Some variations of the stable set polytope of a graph and Gorenstein property of the Ehrhart rings of them*

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1 Introduction

A convex polytope is a subset of a Euclidean space which is a convex hull of finite points. It is known that a convex polytope can be described as the intersection of finete closed halfspaces, i.e., it is described as the set of points satisfying finite inequalities. However, in general, it is difficult to describe a given convex polytope by finite inequalities explicitly.

Let $G = (V, E)$ be a finite simple graph. A stable set S of G is a subset of V with no two elements of S are adjacent. The stable set polytope of G is the convex polytope in $\mathbb{R}^{\#V}$ spanned by the points corresponding to stables sets. It is hard to describe stable set polytope by inequalities. However, there are convex polytopes described by inequalities which are slightly larger than the stable set polytope. We denote them by HSTAB(G), TSTAB(G) and QSTAB(G). These polytopes contain the stable set polytope. In this note, we discuss the Gorenstein property of the Ehrhart rings of these polytopes.

2 Graphs

We denote by $\mathbb{N}$ the set of nonnegative integers, by $\mathbb{Z}$ the set of integers, by $\mathbb{Q}$ the set of rational numbers and by $\mathbb{R}$ the set of real numbers respectively.

*This paper is an announcement of our result and the detailed version will be submitted to somewhere.

For a set X, we denote by $\#X$ the cardinality of X. For sets X and Y, we define $X \setminus Y := \{x \in X \mid x \notin Y\}$. For nonempty sets X and Y, we denote the set of maps from X to Y by Y^X. If X is a finite set, we identify $\mathbb{R}^X$ with the Euclidean space $\mathbb{R}^{\#X}$. For f, f_1, $f_2 \in \mathbb{R}^X$ and $a \in \mathbb{R}$, we define maps $f_1 \pm f_2$ and af by $(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$ and $(af)(x) = a(f(x))$ for $x \in X$. For a subset A of X, we define the characteristic function $\chi_A \in \mathbb{R}^X$ by $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in X \setminus A$. For a nonempty subset $\mathscr{X}$ of $\mathbb{R}^X$, we denote by $\mathrm{conv}\mathscr{X}$ (resp. $\mathrm{aff}\,\mathscr{X}$) the convex hull (resp. affine span) of $\mathscr{X}$. We denote by $\mathrm{relint}\mathscr{X}$ the interior of $\mathscr{X}$ in the topological space $\mathrm{aff}\,\mathscr{X}$.

Definition 2.1 Let X be a finite set and $\xi \in \mathbb{R}^X$. For $B \subset X$, we set $\xi^+(B) := \sum_{b \in B} \xi(b)$. We define the empty sum to be 0, i.e., $\xi^+(\emptyset) = 0$.

A (finite) graph G is a pair of finite set V and a subset E of $\binom{V}{2}$, where $\binom{V}{2}$ is the set of 2 element subsets of V. We denote $G = (V, E)$ or $V = V(G)$ and $E = E(G)$. An element of V (resp. E) is called a vertex (resp. edge) of G. If $\{a, b\} \in E$, where a, $b \in V$, we say that a and b are adjacent. A clique of G is a subset K of V such that any two elements of K are adjacent. Let v_1, v_2, $\ldots$, v_r be distinct vertices of G with $r \geq 3$. If $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq r-1$ and $\{v_r, v_1\} \in E$, then we say that $v_1 v_2 \cdots v_r v_1$ is a cycle (of length r). Suppose that $v_1 v_2 \cdots v_r v_1$ is a cycle. If $\{v_i, v_j\} \in E$ and $2 \leq |i-j| \leq r-2$, we say that $\{v_i, v_j\}$ is a chord of the cycle $v_1 v_2 \cdots v_r v_1$.

Definition 2.2 $S \subset V$ is called a stable set if $\{a, b\} \notin E$ for any a, $b \in S$. And we set

$$\mathrm{STAB}(G) := \mathrm{conv}\{\chi_S \in \mathbb{R}^V \mid S \text{ is a stable set of } G.\}$$

Remark 2.3 It is clear that for $f \in \mathrm{STAB}(G)$,

(1) $f(x) \geq 0$ for any $x \in V$.

(2) $f^+(K) \leq 1$ for any clique K in G.

(3) $f^+(C) \leq \frac{\#C-1}{2}$ for any odd cycle C.

Definition 2.4 We set

$$\mathrm{HSTAB}(G) := \{f \in \mathbb{R}^V \mid f \text{ satisfies (1), (2) and (3) above}\},$$

$$\mathrm{TSTAB}(G) := \left\{ f \in \mathbb{R}^V \,\middle|\, \begin{array}{l} 0 \leq f(x) \leq 1 \text{ for any } x \in V,\ f^+(e) \leq 1 \\ \text{for any } e \in E \text{ and } f^+(C) \leq \frac{\#C-1}{2} \text{ for} \\ \text{any odd cycle } C \end{array} \right\}$$

and

$$\mathrm{QSTAB}(G) := \left\{ f \in \mathbb{R}^V \,\middle|\, \begin{array}{l} f(x) \geq 0 \text{ for any } x \in V \text{ and } f^+(K) \leq \\ 1 \text{ for any clique } K \text{ in } G \end{array} \right\}.$$

If $\mathrm{STAB}(G) = \mathrm{HSTAB}(G)$, then G is called an h-perfect graph, if $\mathrm{STAB}(G) = \mathrm{TSTAB}(G)$, then G is called a t-perfect graph.

It follows from the definition that $\mathrm{STAB}(G) \subset \mathrm{HSTAB}(G) \subset \mathrm{TSTAB}(G)$ and $\mathrm{HSTAB}(G) \subset \mathrm{QSTAB}(G)$.

Definition 2.5 Let $H \subset V$. The induced subgraph $H = (V(H), E(H))$ of G is the graph with $V(H) = H$ and $E(H) = \{\{a, b\} \in E \mid a, b \in H\}$.

Definition 2.6 Let $G' = (V', E')$ be a graph. A coloring of G' is a map f from V' to a finite set C with $f(a) \neq f(b)$ for any $\{a, b\} \in E'$. We set

$$\chi(G') := \min\{\#C \mid \exists f : V' \to C \text{ is a coloring}\}$$

and

$$\omega(G') := \max\{\#K \mid K \text{ is a clique in } G'\}.$$

It is clear from the definition that $\chi(G') \geq \omega(G')$ for any graph G'.

Definition 2.7 G is a perfect graph if $\chi(H) = \omega(H)$ for any induced subgraph H of G.

Fact 2.8 ([Chv, Theorem 3.1]) *G is perfect* $\iff$ $\mathrm{STAB}(G) = \mathrm{QSTAB}(G)$.

Corollary 2.9 *Perfect and t-perfect graphs are h-perfect.*

Fact 2.10 (Strong perfect graph theorem [CRST]) *G is perfect if and only if neither G nor $\overline{G}$ has odd cycle without chord and length at least 5, where $\overline{G} = (V, \binom{V}{2} \setminus E)$.*

3 Ehrhart rings

Let $\mathbb{K}$ be a field, X a finite set and $\mathscr{P}$ a rational convex polytope in $\mathbb{R}^X$, i.e., a convex polytope whose vertices are contained in $\mathbb{Q}^X$. Also let $-\infty$ be a new element with $-\infty \notin X$ and set $X^- := X \cup \{-\infty\}$. Further, let $\{T_x\}_{x \in X^-}$ be a family of indeterminates indexed by X^-.

For $f \in \mathbb{Z}^{X^-}$, we denote the Laurent monomial $\prod_{x \in X^-} T_x^{f(x)}$ in $\mathbb{K}[T_x^{\pm 1} \mid x \in X^-]$ by T^f.

Set $\deg T_x = 0$ for $x \in X$ and $\deg T_{-\infty} = 1$.

Definition 3.1 The Ehrhart ring of $\mathscr{P}$ over a field $\mathbb{K}$ is the subring

$$\mathbb{K}[T^f \mid f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f|_X \in \mathscr{P}]$$

of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in X^-]$, where $f|_X$ is the restriction of f to X. We denote the Ehrhart ring of $\mathscr{P}$ over $\mathbb{K}$ by $E_{\mathbb{K}}[\mathscr{P}]$.

Note that $E_{\mathbb{K}}[\mathscr{P}]$ is an $\mathbb{N}$-graded ring. Further it is known the following (cf. [Hoc]).

Fact 3.2 $E_{\mathbb{K}}[\mathscr{P}]$ *is a Noetherian normal and Cohen-Macaulay domain.*

By Stanley's description of the canonical module of a normal affine semigroup ring [Sta], we see the following.

Fact 3.3 *The ideal*

$$\bigoplus_{f \in \mathbb{Z}^{X^-}, f(-\infty)>0, \frac{1}{f(-\infty)} f|_X \in \mathrm{relint}\,\mathscr{P}} \mathbb{K}T^f$$

of $E_{\mathbb{K}}[\mathscr{P}]$ *is the canonical module of* $E_{\mathbb{K}}[\mathscr{P}]$.

We denote the ideal of Fact 3.3 by $\omega_{E_{\mathbb{K}}[\mathscr{P}]}$ and call the canonical ideal of $E_{\mathbb{K}}[\mathscr{P}]$.

It is also known that the dimension (Krull dimension) of $E_{\mathbb{K}}[\mathscr{P}]$ is equal to $\dim \mathscr{P} + 1$.

Here we recall the definition of the group of divisors (cf. [Fos]). Let R be a Noetherian normal domain with quotient field $Q(R)$. A fractional ideal $I \subset Q(R)$ is called divisorial if $I = R :_{Q(R)} (R :_{Q(R)} I)$. We denote the set of divisorial ideals of R by $\mathrm{Div}(R)$.

Fact 3.4 $\mathrm{Div}(R)$ *form a group by the operation* $I \cdot J := R :_{Q(R)} (R :_{Q(R)} IJ)$.

In fact, $\mathrm{Div}(R)$ is a free abelian group with basis height 1 prime ideals of R.

We denote the n-th power of $I \in \mathrm{Div}(R)$ in this group by $I^{(n)}$.

Fact 3.5 *If R is a Cohen-Macaulay local or graded domain with canonical module, then the canonical module is isomorphic to a divisorial ideal. In particular, the canonical ideal* $\omega_{E_{\mathbb{K}}[\mathscr{P}]}$ *of the Ehrhart ring of a rational convex polytope* $\mathscr{P}$ *is divisorial.*

4 Gorenstein property

First we recall the notion of the trace of a module.

Definition 4.1 *Let R be a commutative ring and M an R-module. We set*

$$\mathrm{tr}(M) := \sum_{\varphi \in \mathrm{Hom}(M,R)} \varphi(M)$$

and call $\mathrm{tr}(M)$ *the trace of* M.

Fact 4.2 ([HHS, Lemma 1.1]) *If an ideal I contains a NZD, then*

$$\mathrm{tr}(I) = I^{-1}I,$$

where $I^{-1} := \{x \in Q(R) \mid xI \subset R\}$.

Note that if R is a Noetherian normal domain and $I \in \mathrm{Div}(R)$, then $I^{-1} = I^{(-1)}$.

Fact 4.3 ([HHS, Lemma 2.1]) *Let R be a Cohen-Macaulay local or graded ring over a field with canonical module ω_R. Then R is Gorenstein if and only if* $\mathrm{tr}(\omega_R) = R$.

Definition 4.4 Set $\mathscr{K} := \{K \subset V \mid K \text{ is a clique of } G \text{ and } \#K \leq 3\}$.

Remark 4.5

$$\mathrm{HSTAB}(G) = \left\{ f \in \mathbb{R}^V \;\middle|\; \begin{array}{l} f(x) \geq 0 \text{ for any } x \in V,\ f^+(K) \leq 1 \\ \text{for any maximal clique } K \text{ of } G \text{ and} \\ f^+(C) \leq \frac{\#C-1}{2} \text{ for any odd cycle } C \\ \text{without chord and length at least 5} \end{array} \right\},$$

$$\mathrm{TSTAB}(G) = \left\{ f \in \mathbb{R}^V \;\middle|\; \begin{array}{l} f(x) \geq 0 \text{ for any } x \in V,\ f^+(K) \leq 1 \\ \text{for any maximal element } K \text{ of } \mathscr{K} \text{ and} \\ f^+(C) \leq \frac{\#C-1}{2} \text{ for any odd cycle } C \\ \text{without chord and length at least 5} \end{array} \right\}$$

and

$$\mathrm{QSTAB}(G) = \left\{ f \in \mathbb{R}^V \;\middle|\; \begin{array}{l} f(x) \geq 0 \text{ for any } x \in V \text{ and } f^+(K) \leq \\ 1 \text{ for any maximal clique } K \text{ of } G \end{array} \right\}.$$

Definition 4.6 For $n \in \mathbb{Z}$, we set

$$\mathcal{U}^{(n)} := \left\{ \mu \in \mathbb{Z}^{V^-} \;\middle|\; \begin{array}{l} \mu(z) \geq n \text{ for any } z \in V,\ \mu^+(K) \leq \\ \mu(-\infty) - n \text{ for any maximal clique } K \\ \text{of } G \text{ and } \mu^+(C) \leq \mu(-\infty)\frac{\#C-1}{2} - n \\ \text{for any odd cycle } C \text{ without chord and} \\ \text{length at least } 5 \end{array} \right\},$$

$$t\mathcal{U}^{(n)} := \left\{ \mu \in \mathbb{Z}^{V^-} \;\middle|\; \begin{array}{l} \mu(z) \geq n \text{ for any } z \in V,\ \mu^+(K) \leq \\ \mu(-\infty) - n \text{ for any maximal element} \\ K \text{ of } \mathscr{K} \text{ and } \mu^+(C) \leq \mu(-\infty)\frac{\#C-1}{2} - n \\ \text{for any odd cycle } C \text{ without chord and} \\ \text{length at least } 5 \end{array} \right\}$$

$$q\mathcal{U}^{(n)} := \left\{ \mu \in \mathbb{Z}^{V^-} \;\middle|\; \begin{array}{l} \mu(z) \geq n \text{ for any } z \in V \text{ and } \mu^+(K) \leq \\ \mu(-\infty) - n \text{ for any maximal clique of} \\ G \end{array} \right\}.$$

By Remark 4.5 and Fact 3.3, we see the following.

Lemma 4.7

$$E_{\mathbb{K}}[\mathrm{HSTAB}(G)] = \bigoplus_{\mu \in \mathcal{U}^{(0)}} \mathbb{K}T^\mu, \qquad \omega_{E_{\mathbb{K}}[\mathrm{HSTAB}(G)]} = \bigoplus_{\mu \in \mathcal{U}^{(1)}} \mathbb{K}T^\mu,$$

$$E_{\mathbb{K}}[\mathrm{TSTAB}(G)] = \bigoplus_{\mu \in t\mathcal{U}^{(0)}} \mathbb{K}T^\mu, \qquad \omega_{E_{\mathbb{K}}[\mathrm{TSTAB}(G)]} = \bigoplus_{\mu \in t\mathcal{U}^{(1)}} \mathbb{K}T^\mu$$

and

$$E_{\mathbb{K}}[\mathrm{QSTAB}(G)] = \bigoplus_{\mu \in q\mathcal{U}^{(0)}} \mathbb{K}T^\mu, \qquad \omega_{E_{\mathbb{K}}[\mathrm{QSTAB}(G)]} = \bigoplus_{\mu \in q\mathcal{U}^{(1)}} \mathbb{K}T^\mu.$$

In fact, the following fact holds.

Proposition 4.8 *Let* $n \in \mathbb{Z}$. *Then*

$$\omega^{(n)}_{E_{\mathbb{K}}[\mathrm{HSTAB}(G)]} = \bigoplus_{\mu \in \mathcal{U}^{(n)}} \mathbb{K}T^\mu,$$

$$\omega^{(n)}_{E_{\mathbb{K}}[\mathrm{TSTAB}(G)]} = \bigoplus_{\mu \in t\mathcal{U}^{(n)}} \mathbb{K}T^\mu$$

and

$$\omega^{(n)}_{E_{\mathbb{K}}[\mathrm{QSTAB}(G)]} = \bigoplus_{\mu \in q\mathcal{U}^{(n)}} \mathbb{K}T^\mu.$$

Corollary 4.9 *$E_{\mathbb{K}}[\mathrm{HSTAB}(G)]$ (resp. $E_{\mathbb{K}}[\mathrm{TSTAB}(G)]$, $E_{\mathbb{K}}[\mathrm{QSTAB}(G)]$) is Gorenstein if and only if there are $\eta \in \mathcal{U}^{(1)}$ (resp. $\eta \in t\mathcal{U}^{(1)}$, $\eta \in q\mathcal{U}^{(1)}$) and $\zeta \in \mathcal{U}^{(-1)}$ (resp. $\zeta \in t\mathcal{U}^{(-1)}$, $\zeta \in q\mathcal{U}^{(-1)}$) with $\eta + \zeta = 0$.*

By estimating the condition in Corollary 4.9, we see the following.

Theorem 4.10 *(1) $E_{\mathbb{K}}[\mathrm{HSTAB}(G)]$ is Gorenstein if and only if*

(a) Sizes of maximal cliques are constant (say n) and

(b) i. $n = 1$,

ii. $n = 2$ and there is no odd cycle without chord and length at least 7 or

iii. $n \geq 3$ and there is no odd cycle without chord and length at least 5.

(2) $E_{\mathbb{K}}[\mathrm{TSTAB}(G)]$ is Gorenstein if and only if

(a) $E = \emptyset$,

(b) G has no isolated vertex nor triangle and there is no odd cycle without chord and length at least 7 or

(c) all maximal cliques of G have size at least 3 and there is no odd cycle without chord and length at least 5.

(3) $E_{\mathbb{K}}[\mathrm{QSTAB}(G)]$ is Gorenstein if and only if sizes of maximal cliques are constant.

Corollary 4.11 *(1) [HT, Theorem 1] Let G be a cycle graph without chord. Then $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is Gorenstein if and only if the length of G is even or less than 7.*

(2) [OH, Theorem 2.1 (b)] Let G be a perfect graph. Then $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is Gorenstein if and only if the size of maximal cliques are the same.

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