On quantum system with time-periodic magnetic fields

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Abstract

Spectrum and scattering theory for the quantum system with time-periodic magnetic field is very important problem not only mathematically but also physically. Nevertheless as for such problems, only few models were considered and many open problems are still remaining. Moreover, in such model, there are many interests in the advanced studies such like nonlinear analysis, resonances and so on. Since the time-periodic magnetic field has the expected to developments in some research fields, we summarize the obtained result and introduce some advances studies.

1 Introduction

The Hamiltonian for Schrödinger operator with time-periodic magnetic fields $H_0(t)$ is written as

$$H_0(t) = \frac{1}{2m}\left(p_1 + \frac{qB(t)}{2}x_2\right)^2 + \frac{1}{2m}\left(p_2 - \frac{qB(t)}{2}x_1\right)^2,$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $p = (p_1, p_2) = -i\nabla$, $q \neq 0$, $m > 0$ are position, momentum, charge and mass of a particle, respectively. Magnetic field is $\mathbb{B}(t) = (0, 0, B(t))$ and $B(t)$ denotes the intense of magnetic field in t. In this paper, we assume the periodic condition $B(t + T) = B(t)$ on magnetic field. In such case, the quantum scattering theory were considered by Korotyaev

[12] and Adachi-Kawamoto [1]. As far as we know, except for these two papers, there are no results associated to the quantum scattering for time-periodic magnetic fields. The key approach of them is to deduce the limiting absorption principle for $\hat{H}_0$, the Floquet Hamiltonian generated by $H_0(t)$, and extend this result to perturbed Hamiltonian $\hat{H} = H_0(t) + V(t)$ by employing the stationary scattering theory due to e.g., Kato-Kuroda [6]. However, only for this approach, one can not prove the non-existence of singular continuous spectrum of $\hat{H}$, and hence, Kawamoto [7] proved the absence of the singular spectrum through by proving the Mourre theory.

In this paper, we let $H(t) = H_0(t) + V(t)$, and assume the following log-decay condition on the potential $V(t,x)$,

Assumption 1.1 *Potential* $V \in L^\infty(\mathbb{R}\,; C^2(\mathbb{R}^2))$ *satisfies* $V(t+T,x) = V(t,x)$ *and*

$$|\partial^\alpha V(t,x)| \leq C_\alpha \left\langle x \right\rangle^{-|\alpha|} (\log(1+|x|))^{-\rho} \tag{1}$$

for some positive constant ρ, C_α *and for all multi-index* $\alpha \in \mathbb{N}^2$, *where* $\langle \cdot \rangle = (1 + \cdot^2)^{1/2}$.

We let $U_0(t,s)$ and $U(t,s)$ are propagators for $H_0(t)$ and $H(t)$, respectively. The aim of this paper is to consider the spectrum and scattering theory for this system. In order to consider such issue, the following lemma acts very important role;

Lemma 1.2 (e.g., Kitada-Yajima [9] and Enss-Veselić [3])

$$L^2(\mathbb{R}^2) = L_{\mathrm{c}}(U(T,0)) \oplus L_{\mathrm{p}}(U(T,0)),$$

where $L_{\mathrm{c}}(U(T,0)) \subset L^2(\mathbb{R}^2)$ *and* $L_{\mathrm{p}}(U(T,0)) \subset L^2(\mathbb{R}^2)$ *are the subspace of continuous spectrum of* $U(T,0)$ *and the subspace of pure point spectrum of* $U(T,0)$, *respectively.*

By this Lemma, in order to consider the spectrum and scattering theory for this system, it is enough to investigate the properties of spectrum of $U(T,0)$. However, such operator is unitary and complex-valued, and it seems difficult to analyze the spectrum of $U(T,0)$ directly. To get over this difficulties, Howland [10] and Yajima [14] considered alternative approach with using Floquet Hamiltonian. In order to introduce Floquet Hamiltonian, we set the energy space by $\mathscr{K} = L^2(\mathbb{T}\,; L^2(\mathbb{R}^2))$ with $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$, and for $f \in \mathscr{K}$ define

$$(\mathcal{L}_{0,\sigma} f)(t) = U_0(t, t-\sigma) f(t-\sigma),$$
$$(\mathcal{L}_\sigma f)(t) = U(t, t-\sigma) f(t-\sigma).$$

Here we notice that for $\sigma_1, \sigma_2 \in \mathbb{R}$,

$$\begin{aligned}(\mathcal{L}_{0,\sigma_1}(\mathcal{L}_{0,\sigma_2} f))(t) &= \mathcal{L}_{0,\sigma_1}(U_0(t, t-\sigma_2) f(t-\sigma_2))(t) \\ &= U_0(t, t-\sigma_1) U_0(t-\sigma_1, t-\sigma_1-\sigma_2) f(t-\sigma_1-\sigma_2) \\ &= U_0(t, t-\sigma_1-\sigma_2) f(t-\sigma_1-\sigma_2) \\ &= (\mathcal{L}_{0,\sigma_1+\sigma_2} f)(t),\end{aligned}$$

and which means $\mathcal{L}_{0,\sigma}$ (resp. $\mathcal{L}_{\sigma}$) is the 1-parameter strongly continuous unitary group on $\mathscr{K}$. Hence Stone's theorem leads there exist selfadjoint operators $\hat{H}_0$ and $\hat{H}$ such that

$$(\mathcal{L}_{0,\sigma}\phi)(t) = e^{-i\sigma \hat{H}_0}\phi(t), \quad (\mathcal{L}_{\sigma}\phi)(t) = e^{-i\sigma \hat{H}}\phi(t).$$

We call $\hat{H}_0$ and $\hat{H}$ *Floquet operator generated by* $H_0(t)$ *and* $H(t)$, respectively. Let $-i\partial_t$ be the derivative operator in t with boundary condition

$$\left\{\psi(t) \in L^2(\mathbb{R}^2) \,\middle|\, \psi(t) \text{ and } (\partial_t\psi)(t) \text{ are absolutely continuous and } \psi(0) = \psi(T)\right\}.$$

Then it is seen (e.g., by Møller [13]) that

$$\hat{H}_0 = -i\partial_t + H_0(t), \quad \hat{H} = \hat{H}_0 + V(t).$$

To consider the spectrum of $U(T, 0)$ the following lemmas are very useful;

Lemma 1.3

$$\mathscr{K} = \mathscr{K}_{\mathrm{c}}(\hat{H}) \oplus \mathscr{K}_{\mathrm{p}}(\hat{H}),$$

where $\mathscr{K}_{\mathrm{c}}(\hat{H}) \subset \mathscr{K}$ *and* $\mathscr{K}_{\mathrm{c}}(\hat{H}) \subset \mathscr{K}$ *are the subspace of continuous spectrum of* $\hat{H}$ *and the subspace of pure point spectrum of* $\hat{H}$*, respectively.*

Lemma 1.4 (e.g., Yajima [15]) *Let* $\hat{H}f = \lambda f$*. Then* $f = f(t)$ *is* $L^2(\mathbb{R}^2)$*-valued continuous function and satisfies* $f(t) = e^{i\lambda t}U(t, 0)f(0)$*. In particular,* $U(T, 0)f(0) = e^{-i\lambda T}f(0)$*. Conversely if* φ *satisfies* $U(T, 0)\varphi = e^{-i\lambda T}\varphi$*, then by letting* $f(t) = e^{i\lambda T}U(t, 0)\varphi$*, we have* $f \in \mathscr{D}(\hat{H})$ *and* $\hat{H}f = \lambda f$*.*

The following lemma is so called *Howland-Yajima method*;

Lemma 1.5 *If the wave operators in the sense of the Floquet Hamiltonian*

$$\mathcal{W}^{\pm} = \mathrm{s}-\lim_{\sigma \to \pm\infty} e^{it\hat{H}} e^{-it\hat{H}_0}$$

exist and complete that is

$$\mathrm{Ran}\left(\mathcal{W}^{\pm}\right) = \mathscr{K}_{\mathrm{ac}}(\hat{H}),$$

where $\mathscr{K}_{\mathrm{ac}}(\hat{H}) \subset \mathscr{K}$ *indicates the subspace of the absolutely continuous spectrum of* $\hat{H}$*. Moreover, the usual wave operators*

$$W^{\pm} = \mathrm{s-}\lim_{t\to\pm\infty} U(t,0)^{*} U_0(t,0)$$

exist. Then the usual wave operators $W^{\pm}$ *are complete that is*

$$\mathrm{Ran}\left(W^{\pm}\right) = L_{\mathrm{ac}}(U(T,0)),$$

$L_{\mathrm{ac}}(U(T,0)) \subset L^2(\mathbb{R}^2)$ *indicates the subspace of the absolutely continuous spectrum of* $U(T,0)$

Thanks to this lemma, one can prove the completeness of $W^{\pm}$ by proving the completeness of $\mathcal{W}^{\pm}$, alternatively.

Constant magnetic field $\mathbb{B} = (0,0,B)$ make the classical trajectory of a quantum particle on the plane $\mathbb{R}^2$ which perpendicular to magnetic filed the circular orbit, and the particle is trapped by constant magnetic field. On the other hand, we oscillate the magnetic field periodically in the time, the particle is not always trapped and under the some suitable condition, scattering states appear, which is characterized by $L_{\mathrm{ac}}(U(T,0))$ or $\mathscr{K}_{\mathrm{ac}}(\hat{H})$. The particle is trapped or not is determined by so-called *discriminant of Hill's equation* D; If $D^2 < 4$, we have $L^2(\mathbb{R}^2) = L_{\mathrm{p}}(U(T,0))$, and if $D^2 \geq 4$, at least one can prove $L_{\mathrm{p}}(U_0(T,0)) = \emptyset$, i.e., $L^2(\mathbb{R}^2) = L_{\mathrm{c}}(U_0(T,0))$. Here we define Hill's equation for this system as follows

$$\zeta_j''(t) + \left(\frac{qB(t)}{2m}\right)^2 \zeta_j(t) = 0, \quad \begin{cases} \zeta_1(0) = 1, \\ \zeta_1'(0) = 0, \end{cases} \quad \begin{cases} \zeta_2(0) = 0, \\ \zeta_2'(0) = 1, \end{cases}$$

and we also define the *discriminant* of Hill's equation as follows

$$D = \zeta_1(T) + \zeta_2'(T).$$

For the special case in the case of $D^2 > 2$, the solution of Hill's equation can be represented as

$$\zeta_1(t) = e^{\lambda t}\chi_1(t), \quad \zeta_2(t) = e^{-\lambda t}\chi_2(t), \tag{2}$$

where χ_1 and χ_2 are periodic or anti-periodic functions, respectively. By using such representation, [12] proved the existence of wave operators and these completeness with the potentials V which satisfies slowly decaying condition, that is, for some $\rho > 0$,

$$|V(t,x)| \le C(1+|x|)^{-\rho}. \tag{3}$$

In the paper of [12], the approach due to [14] (see also Kato [5]) was employed, which uses the following representation of resolvent of $\hat{H}_0$; Let $z \in \mathbb{C}\backslash\mathbb{R}_-$, $\phi \in \mathscr{K}$ and $f \in L^2(\mathbb{T}; C_0^\infty(\mathbb{R}^2))$, and define $X_f(z)\phi := f(\hat{H}_0 - z)^{-1} f\phi$

$$\begin{aligned}(X_f(z)\phi)(t,x) = if(t,x) \sum_{N=1}^{\infty} \int_0^T e^{i(t+NT-s)z} U_0(t+NT,s)(f\phi)(s)ds \\ + if(t,x) \int_0^t e^{i(t-s)z} U_0(t,s)(f\phi)(s)ds.\end{aligned}$$

In the case of time-periodic magnetic fields, in order to use this approach, the integral kernel of $U_0(t,s)$ was found by [12], [1] and [7], and by employing the representation of integral kernel of [7], we get

$$\begin{aligned}&(fU_0(\tau,s)f\phi)(\tau,x) \\ &= \left(\frac{2\pi}{m|\Gamma_1(\tau,s)|}\right) \frac{m^2}{2(\pi i)^2 \zeta_2(\tau)\zeta_2(s)} e^{i(\Omega(\tau)-\Omega(s))L} e^{-ia(\tau)x^2} f(t,x) \int e^{\Gamma_2(\tau,s,x,y)}(g)(s,y)dy,\end{aligned}$$

where $g(s,y) = e^{-ia(s)y^2} f(s,y)\phi(s,y)$ with $a(s) = m(1-\zeta_2'(s))/(2\zeta_2(s))$,

$$\Gamma_1(\tau,s) = \zeta_1(s)/\zeta_2(s) - \zeta_1(\tau)/\zeta_2(\tau)$$

and

$$\Gamma_2(\tau,s,x,y) = \frac{i}{2m\Gamma_1(\tau,s)} \left(\frac{mx}{\zeta_2(\tau)} - \frac{my}{\zeta_2(s)}\right)^2 + \frac{mi}{2}\left(\frac{x^2}{\zeta_2(\tau)} - \frac{y^2}{\zeta_2(s)}\right).$$

Hence the following $L^p - L^q$ estimate can be obtained

$$\|U_0(t,s)\phi\|_{L^p(\mathbb{R}^2)} \le C\,|\zeta_1(t)\zeta_2(s) - \zeta_1(s)\zeta_2(t)|^{-(1/q-1/p)} \|\phi\|_{L^q(\mathbb{R}^2)} \tag{4}$$

for $1/p + 1/q = 1$ with $1 \le q \le 2 \le p \le \infty$, which was firstly obtained by [12] and extended by [7] for more general magnetic fields. However, dealing with the right-hand-side of (4) for general time-periodic fields is very complicated and hence to this easier, [12] assumed the condition (2). Under this condition, one can prove

$$\int_1^\infty \|VU_0(t,0)\phi\|_{L^2(\mathbb{R}^2)}\, dt \le C\,\|V\|_{L^p(\mathbb{R}^2)} \|\phi\|_{L^{q'}(\mathbb{R}^2)} \sum_{N\in\mathbb{Z}} e^{-2|\lambda n|/p}$$

with $1/p + 1/q = 1/2$ and $1/q + 1/q' = 1$ (but even the case of (2), to deduce which demands long and complicated calculations), and this inequality indicates the wave operators exist under the potential satisfying $\|V\|_{L^p(\mathbb{R}^2)} < \infty$. Since the term $\sum_{N\in\mathbb{Z}} e^{-2|\lambda n|/p}$ is summable for any $2 \leq p < \infty$, one can take p enough large. This is the reason why one can prove the existence and completeness of wave operators for the weak decaying potentials such like (3).

After in [1], under the pulsed condition of $B(t)$, the scattering theory was considered. In this case, by the virtue of the pulsed condition, we can obtain the explicit representation of $\zeta_1(t)$ and $\zeta_2(t)$, and by using this representation, asymptotic completeness was proven under the only two conditions that $D^2 > 4$ and $\zeta_2(T) \neq 0$ but with pulsed condition (this condition includes not only the model of [12] but also more generalized model but with pulsed condition).

In papers [12] and [1], to prove the absence of singular spectrum of $\hat{H}$ is difficult for some technical reasons. In the current approaches, the well-used approach for to prove such issue is to deduce the Mourre estimate. The Mourre theory for time-periodic magnetic field was open problem and [7] proved this. As the corollary, the absence of singular spectrum has been proven. The approach of [7] is firstly reducing the Floquet hamiltonian $\hat{H}_0$ to the more simplified form. We let

$$\alpha(t) := \left(\zeta_2(t)p - \zeta_2'(t)x\right)^2$$

and call *pseudo energy*. Then it follows that

$$\mathbb{D}_{H_0(t)}(\alpha(t)) = 0$$

holds, where $\mathbb{D}_\cdot(\cdot)$ indicates the Heisenberg derivative, and that yields

$$i[\hat{H}_0, \alpha(t)] = 0.$$

Hence $\alpha(t)$ can be regarded as the alternative energy of Floquet energy $\hat{H}_0$. Here we remark that $\alpha(t)$ can be rewrite as

$$\alpha(t) = e^{i\zeta_2'(t)x^2/(2\zeta_2(t))}(\zeta_2(t)p)^2 e^{-i\zeta_2'(t)x^2/(2\zeta_2(t))}.$$

Hence we see that the energy $U_0(T,0)$ can be divided into the form such like $U_0(T,0) = e^{i\alpha x^2} e^{-i(\text{some operator })} e^{-i\alpha x^2}$, and [7] found corresponding result in Lemma 1.4. of [7]. From such decomposition, we obtain the unitary operator $\mathscr{J}_D(t)$ which reduces $\hat{H}_0$ to

$$\mathscr{J}_D(t)^* \hat{H}_0 \mathscr{J}_D(t) = -i\partial_t + A^D\left(p^2 - B^D x^2\right) + C^D L + E^D,$$

where A^D and B^D are constants which satisfy $A^D, C^D \neq 0$, $B^D \neq 0$ if $D^2 > 4$ and $B^D = 0$ if $D^2 = 4$, and $E^D = 0$ if $D > 0$ and $E^D = \pi/T$ if $D < 0$, see [7]. Noting the case where $D^2 > 4$, one can see that the reduced operator can be written as the form

$$\hat{H}_0 = -i\partial_t + \alpha^2 p^2 - \beta^2 x^2 + \gamma L$$

with $\alpha, \beta, \gamma \neq 0$ (here we remove E^D for simplicity), and the operator $\alpha p^2 - \beta x^2$ is called *repulsive operator*, the mathematical aspects for which were considered by Bony-Carles-Häfner-Michel [2]. In this paper, they considered the Hamiltonian

$$H_R := p^2 - x^2 + V_R \tag{5}$$

with $V_R = V_R(x) \in L^\infty(\mathbb{R}^n_x)$ satisfies (1) with $|\alpha| = 0$ and $\rho > 1$, and found that the conjugate operator for H_R is

$$\mathscr{A} = \log \left\langle \frac{p+x}{2} \right\rangle - \log \langle p - x \rangle .$$

Indeed by the simple calculation, the commutator $i[H_R, \mathscr{A}]$ satisfies

$$\begin{aligned} i[H_R, \mathscr{A}] &= \left(\frac{p^2}{1+p^2} + \frac{x^2}{1+x^2} \right) + \mathcal{K} \\ &\geq \frac{1}{2} + \mathcal{K}, \end{aligned}$$

where $\mathcal{K}(H_R + i)^{-1}$ is compact. Thanks to the condition

$$i[-i\partial, \mathscr{A}] = i[L, \mathscr{A}] = 0,$$

one can also obtain the positive commutator

$$i[\hat{H}_0, \tilde{\mathscr{A}}] \geq \frac{1}{2} + \mathcal{K}$$

with

$$\tilde{\mathscr{A}} = \alpha\beta \log \left\langle \frac{1}{2}x + \frac{\alpha}{2\beta}p \right\rangle - \alpha\beta \log \left\langle p - \frac{\beta}{\alpha}x \right\rangle$$

By the virtue of this positive commutator, we find the following Mourre estimate

Theorem 1.6 *Define* $\varphi \in C_0^\infty(\mathbb{R}\backslash\sigma_{\rm pp}(\hat{H}))$. *Suppose Assumption 1.1, and suppose also that* $D^2 > 4$ *and* $\zeta_2(T) \neq 0$. *Then there exists a compact operator* K *such that for all* $\psi \in \mathscr{K}$,

$$\left(i[\hat{H}, \tilde{\mathscr{A}}]\varphi(\hat{H})\psi, \varphi(\hat{H})\psi\right) \geq \frac{1}{2}\left\|\varphi(\hat{H})\psi\right\|^2 + \left(K\varphi(\hat{H})\psi, \varphi(\hat{H})\psi\right)$$

holds.

As the sub consequence of this Theorem, we have the following corollary

Corollary 1.7 *Under the same assumptions as in Theorem 1.6,* $\hat{H}$ *has at most countable pure point spectrum and which singular continuous spectrum is empty.*

This theorem and corollary can be proven by employing the approach due to [7]

2 Proof of Theorem 1.6

Lemma 2.1 *Under the assumption 1.1,* $V(t)(\hat{H}_0 + i)^{-1}$ *is the compact one.*

This lemma can be proven as the direct consequence of Theorem 4.1. of [7]. Indeed, by replacing $f(|x|)$ in [7] to $f(t,x) \in L^2(\mathbb{T}\,; C_0^\infty(\mathbb{R}^2))$, we have the operator

$$(X_f(z)\phi)(t,x) := f(t,x)(\hat{H}_0 - z)^{-1} f(t,x)\phi(t,x),$$

is compact one, where $z \in \mathbb{C}_+$. Hence in order to prove Theorem 1.6, it is sufficient to prove

$$i[V, \tilde{\mathscr{A}}]$$

is a relatively compact operator. On the other hand, by the virtue of §5.1 of [7], we have that $\tilde{\mathscr{A}}$ is written as

$$\tilde{\mathscr{A}} = \log\langle\theta_1(t)x + \theta_2(t)p\rangle - \log\langle\theta_3(t)x + \theta_4(t)p\rangle$$

for bounded and periodic functions $\theta_j(t)$, $j \in \{1, 2, 3, 4\}$. Let $F_R \in C_0^\infty(\mathbb{R})$ with $F_R(s) = 1$ for $|s| \leq R$ and $= 0$ for $|s| \geq 2R$. Define

$$\mathscr{B} := (\theta_1 x + \theta_2 p)^2$$

and

$$L_R(\mathscr{B}) := \frac{1}{2} F_R(\mathscr{B}) \log(1+\mathscr{B})).$$

Then for any fixed R, by the Helffer-Sjöstrand formula, we find there exists $l_R(z) \in C_0^\infty(\mathbb{C})$ such that

$$L_R(\mathscr{B}) = \frac{1}{2\pi i} \int_{\mathbb{C}} \bar{\partial}_z l_R(z)(z-\mathscr{B})^{-1} dz d\bar{z}.$$

Then the commutator $i[V, \log(1+\mathscr{B})^{1/2}]$ formally will be

$$\lim_{R\to\infty} \left(\frac{1}{2\pi} \int_{\mathbb{C}} \bar{\partial}_z l_R(z)(z-\mathscr{B})^{-1}[V,\mathscr{B}](z-\mathscr{B})^{-1} dz d\bar{z} \right).$$

Noting $[V,\mathscr{B}] = 2\theta_2 \left((\theta_1 x + \theta_2 p)\cdot \nabla V + \nabla V \cdot (\theta_1 x + \theta_2 p)\right)$ and $(\log(2+x^2))^{-\delta}\varphi(\hat{H})$ for $\delta > 0$ is the compact operator, we can prove the relative compactness for $i[V, \log(1+\mathscr{B})^{1/2}]$ by proving

$$\left\| \nabla V \cdot (\theta_1 x + \theta_2 p)(z-\mathscr{B})^{-1}(\log(1+x^2))^{\delta} \right\| \leq C|\mathrm{Im} z|^{-3/2},$$

and which can be proven just by imitating the approach of [7].

3 Future works

Thanks to the result in [12] and [1], one can see that $\mathrm{Ran}(W^{\pm}) = L_{\mathrm{ac}}(U(T,0))$ under the potential satisfying (3) and specialized $B(t)$. To extend this result to more general $B(t)$ is not so complicated since the important properties for asymptotic behavior of $\zeta_j(t)$ in order to consider the scattering theory has been obtained by [7], and which implies scattering theory can be considered all $B(t)$ with conditions $D^2 > 4$ and $\zeta_2(T) \neq 0$. Hence as the future works, under the above assumption of $B(t)$, to consider the following issues are very interesting and important;

(I). Absence of embedded eigenvalues for $\hat{H}$. Even the strong decay condition (3), this has not been proven yet. Recently Itakura [11] considered the absence of embedded eigenvalues for generalized hamiltonian including H_R with (1). However, to imitate the approach of [11] is not easy. Because, our model demands to deal with the potential $\mathscr{J}_D(t)^{-1} V \mathscr{J}_D(t)$ which is not just multiplication operator but pseudo differential (like) operator.

(II). Spectral and scattering theory for $\hat{H}$ with (1). This is very interesting problem. However to imitate the approach of [12] and [1] is impossible since for all $1 \leq p < \infty$, $\|V\|_{L^p(\mathbb{R}^2)} = \infty$. Hence it is better to deduce the propagation estimate for $e^{-i\sigma \hat{H}}$ by using Mourre estimate. As for H_R, [2] deduced this but in our model, the situation is completely different.

(III). To investigate the lifespan of resonances. As the same reason as above, it is difficult to consider the resonances for $\hat{H}$. However, even for H_R, this issue has not been considered. Hence, as the first step for this, we need to investigate the resonances for H_R. Probably redefining $H_R = p^2 - c_r x^2 + V_R$, the lifespan of resonances is characterized by c_r and such lifespan is completely different from those for $H = -\Delta + V$.

(IV). Nonlinear analysis . Recently, the nonlinear problem for the generalized equations including time-decaying harmonic oscillators (similar to the time-decaying magnetic fields) was considered by Kawamoto-Muramatsu [8] and the asymptotic behavior of solutions to nonlinear equations was investigated. To imitate the approach of [8] directly is difficult since the case $\zeta_j(t)$ with time-decaying $B(t)$ has no 0-point for all $t \gg 1$ but which is not true for time-periodic case. For the periodic solution, Hani-Thomann [4] considered the similar issue for constant magnetic field. Hence combining the results in [8] and [4], one may investigate the asymptotic behavior of the case of time-periodic magnetic fields.

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