Generalize acute point and fixed point and convergence theorems

玉川大学　川崎敏治 (toshiharu.kawasaki@nifty.ne.jp)
(Toshiharu Kawasaki, Tamagawa University)

Abstract

In this paper we generalize the concept of acute point and we introduce some acute point type theorems that holds under the same assumptions as fixed point theorems. Furthermore we show that fixed point theorems are derived from acute point type theorems. Furthermore we introduce some convergence theorems that holds under the same assumptions on parameters as fixed point theorems.

1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is said to be generalized hybrid [16] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. Such a mapping is said to be (α, β)-generalized hybrid. The class of all generalized hybrid mappings is a new class of nonlinear mappings including nonexpansive mappings, nonspreading mappings [18] and hybrid mappings [20]. A mapping T from C into H is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|$$

for any $x, y \in C$; nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$; hybrid if

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$. Any nonexpansive mapping is $(1, 0)$-generalized hybrid; any nonspreading mapping is $(2, 1)$-generalized hybrid; any hybrid mapping is $\left(\frac{3}{2}, \frac{1}{2}\right)$-generalized hybrid.

Motivated these mappings, in [14] Kawasaki and Takahashi introduced a new very wider class of mappings, called widely more generalized hybrid mappings, than the class of all generalized hybrid mappings. A mapping T from C into H is widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. This class includes the class of all generalized hybrid mappings and also the class of all k-pseudocontractions [3] for $k \in [0, 1]$. A mapping T from C into H is called a k-pseudocontraction if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$. Any (α, β)-generalized hybrid mapping is $(\alpha, 1 - \alpha, -\beta, \beta - 1, 0, 0, 0)$-widely more generalized hybrid; any k-pseudocontraction is $(1, 0, 0, -1, 0, 0, -k)$-widely more generalized hybrid. Furthermore they proved some fixed point theorems [5–8, 13–15] and some ergodic theorems [5, 13, 14].

There are some studies on Banach space related to these results. In [22] Takahashi, Wong and Yao introduced the generalized nonspreading mapping and the skew-generalized nonspreading mapping in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is said to be generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ \leq \varepsilon(\phi(Ty, Tx) - \phi(Ty, x)) + \zeta(\phi(y, Tx) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$, where J is the duality mapping on E and

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2.$$

Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading. A mapping T from C into E is said to be skew-generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ \leq \varepsilon(\phi(Ty, Tx) - \phi(y, Tx)) + \zeta(\phi(Ty, x) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-skew-generalized nonspreading. These classes include the class of generalized hybrid mappings in a Hilbert space, however, it does not include the class of widely more generalized hybrid mappings.

Motivated these results, we introduced a new class of mappings [9–11] on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C

into E is called a generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1\phi(Tx,Ty) + \alpha_2\phi(Ty,Tx) + \beta_1\phi(x,Ty) + \beta_2\phi(Ty,x) \\
&\quad +\gamma_1\phi(Tx,y) + \gamma_2\phi(y,Tx) + \delta_1\phi(x,y) + \delta_2\phi(y,x) \\
&\quad +\varepsilon_1\phi(Tx,x) + \varepsilon_2\phi(x,Tx) + \zeta_1\phi(y,Ty) + \zeta_2\phi(Ty,y) \\
&\leq 0
\end{aligned}$$

for any $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction. Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E and let C^* be a nonempty subset of E^*. A mapping T^* from C^* into E^* is called a *-generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1\phi_*(T^*x^*,T^*y^*) + \alpha_2\phi_*(T^*y^*,T^*x^*) + \beta_1\phi_*(x^*,T^*y^*) + \beta_2\phi_*(T^*y^*,x^*) \\
&\quad +\gamma_1\phi_*(T^*x^*,y^*) + \gamma_2\phi_*(y^*,T^*x^*) + \delta_1\phi_*(x^*,y^*) + \delta_2\phi_*(y^*,x^*) \\
&\quad +\varepsilon_1\phi_*(T^*x^*,x^*) + \varepsilon_2\phi_*(x^*,T^*x^*) + \zeta_1\phi_*(y^*,T^*y^*) + \zeta_2\phi_*(T^*y^*,y^*) \\
&\leq 0
\end{aligned}$$

for any $x^*, y^* \in C^*$, where

$$\phi_*(x^*,y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for any $x^*, y^* \in E^*$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction.

On the other hand, in [21] Takahashi and Takeuchi introduced a concept of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. $x \in H$ is called an attractive point of T if

$$\|x - Ty\| \leq \|x - y\|$$

for any $y \in C$. Let

$$A(T) = \{x \in H \mid \|x - Ty\| \leq \|x - y\| \text{ for any } y \in C\}.$$

Furthermore they proved that the Baillon type ergodic theorem [2] for generalized hybrid mappings without convexity of C.

In [22] Takahashi, Wong and Yao introduced some extensions of attractive point and proved some attractive point theorems on Banach spaces. $x \in E$ is an attractive point of T if

$$\phi(x,Ty) \leq \phi(x,y)$$

for any $y \in C$; $x \in E$ is a skew-attractive point of T if

$$\phi(Ty,x) \leq \phi(y,x)$$

for any $y \in C$. Let

$$\begin{aligned} A(T) &= \{x \in E \mid \phi(x, Ty) \le \phi(x, y) \text{ for any } y \in C\}; \\ B(T) &= \{x \in E \mid \phi(Ty, x) \le \phi(y, x) \text{ for any } y \in C\}. \end{aligned}$$

In [1] Atsushiba, Iemoto, Kubota and Takeuchi introduced a concept of acute point as an extension of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H and $k \in [0, 1]$. $x \in H$ is called a k-acute point of T if

$$\|x - Ty\|^2 \le \|x - y\|^2 + k\|y - Ty\|^2$$

for any $y \in C$. Let

$$\mathscr{A}_k(T) = \{x \in H \mid \|x - Ty\|^2 \le \|x - y\|^2 + k\|y - Ty\|^2 \text{ for any } y \in C\}.$$

Furthermore, using a concept of acute point, they proved convergence theorems without convexity of C.

We introduced some extensions of acute point [9–11]. Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell \in \mathbb{R}$. $x \in E$ is called a (k, ℓ)-acute point of T if

$$\phi(x, Ty) \le \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y)$$

for any $y \in C$. $x \in E$ is called a (k, ℓ)-skew-acute point of T if

$$\phi(Ty, x) \le \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y)$$

for any $y \in C$. Let

$$\begin{aligned} &\mathscr{A}_{k,\ell}(T) \\ &= \{x \in E \mid \phi(x, Ty) \le \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}; \\ &\mathscr{B}_{k,\ell}(T) \\ &= \{x \in E \mid \phi(Ty, x) \le \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}. \end{aligned}$$

Furthermore we proved some fixed point and acute point theorems [9], and some convergence theorems [10, 11]. However, acute point theorems require more assumptions on parameters than fixed point theorems.

In this paper we generalize the concept of acute point and we introduce some acute point type theorems that holds under the same assumptions as fixed point theorems. Furthermore we show that fixed point theorems are derived from acute point type theorems. Furthermore we introduce some convergence theorems that holds under the same assumptions on parameters as fixed point theorems.

2 Generalized acute and skew-acute point

Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. $x \in E$ is called a (k, ℓ, s)-generalized acute point of T if

$$s(\phi(x, Ty) - \phi(x, y)) \le k\phi(y, Ty) + \ell\phi(Ty, y) \tag{2.1}$$

for any $y \in C$. $x \in E$ is called a (k, ℓ, s)-generalized skew-acute point of T if

$$s(\phi(Ty, x) - \phi(y, x)) \le k\phi(y, Ty) + \ell\phi(Ty, y) \tag{2.2}$$

for any $y \in C$. Let

$$\begin{aligned}&\mathscr{A}_{k,\ell,s}(T)\\ &= \{x \in E \mid s(\phi(x, Ty) - \phi(x, y)) \le k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\};\\ &\mathscr{B}_{k,\ell,s}(T)\\ &= \{x \in E \mid s(\phi(Ty, x) - \phi(y, x)) \le k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}.\end{aligned}$$

It is obvious that

$$\mathscr{A}_{k_1,\ell_1,s_1}(T) \subset \mathscr{A}_{k_2,\ell_2,s_2}(T),\ \mathscr{B}_{k_1,\ell_1,s_2}(T) \subset \mathscr{B}_{k_2,\ell_2,s_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ and for any $s_1, s_2 \in (0, \infty)$ with $\frac{k_1}{s_1} \le \frac{k_2}{s_2}$ and $\frac{\ell_1}{s_1} \le \frac{\ell_2}{s_2}$;

$$\mathscr{A}_{k_1,\ell_1,s_1}(T) \supset \mathscr{A}_{k_2,\ell_2,s_2}(T),\ \mathscr{B}_{k_1,\ell_1,s_2}(T) \supset \mathscr{B}_{k_2,\ell_2,s_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ and for any $s_1, s_2 \in (-\infty, 0)$ with $\frac{k_1}{s_1} \le \frac{k_2}{s_2}$ and $\frac{\ell_1}{s_1} \le \frac{\ell_2}{s_2}$. Furthermore

$$\mathscr{A}_{k,\ell,0}(T) = \mathscr{B}_{k,\ell,0}(T) = E$$

for any $(k, \ell) \in [0, \infty) \times [0, \infty)$;

$$\mathscr{A}_{k,\ell,0}(T) = \mathscr{B}_{k,\ell,0}(T) = \emptyset$$

for any $(k, \ell) \in (-\infty, 0] \times (-\infty, 0] \setminus \{(0, 0)\}$; otherwise,

$$\mathscr{A}_{k,\ell,0}(T) = E \text{ or } \emptyset,\ \mathscr{B}_{k,\ell,0}(T) = E \text{ or } \emptyset;$$

however, it is generally unknown which case holds. In this way, $\mathscr{A}_{k,\ell,0}(T)$ and $\mathscr{B}_{k,\ell,0}(T)$ may be empty. However, in later discussions, under some assumptions, such cases will be properly ruled out.

The following lemmas are important property characterizing them.

Lemma 2.1. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{A}_{k,\ell,s}(T)$ is closed and convex.*

Lemma 2.2. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{B}_{k,\ell,s}(T)$ is closed.*

Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. $x^* \in E^*$ is called a (k, ℓ, s)-generalized-*-acute point of T^* if

$$s(\phi_*(x^*, T^*y^*) - \phi_*(x^*, y^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \tag{2.3}$$

for any $y^* \in C^*$. $x^* \in E^*$ is called a (k, ℓ, s)-generalized-*-skew-acute point of T^* if

$$s(\phi_*(T^*y^*, x^*) - \phi_*(y^*, x^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \tag{2.4}$$

for any $y^* \in C^*$. Let

$$\begin{aligned}
&\mathscr{A}^*_{k,\ell,s}(T^*) \\
&= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} s(\phi_*(x^*, T^*y^*) - \phi_*(x^*, y^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \\ \text{for any } y^* \in C^* \end{array} \right\}; \\
&\mathscr{B}^*_{k,\ell,s}(T^*) \\
&= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} s(\phi_*(T^*y^*, x^*) - \phi_*(y^*, x^*)) \leq k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \\ \text{for any } y^* \in C^* \end{array} \right\}.
\end{aligned}$$

Lemma 2.3. *Let E^* be the topological dual space of a strictly convex, reflective and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{A}^*_{k,\ell,s}(T^*)$ is closed and convex.*

Lemma 2.4. *Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{B}^*_{k,\ell,s}(T^*)$ is closed.*

Lemma 2.5. *Let E be a strictly convex, reflective and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E, let $T^* = JTJ^{-1}$ and let $k, \ell, s \in \mathbb{R}$. Then*

$$\mathscr{A}^*_{k,\ell,s}(T^*) = J(\mathscr{B}_{\ell,k,s}(T)), \; \mathscr{B}^*_{k,\ell,s}(T^*) = J(\mathscr{A}_{\ell,k,s}(T)).$$

In particular, $J(\mathscr{B}_{k,\ell,s}(T))$ is closed and convex and $J(\mathscr{A}_{k,\ell,s}(T))$ is closed.

Lemma 2.6. *Let E be a strictly convex and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. Then the following hold.*

(1) *If $(k, \ell) \in (-\infty, s] \times (-\infty, 0] \setminus \{(s, 0)\}$, then $C \cap \mathscr{A}_{k,\ell,s}(T)$ is a subset of the set of all fixed points of T;*

(2) *If $(k, \ell) \in (-\infty, 0] \times (-\infty, s] \setminus \{(0, s)\}$, then $C \cap \mathscr{B}_{k,\ell,s}(T)$ is a subset of the set of all fixed points of T.*

Lemma 2.7. *Let E^* be a strictly convex and smooth topological dual space of a Banach space, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell \in \mathbb{R}$. Then the following hold.*

(1) *If $(k, \ell) \in (-\infty, s] \times (-\infty, 0] \setminus \{(s, 0)\}$, then $C \cap \mathscr{A}^*_{k,\ell,s}(T^*)$ is a subset of the set of all fixed points of T^*;*

(2) *If $(k, \ell) \in (-\infty, 0] \times (-\infty, s] \setminus \{(0, 1)\}$, then $C \cap \mathscr{B}^*_{k,\ell,s}(T^*)$ is a subset of the set of all fixed points of T^*.*

3 Generalized acute and skew-acute point theorems

Theorem 3.1. *Let E be a reflexive and smooth Banach space, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + \lambda(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) \geq 0;\\
&\lambda(\alpha_1 + \gamma_1) + (1-\lambda)(\alpha_2 + \beta_2) \geq 0;\\
&\lambda(\beta_1 + \delta_1) + (1-\lambda)(\gamma_2 + \delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \geq 0.
\end{aligned}$$

Then there exists a $(-((1-\lambda)\zeta_1 + \lambda\varepsilon_2), -(\lambda\varepsilon_1 + (1-\lambda)\zeta_2), (1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))$-generalized acute point.

Theorem 3.2. *Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^* and let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction from C^* into itself. Suppose that there exists $z^* \in C^*$ such that $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + \lambda(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) \geq 0;\\
&\lambda(\alpha_1 + \gamma_1) + (1-\lambda)(\alpha_2 + \beta_2) \geq 0;\\
&\lambda(\beta_1 + \delta_1) + (1-\lambda)(\gamma_2 + \delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \geq 0.
\end{aligned}$$

*Then there exists a $(-((1-\lambda)\zeta_1 + \lambda\varepsilon_2), -(\lambda\varepsilon_1 + (1-\lambda)\zeta_2), (1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))$-generalized *-acute point.*

By Theorem 3.2 we obtain the following.

Theorem 3.3. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \geq 0.
\end{aligned}$$

Then there exists a $(-(\lambda\varepsilon_2 + (1-\lambda)\zeta_1), -((1-\lambda)\zeta_2 + \lambda\varepsilon_1), (1-\lambda)(\alpha_2+\beta_2) + \lambda(\alpha_1+\gamma_1))$-generalized skew-acute point.

4 Fixed point theorems

In this section we show that fixed point theorems are derived from generalized acute and skew-acute point theorems.

Theorem 4.1. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty, closed and convex subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0
\end{aligned}$$

and suppose that one of the following holds:

(1) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) > 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 \geq 0$*;*

(2) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) \geq 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 > 0$*.*

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Furthermore, if $(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) > 0$ or $\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1)+(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2) > 0$, then the fixed point of T is unique.

Theorem 4.2. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E satisfying $J(C)$ is closed and convex and let T be an $(\alpha_1, \alpha_2, \beta_1,$*

$\beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$*-generalized pseudocontraction from* C *into itself. Suppose that there exists* $\lambda \in [0,1]$ *such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1)\geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1)\geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1)\geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1\geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1\geq 0
\end{aligned}$$

and suppose that one of the following holds:

(1) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1)>0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1\geq 0$*;*

(2) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1)\geq 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1>0$*.*

Then T *has a fixed point if and only if there exists* $z \in C$ *such that* $\{T^n z \mid n \in \mathbb{N}\cup\{0\}\}$ *is bounded.*

Furthermore, if $(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1)>0$ *or* $\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2)+(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)>0$*, then the fixed point of* T *is unique.*

5 Mean convergence theorems

Theorem 5.1. *Let* E *be a uniformly convex Banach space with a Fréchet differentiable norm, let* C *be a nonempty subset of* E *and let* T *be an* $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$*-generalized pseudocontraction from* C *into itself. Suppose that there exists* $\lambda \in [0,1]$ *such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2)\geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2)\geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2)\geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2\geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2\geq 0,
\end{aligned}$$

and suppose that

$$\mathscr{A}_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T)\subset B(T)\neq\emptyset.$$

Let R *be the sunny generalized nonexpansive retraction of* E *onto* $B(T)$*. Then for any* $x \in C$*,*

$$S_n x \overset{def}{=} \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to an element

$$q \in \mathscr{A}_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T),$$

where $q = \lim_{n\to\infty} RT^n x$.

Additionally, if C *is closed and convex and one of the following holds:*

(1) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) > 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 \geq 0$*;*

(2) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) \geq 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 > 0$*,*

then q *is a fixed point of* T*.*

Theorem 5.2. *Let* E^* *be a uniformly convex topological dual space with a Fréchet differentiable norm, let* C^* *be a nonempty subset of* E^* *and let* T^* *be an* $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2,\varepsilon_1,\varepsilon_2,\zeta_1,\zeta_2)$*-*-generalized pseudocontraction from* C^* *into itself. Suppose that there exists* $\lambda \in [0,1]$ *such that*

$$\begin{aligned}&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\ &\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\ &\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\ &(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\ &\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0,\end{aligned}$$

and suppose that

$$\mathscr{A}^*_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T^*) \subset \mathscr{B}^*_{0,0}(T^*) \neq \emptyset.$$

Let R^* *be the sunny generalized nonexpansive retraction of* E^* *onto* $\mathscr{B}^*_{0,0}(T^*)$*. Then for any* $x^* \in C^*$*,*

$$S_n^* x^* \overset{def}{=} \frac{1}{n}\sum_{k=0}^{n-1}(T^*)^k x^*$$

is weakly convergent to an element

$$q^* \in \mathscr{A}^*_{-((1-\lambda)\zeta_1+\lambda\varepsilon_2),-(\lambda\varepsilon_1+(1-\lambda)\zeta_2),(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}(T^*),$$

where $q^* = \lim_{n\to\infty} R^*(T^*)^n x^*$.

Additionally, if C^* *is closed and convex and one of the following holds:*

(1) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) > 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 \geq 0$*;*

(2) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) \geq 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 > 0$*,*

then q^* *is a fixed point of* T^**.*

By Theorem 5.2 we obtain the following.

Theorem 5.3. *Let E be a strictly convex and reflexive Banach space with Kadec-Klee property and a uniformly Fréchet differentiable norm, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \geq 0,
\end{aligned}$$

suppose that

$$\mathscr{B}_{-(\lambda\varepsilon_2+(1-\lambda)\zeta_1),-((1-\lambda)\zeta_2+\lambda\varepsilon_1),(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}(T) \subset A(T) \neq \emptyset$$

and suppose that J^{-1} is weakly sequentially continuous. Let R^* be the sunny generalized nonexpansive retraction of E^* onto $J(A(T))$. Then for any $x \in C$,

$$S_n x \overset{def}{=} J^{-1}\left(\frac{1}{n}\sum_{k=0}^{n-1} JT^k x\right)$$

is weakly convergent to an element

$$q \in \mathscr{B}_{-(\lambda\varepsilon_2+(1-\lambda)\zeta_1),-((1-\lambda)\zeta_2+\lambda\varepsilon_1),(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}(T),$$

where $q = \lim_{n\to\infty} J^{-1}R^*JT^n x$.

Additionally, if $J(C)$ is closed and convexx and one of the following holds:

(1) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1) > 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1 \geq 0$*;*

(2) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1) \geq 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1 > 0$,

then q is a fixed point of T.

References

[1] S. Atsushiba, S. Iemoto, R. Kubota, and Y. Takeuchi, *Convergence theorems for some classes of nonlinear mappings in Hilbert spaces*, Linear and Nonlinear Analysis **2** (2016), 125–153.

[2] J.-B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B **280** (1975), 1511–1514.

[3] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, Journal of Mathematical Analysis and Applications **20** (1967), 197–228.

[4] T. Ibaraki and W. Takahashi, *A new projection and convergence theorems for the projections in Banach spaces*, J. Approx. Theory **149** (2007), 1–14.

[5] T. Kawasaki, *Fixed points theorems and mean convergence theorems for generalized hybrid self mappings and non-self mappings in Hilbert spaces*, Pacific Journal of Optimization **12** (2016), 133–150.

[6] ______, *Fixed point theorems for widely more generalized hybrid mappings in metric spaces, Banach spaces and Hilbert spaces*, Journal of Nonlinear and Convex Analysis **19** (2018), 1675–1683.

[7] ______, *On convergence of orbits to a fixed point for widely more generalized hybrid mappings*, Nihonkai Mathematical Journal **27** (2016), 89–97.

[8] ______, *Fixed point theorems for widely more generalized hybrid mappings in a metric space, a Banach space and a Hilbert space*, Journal of Nonlinear and Convex Analysis **19** (2018), 1675–1683.

[9] ______, *Fixed point and acute point theorems for new mappings in a Banach space*, Journal of Mathematics **2019** (2019), 12 pages.

[10] ______, *Mean convergence theorems for new mappings in a Banach space*, Journal of Nonlinear and Variational Analysis **3** (2019), 61–78.

[11] ______, *Weak convergence theorems for new mappings in a Banach space*, Linear and Nonlinear Analysis **5** (2019), 147–171.

[12] ______, *Generalized acute point theorems for generalized pseudocontractions in a Banach space*, Linear and Nonlinear Analysis **6** (2020), 73–90.

[13] T. Kawasaki and T. Kobayashi, *Existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces*, Scientiae Mathematicae Japonicae **77 (Online Version: e-2014)** (2014), 13–26 (Online Version: 29–42).

[14] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, Journal of Nonlinear and Convex Analysis **14** (2013), 71–87.

[15] ______, *Fixed point theorems for generalized hybrid demicontinuous mappings in Hilbert spaces*, Linear and Nonlinear Analysis **1** (2015), 125–138.

[16] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese Journal of Mathematics **14** (2010), 2497–2511.

[17] F. Kohsaka and W. Takahashi, *Generalized nonexpansive retractions and a proximal-type algorithm in Banach spaces*, Journal of Nonlinear and Convex Analysis **8** (2007), 197–209.

[18] ______, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Archiv der Mathematik **91** (2008), 166–177.

[19] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[20] ______, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, Journal of Nonlinear and Convex Analysis **11** (2010), 79–88.

[21] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, Journal of Nonlinear and Convex Analysis **12** (2011), 399–406.

[22] W. Takahashi, N.-C. Wong, and J.-C. Yao, *Attractive point and mean convergence theorems for new generalized nonspreading mappings in Banach spaces*, Infinite Products of Operators and Their Applications (S. Reich and A. J. Zaslavski, eds.), Contemporary Mathematics, vol. 636, American Mathematical Society, Providence, 2015, pp. 225–248.