# Identical Duals

## — Gap Function —

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**Abstract**

We consider *identical duals* of two pairs of minimization (primal) problems and maximization (dual) problems from a view point of *gap function*. The identical dual means that both optimum points of a primal problem and its dual one are identical. An identity

$$\text{(CI)} \quad \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n \;=\; x_0\mu_1$$

is called *complementary* [17]. The complementary identity leads to a gap function. We show that the complementary identity and the gap function play a fundamental part in analyzing an identical duality between primal and dual.

# 1 Identical Dual 1

As a pair of primal problem and dual problem, we take $n$-variable optimization problems:

$$(\mathrm{P}_1) \quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + x_n^2 \\ \text{subject to} & \text{(i)} \;\; x \in R^n, \quad \text{(ii)} \;\; x_0 = c \end{array}$$

$$(\mathrm{D}_1) \quad \begin{array}{ll} \text{Maximize} & 2c\mu_1 - \displaystyle\sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k-\mu_{k+1})^2\right] - \mu_n^2 - \mu_n^2 \\ \text{subject to} & \text{(i)} \;\; \mu \in R^n. \end{array}$$

First we present an identity, which plays a fundamental role in analyzing the pair. Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then an identity

$$(\mathrm{C}_1) \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + x_n\mu_n$$

holds true. This identity is called *complementary*. Furthermore the complementary identity implies that

$$
(\mathrm{QI}_1)\quad
\begin{aligned}
&\sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2+x_k^2\right]+\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]+2\mu_n^2-2c\mu_1\\
&=\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2+(x_k-\mu_k+\mu_{k+1})^2\right]+(x_{n-1}-x_n-\mu_n)^2+(x_n-\mu_n)^2.
\end{aligned}
$$

This is an identity on $R^n \times R^n$, which is called *quadratic*.

Now we define three functions $f,\ g : R^n \to R^1,\ \ h : R^n \times R^n \to R^1$ by

$$
\begin{aligned}
f(x) &= \sum_{k=1}^{n}\left[(x_{k-1}-x_k)^2+x_k^2\right]\\
g(\mu) &= 2c\mu_1-\sum_{k=1}^{n-1}\left[\mu_k^2+(\mu_k-\mu_{k+1})^2\right]-2\mu_n^2\\
h(x,\mu) &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2+(x_k-\mu_k+\mu_{k+1})^2\right]+(x_{n-1}-x_n-\mu_n)^2+(x_n-\mu_n)^2.
\end{aligned}
$$

They are called *primal*, *dual* and *gap* functions, respectively. Then $(\mathrm{QI}_1)$ is summarized as follows.

**Lemma 1** *It holds that*

$$(\mathrm{QI}_1)\qquad f(x)-g(\mu) \;=\; h(x,\mu).$$

We consider a linear system of $2n$-equation on $2n$-variable $(x,\ \mu)$:

$$
(\mathrm{EC}_1)\qquad
\begin{aligned}
c-x_1 &= \mu_1, & x_1 &= \mu_1-\mu_2 & &\\
x_{k-1}-x_k &= \mu_k, & x_k &= \mu_k-\mu_{k+1} & &2\le k\le n-1\\
x_{n-1}-x_n &= \mu_n, & x_n &= \mu_n. & &
\end{aligned}
$$

**Lemma 2** *It holds that*

(i) $h(x,\mu) \ge 0 \quad \forall (x,\ \mu) \in R^n \times R^n$

(ii) $h(x,\mu) = 0 \Longleftrightarrow (x,\mu)$ *satisfies* $(\mathrm{EC}_1)$.

**Corollary 1** *It holds that*

(i) $f(x) \ge g(\mu) \quad \forall (x,\ \mu) \in R^n \times R^n$

(ii) $f(x) = g(\mu) \Longleftrightarrow (x,\mu)$ *satisfies* $(\mathrm{EC}_1)$.

**Definition 1** We say that that $(\mathrm{P}_1)$ and $(\mathrm{D}_1)$ are *dual to each other* and $(\mathrm{EC}_1)$ is an *equality condition* (EC) if Corollary 1 (i), (ii) hold. Then we say that one is *dual* of the other. This definition applies for any triplet such as $\{(\mathrm{P}_1),\ (\mathrm{D}_1),\ (\mathrm{EC}_1)\}$.

From Corollary 1, it turns out that both are *dual to each other*, and ($EC_1$) is an *equality condition.*

**Lemma 3** ($EC_1$) *has a unique solution:*

$$\begin{aligned} x &= (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1), \end{aligned} \tag{1}$$

$$\begin{aligned} \mu &= (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n-2k},\ \ldots,\ F_4,\ F_2). \end{aligned} \tag{2}$$

Here $\{F_n\}$ is the *Fibonacci sequence.* This is defined as the solution to the second-order linear difference equation

$$x_{n+2} - x_{n+1} - x_n = 0, \qquad x_1 = 1,\ x_0 = 0. \tag{3}$$

| $n$ | $\cdots$ | $-2$ | $-1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | $\cdots$ | $-1$ | 1 | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |

Table 1 Fibonacci sequence $\{F_n\}$

*Proof.* From ($EC_1$), we have a pair of linear systems of $n$-variable on $n$-equation:

$$(\mathrm{EQ}_1) \qquad \begin{aligned} c &= 3x_1 - x_2 \\ x_1 &= 3x_2 - x_3 \\ &\vdots \\ x_{n-2} &= 3x_{n-1} - x_n \\ x_{n-1} &= 2x_n \end{aligned} \qquad \begin{aligned} c &= 2\mu_1 - \mu_2 \\ \mu_1 &= 3\mu_2 - \mu_3 \\ &\vdots \\ \mu_{n-2} &= 3\mu_{n-1} - \mu_n \\ \mu_{n-1} &= 3\mu_n. \end{aligned}$$

The left system has a solution $x$ in (1), while the right has a solution $\mu$ in (2). □

**Theorem 1** *The primal* ($P_1$) *has a minimum value* $m = c(c - \hat{x}_1) = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\begin{aligned} \hat{x} &= (\hat{x}_1,\ \hat{x}_2,\ \ldots,\ \hat{x}_k,\ \ldots,\ \hat{x}_{n-1},\ \hat{x}_n) \\ &= \frac{c}{F_{2n+1}}(F_{2n-1},\ F_{2n-3},\ \ldots, F_{2n-2k+1},\ \ldots,\ F_3,\ F_1). \end{aligned}$$

*The dual* ($D_1$) *has a maximum value* $M = c\mu_1^* = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\begin{aligned} \mu^* &= (\mu_1^*,\ \mu_2^*,\ \ldots,\ \mu_k^*,\ \ldots,\ \mu_{n-1}^*,\ \mu_n^*) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-2},\ \ldots, F_{2n-2k},\ \ldots,\ F_4,\ F_2). \end{aligned}$$

Let $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ be any two sequences of real number with $x_0 = c$. Then a complementary identity

$$(\mathrm{C}_1) \quad c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1} - x_k)\mu_k + x_k(\mu_k - \mu_{k+1})] + (x_{n-1} - x_n)\mu_n + x_n\mu_n$$

holds true.

Let us define two sequences $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{aligned} & y_1 = c - x_1,\; y_2 = x_1,\; y_3 = x_1 - x_2,\; y_4 = x_2,\; y_5 = x_2 - x_3 \\ & \quad \ldots,\; y_{2n-2} = x_{n-1},\; y_{2n-1} = x_{n-1} - x_n,\; y_{2n} = x_n \\ & \nu_1 = \mu_1,\; \nu_2 = \mu_1 - \mu_2,\; \nu_3 = \mu_2,\; \nu_4 = \mu_2 - \mu_3,\; \nu_5 = \mu_3 \\ & \quad \ldots,\; \nu_{2n-2} = \mu_{n-1} - \mu_n,\; \nu_{2n-1} = \mu_n,\; \nu_{2n} = \mu_n \end{aligned} \tag{4}$$

, respectively. Then an identity

$$(\mathrm{C}_1^*) \qquad c\nu_1 \;=\; \sum_{k=1}^{2n} y_k\nu_k$$

holds *under a constraint* – a linear system of $4n$-variables $(y, \nu)$ on $2n$-equations – :

$$(\mathrm{C}^{y\nu}) \qquad \begin{aligned} c &= y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\ y_2 &= y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\ &\vdots & &\vdots \\ y_{2n-4} &= y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &= y_{2n-1} + y_{2n} & \nu_{2n-1} &= \nu_{2n}. \end{aligned}$$

An equality $(\mathrm{C}_1^*)$ with constraint $(\mathrm{C}^{y\nu})$ is called a $2n$-variable *conditional complementarity.* This is simply written as $(\mathrm{C}_1^*)$ under $(\mathrm{C}^{y\nu})$.

Now let $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ satisfy $(\mathrm{C}_1^{y\nu})$. Then an elementary inequality with equality

$$2xy \;\le\; x^2 + y^2 \quad \text{on } R^2 \;;\; x = y \tag{5}$$

yields

$$2c\nu_1 \;\le\; \sum_{k=1}^{2n}(y_k^2 + \nu_k^2).$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n}\nu_k^2 \;\le\; \sum_{k=1}^{2n} y_k^2.$$

The sign of equality holds iff

$$(\mathrm{EC}_1) \quad y_k = \nu_k \quad 1 \le k \le 2n. \tag{6}$$

Hence we have a pair of conditional optimization problems:

$$(\mathrm{P}_1^*) \quad \begin{array}{rl} \text{minimize} & y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 + y_{2n}^2 \\ \text{subject to} & (1) \ \ y_1 + y_2 = c \\ & (2) \ \ y_3 + y_4 = y_2 \\ & \vdots \\ & (n-1) \ \ y_{2n-3} + y_{2n-2} = y_{2n-4} \\ & (n) \ \ y_{2n-1} + y_{2n} = y_{2n-2} \\ & (n+1) \ \ y \in R^{2n} \end{array}$$

$$(\mathrm{D}_1^*) \quad \begin{array}{rl} \text{Maximize} & 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 + \nu_{2n}^2) \\ \text{subject to} & [1] \ \ \nu_2 + \nu_3 = \nu_1 \\ & [2] \ \ \nu_4 + \nu_5 = \nu_3 \\ & \vdots \\ & [n-1] \ \ \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ & [n] \ \ \nu_{2n} = \nu_{2n-1} \\ & [n+1] \ \ \nu \in R^{2n}. \end{array}$$

Let $(\mathrm{AC}_1)$ be an *augmentation* of the system $(\mathrm{C}_1^{y\nu})$ with the additional equality condition $(\mathrm{EC}_1)$:

$$(\mathrm{AC}_1) \quad \begin{array}{rclrcl} c &=& y_1 + y_2 & \nu_1 &=& \nu_2 + \nu_3 \\ y_2 &=& y_3 + y_4 & \nu_3 &=& \nu_4 + \nu_5 \\ &\vdots& & &\vdots& \\ y_{2n-4} &=& y_{2n-3} + y_{2n-2} & \nu_{2n-3} &=& \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &=& y_{2n-1} + y_{2n} & \nu_{2n-1} &=& \nu_{2n} \end{array}$$

$$y_k = \nu_k \quad 1 \le k \le 2n.$$

The linear system $(\mathrm{AC}_1)$ is of $4n$-variables on $4n$-equations. Let $(y, \nu)$ satisfy $(\mathrm{AC}_1)$. Then both sides become a common value with five expressions:

$$(5\mathrm{V}_1) \quad \begin{aligned} & y_1^2 + y_2^2 + \cdots + y_{2n}^2 \\ &= cy_1 \\ &= 2c\nu_1 - (\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2) \\ &= \nu_1^2 + \nu_2^2 + \cdots + \nu_{2n}^2 \\ &= c\nu_1. \end{aligned}$$

The system ($\mathrm{AC}_1$) has indeed a unique common solution:

$$y = (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-1},\ y_{2n})$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1),$$
$$\nu = (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-1},\ \nu_{2n})$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1).$$

**Theorem 2** *The primal* ($\mathrm{P}_1$) *has a minimum value* $m = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\hat{y} = (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_k,\ \ldots,\ \hat{y}_{2n-1},\ \hat{y}_{2n})$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1).$$

*The dual* ($\mathrm{D}_1$) *has a maximum value* $M = \dfrac{F_{2n}}{F_{2n+1}}c^2$ *at a path*

$$\nu^* = (\nu_1^*,\ \nu_2^*,\ \ldots,\ \nu_k^*,\ \ldots,\ \nu_{2n-1}^*,\ \nu_{2n}^*)$$
$$= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_2,\ F_1).$$

*Both optimal solutions (point and value) are identical:*

$$\hat{x} = \mu^*, \quad m = M.$$

*Further both are Fibonacci.*

Thus *Fibonacci Identical Duality* (FID) holds between ($\mathrm{P}_1$) and ($\mathrm{D}_1$) [15–17].

We remark that the $2n$*-variable* pair is a transliteration from $n$*-variable* one ($\mathrm{P}_1$), ($\mathrm{D}_1$).

## 2 Identical Dual 2

Next we consider the following pair

$$(\mathrm{P}_m)\quad \begin{array}{ll} \text{minimize} & \displaystyle\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + \frac{F_{m+1}}{F_m}x_n^2 \\ \text{subject to} & \text{(i)}\ \ x \in R^n, \quad \text{(ii)}\ \ x_0 = c \end{array}$$

$$(\mathrm{D}_m)\quad \begin{array}{ll} \text{Maximize} & \displaystyle 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] - \mu_n^2 - \frac{F_m}{F_{m+1}}\mu_n^2 \\ \text{subject to} & \text{(i)}\ \ \mu \in R^n, \end{array}$$

where $\{F_n\}$ is the *Fibonacci sequence*. The identity ($\mathrm{C}_1$) is enhanced to

$$(\mathrm{C}_m)\ c\mu_1 \;=\; \sum_{k=1}^{n-1}[(x_{k-1}-x_k)\mu_k + x_k(\mu_k-\mu_{k+1})] + (x_{n-1}-x_n)\mu_n + \sqrt{\frac{F_{m+1}}{F_m}}\,x_n\sqrt{\frac{F_m}{F_{m+1}}}\,\mu_n$$

where $m \geq 1$. This identity is called $F_m$-*complementary*.

Furthermore the complementary identity implies that

$$(\mathrm{QI}_m)\quad \begin{aligned} &\sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + \frac{F_{m+1}}{F_m}x_n^2 \\ &\qquad + \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k-\mu_{k+1})^2\right] + \mu_n^2 + \frac{F_m}{F_{m+1}}\mu_n^2 - 2c\mu_1 \\ &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2 + (x_k-\mu_k+\mu_{k+1})^2\right] \\ &\qquad + (x_{n-1}-x_n-\mu_n)^2 + \left(\sqrt{\frac{F_{m+1}}{F_m}}\,x_n - \sqrt{\frac{F_m}{F_{m+1}}}\,\mu_n\right)^2. \end{aligned}$$

This is an identity on $R^n\times R^n$, which is called *quadratic*.

Now we define three functions $f,\ g : R^n \to R^1,\ \ h : R^n\times R^n \to R^1$ by

$$\begin{aligned} f(x) &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k)^2 + x_k^2\right] + (x_{n-1}-x_n)^2 + \frac{F_{m+1}}{F_m}x_n^2 \\ g(\mu) &= 2c\mu_1 - \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k-\mu_{k+1})^2\right] - \mu_n^2 - \frac{F_m}{F_{m+1}}\mu_n^2 \\ h(x,\mu) &= \sum_{k=1}^{n-1}\left[(x_{k-1}-x_k-\mu_k)^2 + (x_k-\mu_k+\mu_{k+1})^2\right] \\ &\qquad + (x_{n-1}-x_n-\mu_n)^2 + \left(\sqrt{\frac{F_{m+1}}{F_m}}\,x_n - \sqrt{\frac{F_m}{F_{m+1}}}\,\mu_n\right)^2. \end{aligned}$$

They are called *primal*, *dual* and *gap* functions, respectively. Then ($\mathrm{QI}_m$) is summarized as follows.

**Lemma 4** *It holds that*

$$(\mathrm{QI}_m)\qquad f(x) - g(\mu) \;=\; h(x,\mu).$$

We consider a linear system of $2n$-equation on $2n$-variable $(x,\ \mu)$:

$$(\mathrm{EC}_m)\qquad \begin{aligned} c - x_1 &= \mu_1, & x_1 &= \mu_1 - \mu_2 & & \\ x_{k-1} - x_k &= \mu_k, & x_k &= \mu_k - \mu_{k+1} & & 2 \leq k \leq n-1 \\ x_{n-1} - x_n &= \mu_n, & \frac{F_{m+1}}{F_m}x_n &= \mu_n. & & \end{aligned}$$

**Lemma 5** *It holds that*

(i) $h(x,\mu) \geq 0 \quad \forall (x,\,\mu) \in R^n \times R^n$

(ii) $h(x,\mu) = 0 \Longleftrightarrow (x,\mu)$ *satisfies* $(\mathrm{EC}_m)$.

**Corollary 2** *It holds that*

(i) $f(x) \geq g(\mu) \quad \forall (x,\,\mu) \in R^n \times R^n$

(ii) $f(x) = g(\mu) \Longleftrightarrow (x,\mu)$ *satisfies* $(\mathrm{EC}_m)$.

From Corollary 2, it turns out that $(\mathrm{P}_m)$ and $(\mathrm{D}_m)$ are dual to each other, and $(\mathrm{EC}_m)$ is an equality condition. The equality condition $(\mathrm{EC}_m)$ is a linear system of $2n$-equations on $2n$-variables $(x,\mu)$.

**Lemma 6** *Let* $(x,\mu)$ *satisfy* $(\mathrm{EC}_m)$. *Then both sides become a common value with five expressions:*

$$(5\mathrm{V}_m) \qquad \begin{aligned} f(x) &= c(c - x_1) = g(\mu) \\ &= \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 + \frac{F_m}{F_{m+1}}\mu_n^2 = c\mu_1. \end{aligned}$$

*The primal* $(\mathrm{P}_m)$ *has a minimum value*

$$m = f(x) = c(c - x_1)$$

*at* $x$, *while the dual* $(\mathrm{D}_m)$ *has a maximum value*

$$M = g(\mu) = \sum_{k=1}^{n-1}\left[\mu_k^2 + (\mu_k - \mu_{k+1})^2\right] + \mu_n^2 + \frac{F_m}{F_{m+1}}\mu_n^2 = c\mu_1$$

*at* $\mu$.

**Lemma 7** $(\mathrm{EC}_m)$ *has indeed a unique solution:*

$$\begin{aligned} x &= (x_1,\ x_2,\ \ldots,\ x_k,\ \ldots,\ x_{n-1},\ x_n) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-2},\ F_{m+2n-4},\ \ldots, F_{m+2n-2k},\ \ldots,\ F_{m+2},\ F_m), \end{aligned} \tag{7}$$

$$\begin{aligned} \mu &= (\mu_1,\ \mu_2,\ \ldots,\ \mu_k,\ \ldots,\ \mu_{n-1},\ \mu_n) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1},\ F_{m+2n-3},\ \ldots, F_{m+2n-2k+1},\ \ldots,\ F_{m+3},\ F_{m+1}). \end{aligned} \tag{8}$$

*Proof.* From $(\mathrm{EC}_m)$, we have a pair of linear systems of $n$-variable on $n$-equation:

$$(\mathrm{EQ}_m) \qquad \begin{aligned} c &= 3x_1 - x_2 & c &= 2\mu_1 - \mu_2 \\ x_1 &= 3x_2 - x_3 & \mu_1 &= 3\mu_2 - \mu_3 \\ &\ \vdots & &\ \vdots \\ x_{n-2} &= 3x_{n-1} - x_n & \mu_{n-2} &= 3\mu_{n-1} - \mu_n \\ x_{n-1} &= \frac{F_{m+2}}{F_m}x_n & \mu_{n-1} &= \frac{F_{m+3}}{F_{m+1}}\mu_n. \end{aligned}$$

The left system has a solution $x$ in (7), while the right has a solution $\mu$ in (8). $\square$

Let us define two sequences $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ from $x = \{x_k\}_0^n$, $\mu = \{\mu_k\}_1^n$ through

$$\begin{aligned} &y_1 = c - x_1,\ y_2 = x_1,\ y_3 = x_1 - x_2,\ y_4 = x_2,\ y_5 = x_2 - x_3 \\ &\quad \ldots,\ y_{2n-2} = x_{n-1},\ y_{2n-1} = x_{n-1} - x_n,\ y_{2n} = x_n \\ &\nu_1 = \mu_1,\ \nu_2 = \mu_1 - \mu_2,\ \nu_3 = \mu_2,\ \nu_4 = \mu_2 - \mu_3,\ \nu_5 = \mu_3 \\ &\quad \ldots,\ \nu_{2n-2} = \mu_{n-1} - \mu_n,\ \nu_{2n-1} = \mu_n,\ \nu_{2n} = \mu_n \end{aligned} \tag{9}$$

, respectively. Then an identity

$$(\mathrm{C}_m^*) \qquad c\nu_1 \;=\; \sum_{k=1}^{2n-1} y_k\nu_k + \sqrt{\frac{F_{m+1}}{F_m}}\, y_{2n}\sqrt{\frac{F_m}{F_{m+1}}}\, \nu_{2n}$$

holds *under a constraint* – a linear system of $4n$-variables $(y, \nu)$ on $2n$-equations – :

$$(\mathrm{C}^{y\nu}) \qquad \begin{aligned} c &= y_1 + y_2 & \nu_1 &= \nu_2 + \nu_3 \\ y_2 &= y_3 + y_4 & \nu_3 &= \nu_4 + \nu_5 \\ &\vdots & &\vdots \\ y_{2n-4} &= y_{2n-3} + y_{2n-2} & \nu_{2n-3} &= \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &= y_{2n-1} + y_{2n} & \nu_{2n-1} &= \nu_{2n}. \end{aligned}$$

An equality $(\mathrm{C}_m^*)$ with constraint $(\mathrm{C}^{y\nu})$ is called a $2n$-variable *conditional complementarity*. This is simply written as $(\mathrm{C}_m^*)$ under $(\mathrm{C}^{y\nu})$.

Now let $y = \{y_k\}_1^{2n}$, $\nu = \{\nu_k\}_1^{2n}$ satisfy $(\mathrm{C}^{y\nu})$. Then the elementary inequality with equality yields

$$2c\nu_1 \;\le\; \sum_{k=1}^{2n-1}(y_k^2 + \nu_k^2) + \frac{F_{m+1}}{F_m}y_{2n}^2 + \frac{F_m}{F_{m+1}}\nu_{2n}^2.$$

Thus we have an inequality

$$2c\nu_1 - \sum_{k=1}^{2n-1}\nu_k^2 - \frac{F_m}{F_{m+1}}\nu_{2n}^2 \;\le\; \sum_{k=1}^{2n-1} y_k^2 + \frac{F_{m+1}}{F_m}y_{2n}^2.$$

The sign of equality holds iff

$$(\mathrm{EC}_m) \qquad y_k \;=\; \nu_k \quad 1 \le k \le 2n-1, \quad F_{m+1}y_{2n} \;=\; F_m\nu_{2n}. \tag{10}$$

We remark that an equivalence

$$\sqrt{\frac{F_{m+1}}{F_m}}\, y_{2n} \;=\; \sqrt{\frac{F_m}{F_{m+1}}}\, \nu_{2n} \Longleftrightarrow \frac{F_{m+1}}{F_m}y_{2n} \;=\; \nu_{2n}$$

yields the last equality.

Hence we have a pair of conditional optimization problems:

$$(\mathrm{P}^*_m)\qquad \begin{array}{rl} \text{minimize} & y_1^2 + y_2^2 + \cdots + y_{2n-1}^2 + \dfrac{F_{m+1}}{F_m} y_{2n}^2 \\ \text{subject to} & (1)\ \ y_1 + y_2 = c \\ & (2)\ \ y_3 + y_4 = y_2 \\ & \vdots \\ & (n-1)\ \ y_{2n-3} + y_{2n-2} = y_{2n-4} \\ & (n)\ \ y_{2n-1} + y_{2n} = y_{2n-2} \\ & (n+1)\ \ y \in R^{2n} \end{array}$$

$$(\mathrm{D}^*_m)\qquad \begin{array}{rl} \text{Maximize} & 2c\nu_1 - \left(\nu_1^2 + \nu_2^2 + \cdots + \nu_{2n-1}^2 + \dfrac{F_m}{F_{m+1}} \nu_{2n}^2\right) \\ \text{subject to} & [1]\ \ \nu_2 + \nu_3 = \nu_1 \\ & [2]\ \ \nu_4 + \nu_5 = \nu_3 \\ & \vdots \\ & [n-1]\ \ \nu_{2n-2} + \nu_{2n-1} = \nu_{2n-3} \\ & [n]\ \ \nu_{2n} = \nu_{2n-1} \\ & [n+1]\ \ \nu \in R^{2n}. \end{array}$$

Let $(\mathrm{AC}_m)$ be an *augmentation* of the system $(\mathrm{C}^{y\nu}_m)$ with the additional equality condition $(\mathrm{EC}_m)$:

$$(\mathrm{AC}_m)\qquad \begin{array}{rclcrcl} c &=& y_1 + y_2 & & \nu_1 &=& \nu_2 + \nu_3 \\ y_2 &=& y_3 + y_4 & & \nu_3 &=& \nu_4 + \nu_5 \\ &\vdots& & & &\vdots& \\ y_{2n-4} &=& y_{2n-3} + y_{2n-2} & & \nu_{2n-3} &=& \nu_{2n-2} + \nu_{2n-1} \\ y_{2n-2} &=& y_{2n-1} + y_{2n} & & \nu_{2n-1} &=& \nu_{2n} \end{array}$$

$$y_k = \nu_k \quad 1 \le k \le 2n-1, \quad F_{m+1} y_{2n} = F_m \nu_{2n}.$$

The linear system $(\mathrm{AC}_m)$ is of $4n$-variables on $4n$-equations. Let $(y, \nu)$ satisfy $(\mathrm{AC}_m)$.

The system $(\mathrm{AC}_m)$ has indeed a unique solution:

$$\begin{aligned} & y = (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-2},\ y_{2n-1},\ y_{2n}) \\ = & \frac{c}{F_{m+2n}} (F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_m}), \\ & \nu = (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-2},\ \nu_{2n-1},\ \nu_{2n}) \\ = & \frac{c}{F_{m+2n}} (F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_{m+1}}). \end{aligned}$$

Note that only the last elements are different, as underlined. However, in Case $m = 1$, both solutions are identical:

$$\begin{aligned} y &= (y_1,\ y_2,\ \ldots,\ y_k,\ \ldots,\ y_{2n-2},\ y_{2n-1},\ y_{2n}) \\ = \nu &= (\nu_1,\ \nu_2,\ \ldots,\ \nu_k,\ \ldots,\ \nu_{2n-2},\ \nu_{2n-1},\ \nu_{2n}) \\ &= \frac{c}{F_{2n+1}}(F_{2n},\ F_{2n-1},\ \ldots, F_{2n-k+1},\ \ldots,\ F_3,\ F_2,\ \underline{F_1}). \end{aligned}$$

We note that $F_2 = F_1 = 1$.

**Theorem 3** *The primal* $(\mathrm{P}_m)$ *has a minimum value* $m = \dfrac{F_{m+2n-1}}{F_{m+2n}}c^2$ *at a path*

$$\begin{aligned} \hat{y} &= (\hat{y}_1,\ \hat{y}_2,\ \ldots,\ \hat{y}_k,\ \ldots,\ \hat{y}_{2n-2},\ \hat{y}_{2n-1},\ \hat{y}_{2n}) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_m}). \end{aligned}$$

*The dual* $(\mathrm{D}_m)$ *has a maximum value* $M = \dfrac{F_{m+2n-1}}{F_{m+2n}}c^2$ *at a path*

$$\begin{aligned} \nu^* &= (\nu_1^*,\ \nu_2^*,\ \ldots,\ \nu_k^*,\ \ldots,\ \nu_{2n-2}^*,\ \nu_{2n-1}^*,\ \nu_{2n}^*) \\ &= \frac{c}{F_{m+2n}}(F_{m+2n-1},\ F_{m+2n-2},\ \ldots, F_{m+2n-k},\ \ldots,\ F_{m+2},\ F_{m+1},\ \underline{F_{m+1}}). \end{aligned}$$

*Both optimal solutions (point and value) are identical except for the last element:*

$$\hat{y}_k = \nu_k^* \quad 1 \le k \le 2n-1, \quad m = M.$$

*Further both are Fibonacci:*

$$\hat{y}_k = \nu_k^* = \frac{F_{m+2n-k}}{F_{m+2n}}c \quad 1 \le k \le 2n-1, \quad \hat{y}_{2n} = \frac{F_m}{F_{m+2n}}c, \ \nu_{2n}^* = \frac{F_{m+1}}{F_{m+2n}}c$$
$$m = M = \frac{F_{m+2n-1}}{F_{m+2n}}c^2.$$

Thus *Fibonacci Identical* [1] *Duality* (FID) holds between $(\mathrm{P}_m)$ and $(\mathrm{D}_m)$ [15–17].

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[1] *Identical* means *identical except for the last element.*

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