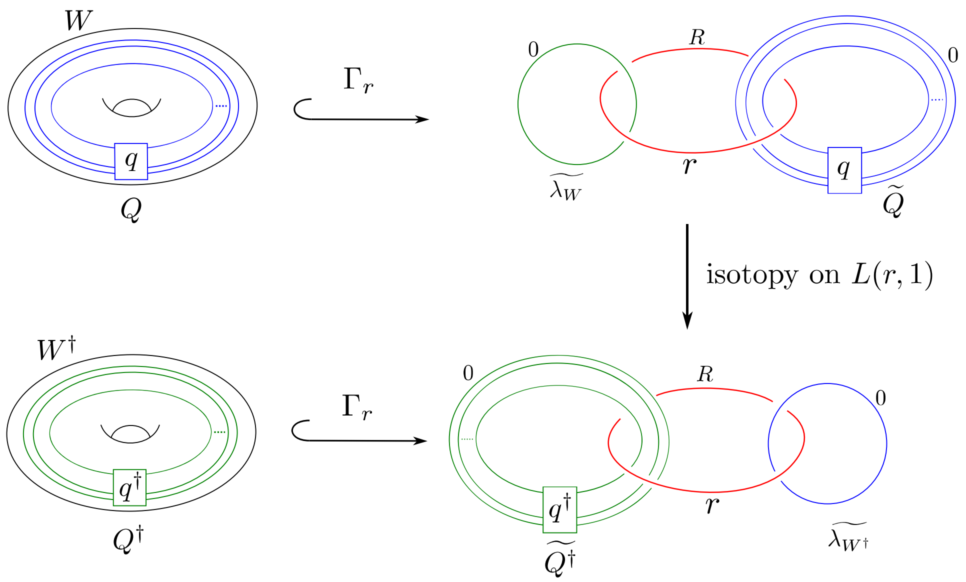

Figure 10: (color online) An r-dualizable pattern Q and its dual $Q^\dagger$. The framed unknot R with framing r represents the Lens space $L(r,1)$. The boxes labeled by q and $q^\dagger$ are the tangles corresponding to Q and $Q^\dagger$, respectively.

there are natural homeomorphisms

$$M_{Q(U)}(0) \cong M_{R\cup\widetilde{Q}\cup\widetilde{\lambda_W}}(r,0,0) \cong M_{R\cup\widetilde{\lambda_{W^\dagger}}\cup\widetilde{Q^\dagger}}(r,0,0) \cong M_{Q^\dagger(U)}(0).$$

Here we regard $\widetilde{Q}, \widetilde{\lambda_W}, \widetilde{Q^\dagger}$ and $\widetilde{\lambda_{W^\dagger}}$ as knots in $\mathbf{S}^3 \setminus \nu(R)$ by abuse of notations. Denote the composition of the homeomorphisms by $\phi^{(r)} \colon M_{Q(U)}(0) \to M_{Q^\dagger(U)}(0)$.

We can check that for an r-dualizable pattern $Q \subset W$, the 3-component link $R\cup\widetilde{Q}\cup\widetilde{\lambda_W}$ with framing $(r,0,0)$ satisfies the conditions of "RBG link" introduced by Manolescu and Piccirillo [4]. By the same argument in [4, Theorem 3.7 and Lemma 4.2], we obtain the following result on the existence of an extension of $\phi^{(r)}$ to a homeomorphism between 0-traces.

Theorem 5.1 (e.g. [4, Theorem 3.7 and Lemma 4.2])**.** *The homeomorphism*

$$\phi^{(r)} \colon M_{Q(U)}(0) \to M_{Q^\dagger(U)}(0)$$

extends to a homeomorphism $\Phi \colon X_{Q(U)}(0) \to X_{Q^\dagger(U)}(0)$ *if and only if* r *is even.*

Question 5.2. *For $r \in 2\mathbf{Z} \setminus \{0\}$, give a sufficient condition for $\phi^{(r)}$ to extend to a diffeomorphism $\Phi\colon X_{Q(U)}(0) \to X_{Q^\dagger(U)}(0)$.*

Remark 5.3. A pattern of geometric winding number one is (0-)dualizable. However, such a pattern may not be r-dualizable for some $r \neq 0$. For example, a pattern Q of geometric winding number one with $Q(U) \neq U$ is not ± 1-dualizable.

On the other hand, any r-dualizable pattern has algebraic winding number one because of homological reason.

5.2 Special RBG link

Manolescu and Piccirillo [4, Section 3] showed that any pair of knots admitting the same 0-surgery can be explained in terms of "RBG links". In particular, r-dualizable patterns are explained by using "special" RBG links in their sentences.

A 3-component framed link $R \cup B \cup G$ with framing $(r, b, g) \in \mathbf{Q}^3$ in $\mathbf{S}^3$ is a *special RBG link* if

- B and G are meridians of R,
- the linking number $l = \mathrm{lk}(B, G)$ of B and G satisfies $l = 0$ or $rl = 2$, where we give B and G homologous orientations in $\mathbf{S}^3 \setminus \nu(R)$,
- $r \in \mathbf{Z}$ and $b = g = 0$.

Note that B and G are unknotted but R may be knotted. A special RBG link is *unknotted* if R is unknotted.

For an unknotted special RBG link $R \cup B \cup G$, we can obtain an r-dualizable pattern as follows; Take a disk Δ_G bounded by G. Slide B along R until B no longer intersects Δ_G and denote the resulting framed knot by K_B. Then, the framing b' of K_B is zero because of homological reason. In fact, we have

$$\mathbf{Z}/b'\mathbf{Z} \cong H_1(M_{K_B}(b')) \cong H_1(M_{R\cup B\cup G}(r, 0, 0)) \cong \mathbf{Z}/(\det A)\mathbf{Z},$$

where A is the linking (or framing) matrix of $R \cup B \cup G$

$$A = \begin{pmatrix} r & 1 & 1 \\ 1 & 0 & l \\ 1 & l & 0 \end{pmatrix}.$$

Since the determinant $\det A$ is $l(2 - rl) = 0$, we have $b' = 0$. Put $Q_B = K_B \subset \mathbf{S}^3 \setminus \nu(R) = \mathbf{S}^1 \times D^2$. By replacing the roles of B and G, define K_G and Q_G. Then, we can check that Q_B and Q_G are r-dualizable and $Q_B^\dagger = Q_G$ (for example, the 0-framing of K_B is isotopic to the 0-framing of B in $M_R(r) = L(r, 1)$).

Conversely, for any r-dualizable pattern $Q \subset W$, we can construct an unknotted special RBG link $R \cup B \cup G$ such that $Q = K_B \subset \mathbf{S}^3 \setminus \nu(R)$ and $Q^\dagger = K_G \subset \mathbf{S}^3 \setminus \nu(R)$. In fact, for the 3-component link $R \cup \widetilde{Q} \cup \mu_R$ with framing $(r, 0, 0)$, there is a sequence of slidings G along R which changes the framed link into $R \cup \widetilde{\lambda_W} \cup \mu_R$ with framing $(r, 0, 0)$ because of the definition of r-dualizable pattern. Note that μ_R is a meridian of R, where $\widetilde{Q} \cup \mu_R$ is a split link and $\widetilde{\lambda_W} \cup \mu_R$ may not be split. Then we can compute that

$$\mathbf{Z} \cong H_1(M_{Q(U)}(0)) \cong H_1(M_{R\cup\widetilde{Q}\cup\mu_R}(r,0,0)) \cong H_1(M_{R\cup\widetilde{\lambda_W}\cup\mu_R}(r,0,0)) \cong \mathbf{Z}/(\det A')\mathbf{Z},$$

where A' is the linking (or framing) matrix of $R \cup \widetilde{\lambda_W} \cup \mu_R$

$$A' = \begin{pmatrix} r & 1 & 1 \\ 1 & 0 & l' \\ 1 & l' & 0 \end{pmatrix}, \ l' = \mathrm{lk}(\widetilde{\lambda_W}, \mu_R).$$

Note that we give $\widetilde{\lambda_W}$ and μ_R homologous orientations in $\mathbf{S}^3 \setminus \nu(R)$. The determinant $\det A'$ is $l'(2 - rl')$. Since the determinant is zero, we have $l' = 0$ or $rl' = 2$. Hence, $R \cup \widetilde{\lambda_W} \cup \mu_R$ is an unknotted special RBG link and this is the desired one (compare with [7, Proposition 4.2]).

Remark 5.4. A special RBG link $R \cup B \cup G$ satisfies

- there are homeomorphisms $\phi_B \colon M_{R\cup B}(r,b) \to \mathbf{S}^3$ and $\phi_G \colon M_{R\cup G}(r,g) \to \mathbf{S}^3$,
- $H_1(M_{R\cup B\cup G}(r,b,g); \mathbf{Z}) \cong \mathbf{Z}$.

Generally, a rationally framed 3-component link $R \cup B \cup G$ satisfying the above two conditions is called an *RBG link* (for more detail, see [4]).

Remark 5.5. RGB diagrams defined by Piccirillo [7] and named by the author [9] can be regarded as special RBG links (after some deformations). The etymology of "RBG" and "RGB" are the RGB-color. In fact, the components R, G and B are colored by red, green, and blue, respectively in [7, 9].

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