Quasi-symmetric numerical semigroups on triple covers of curves [1]

神奈川工科大学・基礎・教養教育センター　米田 二良
Jiryo Komeda
Center for Basic Education and Integrated Learning
Kanagawa Institute of Technology

Abstract

We study quasi-symmetric numerical semigroups through the map dividing by 3. We give quasi-symmetric numerical semigroups which are the Weierstrass semigroups of ramification points of triple cyclic covers of the projective line. Moreover, we find examples of quasi-symmetric Weierstrass numerical semigroups which cannot be attained by any ramification point of a triple cyclic cover of the projective line. We also construct many quasi-symmetric non-Weierstrass numerical semigroups

1 Introduction

Let $\mathbb{N}_0$ be the additive monoid of non-negative integers. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if the complement $\mathbb{N}_0 \backslash H$ is finite. The cardinality of $\mathbb{N}_0 \backslash H$ is called the *genus* of H, denoted by $g(H)$. We set

$$c(H) := \min\{c \in \mathbb{N}_0 \mid c + \mathbb{N}_0 \subseteqq H\},$$

which is called the *conductor* of H. It is well-known that $c(H) \leqq 2g(H)$. H is said to be *symmetric* if $c(H) = 2g(H)$. H is said to be *quasi-symmetric* if $c(H) = 2g(H) - 1$. A *curve* means a projective non-singular irreducible algebraic curve over an algebraically closed field k of characteritic 0. For a pointed curve (C, P) we set

$$H(P) = \{\alpha \in \mathbb{N}_0 \mid \exists f \in k(C) \text{ such that } (f)_\infty = \alpha P\},$$

where $k(C)$ is the field of rational functions on C. $H(P)$ is a numerical semigroup of genus $g(C)$ where $g(C)$ is the genus of C, which is called the *Weierstrass semigroup* of P. Let d_2 be the map from the set $\mathcal{H}$ of numerical semigroups to $\mathcal{H}$ defined by

$$d_2(H) = \{h' \in \mathbb{N}_0 \mid 2h' \in H\},$$

which is a numerical semigroup. Let $\pi : \tilde{C} \longrightarrow C$ be a double covering of curves with a ramification point $\tilde{P}$. Then $d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$. For any integer $t \geqq 3$ we set

$$d_t(H) = \{h' \in \mathbb{N}_0 \mid th' \in H\},$$

[1]This paper is an extended abstract and the details were published (see [5])
This work was supported by JSPS KAKENHI Grant Number18K03228.

which is a numerical semigroup. Let t be an integer $\geqq 3$. Let $\pi : C \longrightarrow C'$ be a cyclic covering of degree t with a totally ramification point P over P'. Then $d_t(H(P)) = H(P')$.

In this article we are devoted to study quasi-symmetric numerical semigroups through its image of the map d_3. Oliveira-Stöhr [6] constructed quasi-symmetric numerical semigroups H from any numerical semigroup H' with $d_3(H) = H'$. We generalize their method in Section 2. In Section 3 we obtain many non-Weierstrass quasi-symmetric numerical semigroups using the proof of Theorem 5.1 in [6] where a numerical semigroup H said to be *Weierstrass* if there exists a pointed curve (C, P) with $H = H(P)$. We give quasi-symmetric numerical semigroups gained by the Weierstrass semigroups of ramification points of triple cyclic covers of curves. Moreover, we show that some Weierstrass quasi-numerical semigroups cannot be gained by the above way.

2 Description of a quasi-symmetric numerical semigroup through d_3

Remark 2.1 *([2]) Let H be a quasi-symmetric numerical semigroup.*
(1) If $g(H)$ is even, then $d_2(H)$ is a symmetric numerical semigroup of genus $\dfrac{g(H)}{2}$.
(2) If $g(H)$ is odd, then $d_2(H)$ is a quasi-symmetric numerical semigroup of genus $\dfrac{g(H)+1}{2}$.

Remark 2.2 *([3]) If H is a symmetric numerical semigroup, then we have*

$$H = 2d_2(H) \cup \{2g(H) - 1 - 2t \mid t \in \mathbb{Z}\backslash d_2(H)\}.$$

Theorem 2.3 *([5]) Let H be a quasi-symmetric numerical semigroup with $g(H) \equiv 1$ mod 3. Then $d_3(H)$ is also a quasi-symmetric numerical semigroup of genus $\dfrac{g(H)+2}{3}$, that is to say, $g(H) = 3g(d_3(H)) - 2$.*

To describe a numerical semigroup we use the following notation: For any non-negative integers $a_1, a_2, \cdots, a_n$ we denote by

$$\langle a_1, a_2, \cdots, a_n \rangle$$

the additive monoid generated by $a_1, a_2, \cdots, a_n$.

Example. Let $H = \langle 4, 11, 13 \rangle$. Then we have $\mathbb{N}_0 \backslash H = \{1, 2, 3, 5, 6, 7, 9, 10, 14, 18\}$, which implies that $g(H) = 10$ and $c(H) = 19 = 2g(H) - 1$, Hence, H is quasi-symmetric and $d_3(H) = \langle 4, 5, 7 \rangle$, whose genus is $4 = \dfrac{g(H)+2}{3}$.

To state the theorem we need the following lemma:

Lemma 2.4 *([5]) Let H be a quasi-symmetric numerical semigroup with $g(H) \not\equiv 1$ mod 3. Then we have*

$$H = 3d_3(H) \cup \{2g(H) - 2 - 3t \mid t \in \mathbb{Z}\backslash d_3(H)\} \cup \{h \in H \mid h \equiv g(H) + 2 \text{ mod } 3\}.$$

Theorem 2.5 *([5]) Let H be a quasi-symmetric numerical semigroup of genus g with $g \not\equiv 1$ mod 3 and $g \geqq \frac{3c(d_3(H))}{2} + 1$. Then H is one of the following:*
(1) $H = 3d_3(H) \cup \{2g - 2 - 3r \mid r \in \mathbb{Z} \backslash d_3(H)\} \cup ((g + 2) + 3\mathbb{N}_0)$.
(2) There exists a non-empty set

$$\{t_1, \cdots, t_u\} \subseteqq \left\{2, 3, \cdots, \left[\frac{c(d_3(H)) + 1}{2}\right]\right\}.$$

such that

$$H = 3d_3(H) \cup \{2g - 2 - 3r \mid r \in \mathbb{Z} \backslash d_3(H)\} \cup \{g + 2 - 3t \mid t \in \{t_1, \cdots, t_u\}\}$$
$$\cup((g + 2) + 3\mathbb{N}_0) \backslash \{g - 4 + 3t \mid t \in \{t_1, \cdots, t_u\}\}.$$

The converse of Theorem 2.5 holds in the following case:

Remark 2.6 *([6]) Let H' be a numerical semigroup with $H' \neq \mathbb{N}_0$. Let $g \geqq 2c(H')$ with $g \not\equiv 1$ mod 3. We set*

$$H = 3H' \cup \{2g - 2 - 3r \mid r \in \mathbb{Z} \backslash H'\} \cup ((g + 2) + 3\mathbb{N}_0).$$

Then H is a quasi-symmetric numerical semigroup of genus g with $d_3(H) = H'$.

Example. Let $H' = \langle 2, 3\rangle$. Then $c(H') = 2$ and $m(H') = 2$. Take $g = 5 \equiv 2$ mod 3. We set

$$H = 3\langle 2, 3\rangle \cup \{10 - 2 - 3r \mid r \in \mathbb{Z} \backslash \langle 2, 3\rangle\} \cup ((5 + 2) + 3\mathbb{N}_0)$$
$$= \langle 6, 9\rangle \cup \{5, 11, 14, \cdots\} \cup \{7, 10, 13, \cdots\} = \langle 5, 6, 7, 9\rangle.$$

Then $g(H) = 5$ and $c(H) = 13 - 5 + 1 = 9 = 2g(H) - 1$. Hence, H is a quasi-symmetric numerical semigroup with $d_3(H) = H'$.

The converse of Theorem 2.5 also holds in the cases which are different from the one in Remark 2.6.

Theorem 2.7 *Let H' be a numerical semigroup with $H' \neq \mathbb{N}_0$. Let $g \geqq 2c(H') + \frac{m(H') + 1}{2}$ with $g \not\equiv 1$ mod 3 where $m(H')$ is the minimum of positive integers in H'. Let $t \in \mathbb{Z}$ with $2 \leqq t \leqq \frac{m(H') + 1}{2}$. We set*

$$H = 3H' \cup \{2g - 2 - 3r \mid r \in \mathbb{Z} \backslash H'\} \cup \{g + 2 - 3t\} \cup ((g + 2) + 3\mathbb{N}_0) \backslash \{g - 4 + 3t\}.$$

Then H is a quasi-symmetric numerical semigroup of genus g with $d_3(H) = H'$.

Example. Let $H' = \langle 3, 4, 5\rangle$. Then $c(H') = 3$ and $m(H') = 3$. Take $g = 8 \equiv 2$ mod 3 and $t = 2$. We set

$$H = 3\langle 3, 4, 5\rangle \cup \{16 - 2 - 3r \mid r \in \mathbb{Z} \backslash \langle 3, 4, 5\rangle\} \cup \{10 - 3 \times 2\} \cup ((8 + 2) + 3\mathbb{N}_0) \backslash \{8 - 4 + 6\}$$
$$= \langle 9, 12, 15\rangle \cup \{8, 11, 17, 20, \cdots\} \cup \{4, 13, 16, \cdots\} = \langle 4, 9, 11\rangle.$$

Then $g(H) = 8$ and $c(H) = 18 - 4 + 1 = 15 = 2g(H) - 1$. Hence, H is a quasi-symmetric numerical semigroup with $d_3(H) = H'$.

3 Three types of quasi-symmetric semigroups

Remark 3.1 *Let H' be a non-Weierstrass numerical semigroup. Take $g \geqq 15g(H') + 11$. Let H be a numerical semigroup of genus g with $d_3(H) = H'$. Then H is also a non-Weierstrass numerical semigroup. (See the proof of Theorem 5.1 in [6]).*

Using the following theorem, which follows from Remark 3.1, we can give a lot of non-Weierstrass quasi-symmetric numerical semigroups.

Theorem 3.2 *Let H' be a non-Weierstrass numerical semigroup. Take $g \geqq 15g(H') + 11$ with $g \not\equiv 1 \bmod 3$. We set $T(H') = \left[\frac{m(H')+1}{2}\right]$. Then there are at least $T(H')$ non-Weierstrass quasi-symmetric numerical semigroups H of genus g with $d_3(H) = H'$. In fact, $H_1, H_2, \cdots, H_{T(H')}$ are such numerical semigroups where*

$$H_1 = 3H' \cup \{2g-2-3r \mid r \in \mathbb{Z}\backslash H'\} \cup ((g+2) + 3\mathbb{N}_0)$$

and for any integer t with $2 \leqq t \leqq T(H')$ we set

$$H_t = 3H' \cup \{2g-2-3r \mid r \in \mathbb{Z}\backslash H'\} \cup \{g+2-3t\} \cup ((g+2)+3\mathbb{N}_0)\backslash\{g-4+3t\}.$$

Example. Let $H' = \langle 13, 14, 15, 16, 17, 18, 20, 22, 23\rangle$, which is a non-Weierstrass nunmerical semigroup of genus 16 ([1]). Let $g = 15 \times 16 + 11 = 251$. Then there are seven non-Weierstrass quasi-symmetric numerical semigroups $H_1, H_2, \cdots, H_7$ of genus 251 with $d_3(H_i) = H'$.

A numerical semigroup H is said to be *triple cyclic covering type* , which is abbreviated to *TC* if there exists a triple cyclic cover of curves with a ramification point whose Weierstrass semigroup is H.

Theorem 3.3 *If H is a quasi-symmetric numerical semigroup of genus g with $g \equiv 1 \bmod 3$, then it is not TC.*

Proof. Since $g \equiv 1 \bmod 3$, we obtain $g(H) = 3g(d_3(H)) - 2$ from Theorem 2.3. Assume that H were TC. Then it would follow from Riemann-Hurwitz formula that $g(H) \geqq 3g(d_3(H)) - 1$. This is a contradiction.

Example. Let n be a positive integer with $n \equiv 1 \bmod 3$. We set $H_n = \langle 5, 5n+3, 5n+4, 5(n+1)+1\rangle$. Then we have $g(H_n) = 5n+2 \equiv 1 \bmod 3$ and $c(H_n) = 5(2n+1)+2 = 2g(H_n) - 1$. Hence, H_n is a quasi-symmetric numerical semigroup, which is not TC. But, H_n is Weierstrass, because the minimum positive integer in H_n is 5 (See [4]).

Theorem 3.4 *([5]) Let H be a quasi-symmetric Weierstrass numerical semigroup of genus g. Take a pointed curve (C, P) such that $H(P) = H$. Let Q be a point of C with $Q \neq P$ such that $K_C \sim (2g-3)P + Q$, where K_C is a canonical divisor on C. Let d be an integer with $d \geqq g$. Consider a triple cyclic cover*

$$\tilde{C} = \mathrm{Spec}(O_C \oplus O_C(-dP) \oplus O_C(-2dP - Q)) \longrightarrow C$$

which has a ramification point $\tilde{P}$ over P. Then

$$H(\tilde{P}) = 3H + \langle 3d-1, 2(3d-1) + 3(g-1)\rangle,$$

which is quasi-symmetric. Hence this quasi-symmetric numerical semigroup is TC.

Example. Let $\tilde{H} = \langle 5, 9, 12, 13\rangle$. Then $\tilde{H}$ is a TC numerical semigroup which is quasi-symmetric. Indeed, in Theorem 3.4 we set $H = \langle 3, 4, 5\rangle$ and $d = 2$. Then we get

$$H(\tilde{P}) = 3\langle 3, 4, 5\rangle + \langle 5, 10+3\rangle = \langle 5, 9, 12, 13\rangle.$$

References

[1] R.O. Buchweitz, *On Zariski's criterion for equisingularity and non-smoothable monomial curves*, Preprint 113, University of Hannover (1980).

[2] J. Komeda, *Quasi-symmetric numerical semigroups and double covers of curves*, RIMS Kokyuroku **1964** (2015) 55–60.

[3] J. Komeda, *On quasi-symmetric numerical semigroups*, Research Reports of Kanagawa Institute of Technology **B-35** (2011) 17–21.

[4] J. Komeda, *On the existence of Weierstrass points whose first non-gaps are five*, Manuscripta Math. **76** (1992) 193–211.

[5] J. Komeda, *A classification of the quasi-symmetric numerical semigroups through their division by* 3, In preparation.

[6] G. Oliveira and K.-O. Stöhr, *Gorenstein curves with quasi-symmetric Weierstrass semigroups*, Geometriae Dedicata, **67** (1997), 45–63.