

On lattice points which become vertices of Heronian triangles II *

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1 Introduction

Heronian triangles are triangles whose side lengths and area are all integers. For example, the triangles with side lengths 3, 4, 5, area 6, and with side lengths 5, 5, 6, area 12, and others. Specially the triangle whose side lengths 3, 4 and 5 is *Pythagorean triangle*, that is, a right triangle whose side lengths are all integers. We discuss the lattice points which become vertices of Heronian triangles.

In section 2, we state some results previously clarified in [1]. In section 3, we will describe the contents of the subsequent research. In this article, we will explain our result that every Heronian triangle can be realized as a lattice triangle. P.Yiu's result[2] is another approach to a similar problem. We will consider some properties about lattice points which become vertices of Heronian triangles.

2 Some results previously clarified by us

We get the following theorems.

Theorem 2.1 [1, Theorem 3.1] *There exist circles which contain n lattice points such that the distance between any pair of them is an integer. Specially, in case $n = 3$, there are three lattice points on circles which become vertices of Heronian triangles.*

Theorem 2.2 [1, Theorem 3.3] *Not on the line three points which belong to the following set consists of lattice points*

$$\{(0, \pm y), (0, 0), (\pm x_1, 0), \dots, (\pm x_n, 0)\}$$

*This paper is a preliminary version and a final version will be submitted to elsewhere.

become vertices of Heronian triangles, where let p_i ($i = 1, \dots, n$) be odd prime number satisfying $p_i > p_j$ (for $i < j$), and $p_1 > 2p_2 \cdots p_n$, we can write $x_k = (p_1 \cdots p_k)^2 - 4(p_{k+1} \cdots p_n)^2$ ($k = 1, \dots, n$), $y = 4p_1 \cdots p_n$.

3 About the realization of a lattice triangle from a Heronian triangle

Lattice triangles (lattice Heronian triangles) are Heronian triangles whose all vertices are on lattice points. In this section, we consider a lattice triangle that is realized from a Heronian triangle.

3.1 Main claim

We describe one way to solve this problem. And we reveal the following proposition.

Proposition 3.1 *All Heronian triangles can be realized as lattice triangles.*

Outline of the proof. Every Heronian triangles can be put the coordinate plane by performing parallel translation and rotation.

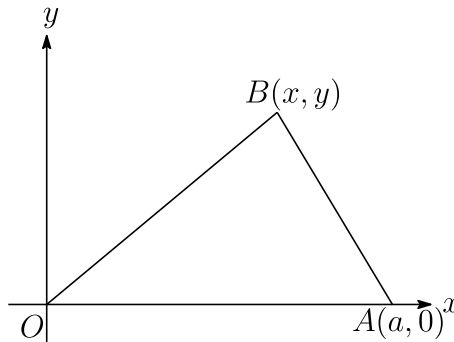


Figure 3.A

In this situation, x, y are both rational numbers. Therefore, by performing an appropriate similar transformation, we can x, y are both integers, that is, a point (x, y) is a lattice point. Therefore three points $(0, 0), (x, y), (a, 0)$ are all lattice points, and a Heronian $\triangle OAB$ is a lattice triangle. ■

3.2 Related results

Another approach to the same problem is P.Yiu's result[2]. P.Yiu apply elementary number theory such as integer solution of quartic equations.

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References

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