Schrödinger operator with constant magnetic field and slowly varying perturbation on a multidimensional strip region

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1 Introduction

In this paper we are concerned with the magnetic Schrödinger operator with slowly varying external electric potential :

$$H(\epsilon) := D_x^2 + (D_y + \mu x)^2 + V(\epsilon x, \epsilon y), \ D_\nu = \frac{1}{i}\partial_\nu,$$

where $x = (x_1, \cdots, x_d) \in \Lambda_d := \Pi_{j=1}^d [-a_j, a_j]$, $y \in \mathbb{R}^d$, $\mu = (\mu_1, \cdots, \mu_d)$ with $\epsilon, a_j, \mu_j > 0$.

The non-perturbed operator

$$H = D_x^2 + (D_y + \mu x)^2 = \sum_{j=1}^d D_{x_j}^2 + (D_{y_j} + \mu_j x_j)^2$$

is defined on $\mathcal{H}_{\Omega_d}^D := \{u \in H^2(\Omega_d); u|_{\partial\Omega_d} = 0\}$, where $H^2(\Omega_d)$ stands for the second order Sobolev space on $\Omega_d := \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d; -a_j \le x_j \le a_j\} = \Lambda_d \times \mathbb{R}^d$. The Fourier transformation with respect to y reduces the spectral problem of H to an analysis of the eigenvalues $\{e_l(k)\}_{l=0}^\infty$ depending on $k = (k_1, \cdots, k_d)$ of the operator

$$H_0(k) = D_x^2 + (k + \mu x)^2 = \sum_{j=1}^d D_{x_j}^2 + (k_j + \mu_j x_j)^2,$$

on Λ_d with Dirichlet boundary condition.

The spectrum of H is absolutely continuous, and coincides with $[e_0(0), +\infty[$. The points $e_j(0)$ are thresholds in $\sigma(H)$. By the Weyl criterion, the essential spectra of $H(\epsilon)$ and H are the same, and discrete eigenvalues with finite multiplicities can arise in $] - \infty, e_0(0)[$. Moreover, It is reasonable to expect that the electric field creates embedded eigenvalues and resonances on the second sheet. The principal topic of this paper centers around the effect of the slowly varying decaying perturbation $V(\epsilon x, \epsilon y)$ on the non-perturbed operator H. Particular attention will be paid to the asymptotic behavior of the spectrum near the thresholds $e_j(0)$.

The Schrödinger operator with magnetic and electric potentials on a domain Ω of $\mathbb{R}^2$, received considerable attention in the past. The spectrum of the non-perturbed Hamiltonian H on a bounded domain $\Omega \subset \mathbb{R}^2$ were considered by many others. In particular the asymptotic behavior of the bottom of the spectrum of H as μ tends to infinity has been treated for different geometry of Ω (see [23] and the references cited therein). In the case where Ω is the semi-infinite plane or the disk, the WKB approximations of the energies and the eigenfunctions are obtained in [8, 3]. In the case $\Omega = \mathbb{R}^2$, the literature is so voluminous that we cannot possibly describe individual references and hence we primarily refer to the monographs [21, 23] and the references given there.

S. De Bièvre and J. F. Pulé [2] studied the perturbed operator $H(1)$ on the half plane with Dirichlet boundary condition. They showed that the spectrum of $H(1)$ is purely absolutely continuous in a spectral interval of size $\gamma\mu$ (for some $\gamma < 1$) between the Landau levels of the operator H_0. A similar problem has been considered in [5, 6, 7] for $H(1)$ on a strip Ω_1 of $\mathbb{R}^2$. Moreover, Mourre's theory and the spectral shift function near the thresholds $e_j(0)$ were considered in [5].

In [11], the first author uses WKB approximations to study the dynamics and the bottom of the spectrum of the operator $H(\epsilon)$ on Ω_1. With this method it is difficult to recover all the spectrum of $H(\epsilon)$. On the other hand, the multi-dimensional case (i.e., Ω_d with $d > 1$) is more complicated, since the thresholds $e_j(0)$ are in general degenerates when $d > 1$. Our goal in this paper is to give a rigorous way to recover the spectrum of $H(\epsilon)$ on Ω_d, $(d \geq 1)$ near any energy level λ, by studying systems of pseudodifferential operators which have a principal symbol quite close to one of $e_j(\epsilon D_y) + V(0, y) - z$, where z is the spectral parameter. First, we give a complete asymptotic expansion in powers of ϵ of $\mathrm{tr}(\Psi f(H(\epsilon)))$ where $f \in C_0^\infty(\mathbb{R})$ and Ψ is a multiplication operator by a real integrable function $\Psi(y) \in L^1(\mathbb{R}^d)$. In particular, we obtain a Weyl type asymptotics with optimal remainder estimates of the counting function of eigenvalues of $H(\epsilon)$ in any closed interval in $]-\infty, e_0(0)[$. To investigate the effect of the perturbation on the continuous spectrum of H, it is natural to study the spectral shift function (SSF for short). When V vanishes as $\|y\| \to \infty$ (see (12)), the SSF $\xi(\mu; \epsilon)$ related to $H(\epsilon)$ and H is well defined in the sense of distribution :

$$\mathrm{tr}\big[f(H(\epsilon)) - f(H)\big] = -\langle \xi'(\cdot; \epsilon), f(\cdot)\rangle = \int_{\mathbb{R}} \xi(\mu; \epsilon) f'(\mu) d\mu, \; f \in C_0^\infty(\mathbb{R}). \tag{1}$$

The function $\xi(\mu; \epsilon)$ is fixed up to a constant by the formula (1), and we normalize $\xi(\mu; \epsilon)$ so that $\xi(\mu; \epsilon) = 0$ for $\mu < \inf(\sigma(H(\epsilon))$. The operator $H(\epsilon)$ could have embedded eigenvalues and then the derivative of the SSF could be locally a Dirac distribution. The spectral shift function may be considered as a generalization of the eigenvalues counting function. It is one of important physical quantities in scattering theory, and it plays an important role in the study of the location of resonances in various scattering problems. We refer to [28] and references cited there for comprehensive information on related subjects.

Under the assumption (12), we give a complete asymptotic expansion in powers of ϵ

of the left hand side of (1) in Theorem 3.3 and moreover in Theorem 3.4, we establish a complete asymptotic expansions in powers of ϵ for $\xi(\mu;\epsilon)$.

The paper is organized as follows: Section 2 is devoted to the study of the fibered operator $H_0(k)$ on Λ_d. In Section 3 (respectively section 6), we give the main results (respectively the proofs) of this paper. In sections 4 and 5 we describe general mainly well-known results on the effective Hamiltonian and on the ϵ-pseudodifferential operators with operator-valued symbol.

Notations : We shall employ the following standard notations. Given a complex function f_h depending on a small positive parameter h, the relation $f_h = \mathcal{O}(h^N)$ means that there exist $C_N, h_N > 0$ such that $|f_h| \leq C_N h^N$ for all $h \in]0, h_N[$. The relation $f_h = \mathcal{O}(h^\infty)$ means that, for all $N \in \mathbb{N} := \{0, 1, 2, \ldots\}$, we have $f_h = \mathcal{O}(h^N)$. We write $f_h \sim \sum_{j=0}^{\infty} a_j h^j$ if, for each $N \in \mathbb{N}$, we have $f_h - \sum_{j=0}^{N} a_j h^j = \mathcal{O}(h^{N+1})$. We adopt the notation $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Let $\mathcal{H}$ be a Hilbert space. The scalar product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle$. The set of linear bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{L}\ (\mathcal{H}_1)$ in the case where $\mathcal{H}_1 = \mathcal{H}_2$.

2 The non-perturbed Hamiltonian H

In this section we establish the basic spectral properties for the non-perturbed operator H. We focus on a diagonalization of $H_0(k)$ and the corresponding generalized eigenfunction. Moreover, we introduce an integrated density of states, ρ, corresponding to H.

The operator H is unitarily equivalent to

$$\mathcal{F} H \mathcal{F}^* = \int_{\mathbb{R}^d}^{\oplus} H_0(k) dk, \tag{2}$$

where $\mathcal{F}$ is the partial Fourier transform with respect to y given by

$$(\mathcal{F}u)(x,k) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-iyk} u(x,y) dy,$$

and

$$H_0(k) = D_x^2 + (k + \mu x)^2, \tag{3}$$

is the operator defined on $\mathcal{H}_{\Lambda_d} := \{u \in H^2(\Lambda_d);\ u|_{\partial\Lambda_d} = 0\}$. In what follows, we will consider $\mathcal{H}_{\Lambda_d}$ as a Hilbert space equipped with the standard scalar product of $H^2(\Lambda_d)$.

We first examine the two dimensional case (i.e, $d = 1$, $\Omega_1 = [-a, a] \times \mathbb{R}$). From the Sturm-Liouville theory (see for instance [25]), it is well-known that $H_0(k)$ has a simple discrete spectrum : $e_0(k) < e_1(k) < \cdots$. The change of variable $x \mapsto -x$ implies that $e_l(k) = e_l(-k)$. Since the eigenvalues are simple, an ordinary analytic perturbation theory shows that $e_l(k)$ (and the corresponding eigenfunction) are analytic functions in k (see [22, 27]).

Theorem 2.1. *The eigenvalue $e_j(k)$ satisfies :*

$$ke_j'(k) > 0 \quad (k \neq 0), \quad \text{and } \; e_j'(0) = 0, \quad e_j''(0) > 0. \tag{4}$$

Moreover, for every fixed $j \in \mathbb{N}$ and any $a, \mu > 0$, the following properties hold :

$$e_j(k) = e_j(0) + \sum_{l=1}^{\infty} \alpha_{j,l} k^{2l} \; (k \to 0), \qquad \alpha_{j,1} > 0, \tag{5}$$

$$e_j(k) = k^2 - 2a\mu k + \nu_j (2\mu k)^{2/3}(1 + o(1)), \quad (k \to +\infty), \tag{6}$$

where $0 < \nu_0 < \nu_1 < \cdots < \nu_j < \cdots$ are the eigenvalues of the operator $D_x^2 + x$ on $\mathbb{R}^+$. The normalized eigenfunctions $\Psi_n(\cdot, k)$ corresponding to $e_n(k)$ can be chosen real-valued and analytic with respect to k satisfying :

$$\forall p \in \mathbb{N}, \; \exists C_p, \; \text{such that} \; \int_{-a}^{a} \left(\partial_k^p \Psi_n(x,k)\right)^2 dx \leq C_p, \quad \|\Psi_n(\cdot,k)\|_{L^2(-a,a)} = 1. \tag{7}$$

Proof. (4) is proved in [15] (see Theorem 2 in [15]). Formula (5) follows from the fact that $e_j(k)$ is an even real analytic function with $e_j''(0) > 0$.

To prove (6), consider the operator $\tilde{H}(k) = D_x^2 + 2\mu x k + k^2$. Replacing x by $t = \mu(x+a)$ and rescaling $t \mapsto \lambda t/\mu$ (with $\lambda = (2\mu k)^{1/3}$) we transform $\tilde{H}(k)$ into $\lambda^2 G - 2a\mu k + 2k^2$, where

$$G = D_t^2 + t : L^2([0,b]) \to L^2([0,b]), \quad b = 2\lambda a,$$

is the Airy operator with Dirichlet boundary condition. The general solution of the equation $D_t^2 u(t) + tu(t) = 0$ can be written as a linear combination of the Airy functions:

$$u(t) = C_+ \mathrm{Ai}(t) + C_- \mathrm{Bi}(t).$$

We recall that $\mathrm{Bi}(t) = \mathrm{Ai}(e^{2\pi i/3} x)$. Thus, the eigenvalues ν_j of the operator G are the roots of the equation

$$\mathrm{Ai}(-\nu_j) = \mathrm{Bi}(-\nu_j) \frac{\mathrm{Ai}(-\nu_j + b)}{\mathrm{Bi}(-\nu_j + b)}.$$

Since the right-hand side of the above equality tends to zero as b tends to $+\infty$, $-\nu_j$ are approximated (when $k \to +\infty$) by the zeros of the Airy function $\mathrm{Ai}(x)$. Consequently, the eigenvalues $\lambda_0(k) < \lambda_1(k) < \cdots$ of $\tilde{H}(k)$ satisfies

$$\lambda_j(k) = k^2 - 2a\mu k + \nu_j (2\mu k)^{2/3}(1 + o(1)) \quad (k \to +\infty). \tag{8}$$

On the other hand, since $-a \leq x \leq a$, it follows that [1]

$$H_0(k) - \mu^2 a^2 \leq \tilde{H}(k) = H_0(k) - \mu^2 x^2 \leq H_0(k).$$

which together with Theorem XIII.1 in [27] yields

$$e_j(k) - \mu^2 a^2 \leq \lambda_j(k) \leq e_j(k).$$

[1]Let A and B be selfadjoint operators that are bounded from below. We write $A \leq B$ if and only if $D(B) \subset D(A)$ and $(Au, u) \leq (Bu, u) \; \forall u \in D(B)$.

Thus (6) follows from (8) and the above inequality.

The only point remaining concerns the estimate (7). Let $\Psi_n(\cdot,k)$ be the normalized real-valued analytic function corresponding to $e_n(k)$. Since Ψ_n is real and $\|\Psi_n(\cdot,k)\| = 1$, it follows that

$$\frac{\partial}{\partial k}\int_{-a}^{a}\Psi_n(x,k)^2 dx = 0 = 2\int_{-a}^{a}\Psi_n(x,k)\frac{\partial}{\partial k}\Psi_n(x,k)dx. \tag{9}$$

Put $\widehat{H}(k) = H_0(k) - k^2$, and let Γ_n be a simple closed contour around $e_n(k) - k^2$ such that $\text{dist}(\Gamma_n, \sigma(\widehat{H}(k))) \geq C > 0$ uniformly on k. Let $P_n(k)$ be the orthogonal projection onto the eigenspace spanned by $\Psi_n(\cdot,k)$, that is for $u(x) \in \mathcal{H}_{\Lambda_2}$

$$P_n(k)u(x) = \frac{1}{2\pi i}\int_{\Gamma_n}(\widehat{H}(k) - z)^{-1}dz = \langle\, u(\cdot)\,, \Psi_n(\cdot,k)\rangle\, \Psi_n(x,k). \tag{10}$$

From (9) we deduce that $P_n(k)\partial_k\Psi_n(\cdot,k) = 0$. Combining this with the fact that $P_n(k)\Psi_n(\cdot,k) = \Psi_n(\cdot,k)$ and using (10) as well as the fact that $\partial_k\widehat{H}(k) = 2\mu x$, we get

$$\partial_k\Psi_n(x,k) = \partial_k P_n(k)\Psi_n(x,k) = \frac{-1}{2\pi i}\int_{\Gamma_n}(\widehat{H}(k) - z)^{-1}2\mu x(\widehat{H}(k) - z)^{-1}dz\Psi_n(x,k), \tag{11}$$

which yields

$$\|\partial_k\Psi_n(\cdot,k)\| = \mathcal{O}(1)\|\Psi_n(\cdot,k)\| = \mathcal{O}(1).$$

We now proceed by induction using (11). □

Let $\big(e_l^j(k_j)\big)_{l\in\mathbb{N}}$ and $\big(\Psi_l^j(x_j,k_j)\big)_{l\in\mathbb{N}}$ be the eigenvalues and eigenvectors of the operator $D_{x_j}^2 + (k_j + \mu_j x_j)^2$ given by Theorem 2.1. For $J = (j_1,\cdots,j_d) \in \mathbb{N}^d$ and $k = (k_1,\cdots,k_d) \in \mathbb{R}^d$, we denote

$$e_J(k) = e_{j_1}^1(k_1) + \cdots + e_{j_d}^d(k_d), \quad \Psi_J(x,k) = \Psi_{j_1}^1(x_1,k_1) \times \cdots \times \Psi_{j_d}^d(x_d,k_d).$$

By Theorem 2.1, we have

Corollary 2.2. *The spectrum of the operator $H_0(k)$ on $\{u \in H^2(\Lambda_d); u|_{\partial\Lambda_d} = 0\}$ is discrete and coincides with $\{e_J(k); J \in \mathbb{N}^d\}$. The family $(\Psi_J(\cdot,k))_{J\in\mathbb{N}^d}$ is an orthonormal basis in $L^2(\Lambda_d)$.*

Now let us return to the non-perturbed operator $H = D_x^2 + (D_y + \mu x)^2$ as an unbounded operator on $\mathcal{H}_{\Omega_d}^D$. According to Theorem 2.1, Corollary 2.2, and the theory of decomposable operators (see Theorem XIII. 85 in [27]) the spectrum of H is absolutely continuous, and given by

$$\sigma(H) = \bigcup_{J\in\mathbb{N}^d}\bigcup_{k\in\mathbb{R}^d} e_J(k) = [e_0(0), +\infty[.$$

The points $e_J(0)$ are thresholds in $\sigma(H)$. From now on we denote this set by

$$\Sigma := \bigcup_{j\in\mathbb{N}^d} e_J(0) = \sigma(H_0(0)).$$

For $t_0 \in \Sigma$, we let $\mathcal{S}_{t_0} := \{J \in \mathbb{N}^d; e_J(0) = t_0\}$ and $m_{t_0} := \#\mathcal{S}_{t_0}$ be its multiplicity. To end this section, let us introduce the function $\rho : \mathbb{R} \to \mathbb{R}$ related to the non-perturbed H by

$$\rho(t) = \sum_{J\in\mathbb{N}^d} \int_{\{e_J(k)\le t\}} \frac{dk}{(2\pi)^d}.$$

Obviously, $\rho(t) = 0$ for $t < e_0(0) = \inf\sigma(H)$. In an appendix, we shall prove that the function $\rho(t)$ is analytic except near Σ. More precisely, we have

Theorem 2.3. *The function ρ is analytic except at Σ. Moreover, near any point $t_0 = e_J(0) \in \Sigma$, there exists analytic functions f and g such that :*

$$\rho(t) = f(t - t_0) + Y(t - t_0)g(t - t_0),$$

for $|t - t_0|$ small enough with

$$g(t) \sim_{t\to 0} \sum_{J\in\mathcal{S}_{t_0}} \frac{\mathrm{vol}(S^{d-1})}{d\sqrt{\det(\frac{\nabla^2 e_J(0)}{2})}} t^d.$$

Here $Y(t)$ is the Heaviside function and S^{d-1} stands for the unit sphere in $\mathbb{R}^d$.

3 Perturbed Hamiltonian

In this section, we investigate the effect of the slowly varying potential on the spectrum of the non-perturbed operator H_0. First, we give a complete asymptotic expansion in powers of ϵ of $\mathrm{tr}(\Psi f(H(\epsilon)))$ where $f \in C_0^\infty(\mathbb{R})$ and Ψ is an $L^1(\mathbb{R}^d_y)$-function. In particular, we obtain a Weyl type asymptotics with optimal remainder estimates of the counting function of eigenvalues of $H(\epsilon)$ below the essential spectra. Finally, we give a complete asymptotic expansion in powers of ϵ of the spectral shift function corresponding to $(H(\epsilon), H)$.

We suppose that V is regular, and there exists $\delta \ge 0$ such that :

$$\forall\alpha, \beta \in \mathbb{N}^d,\ \exists C_{\alpha,\beta} \text{ s.t } \sup_{x\in\Lambda_d} |\partial_x^\beta \partial_y^\alpha V(x, y)| \le C_{\alpha,\beta}\langle y\rangle^{-\delta}. \tag{12}$$

First, we derive a local trace formula.

Theorem 3.1. *Assume* (12) *with $\delta \ge 0$, and let Ψ be a regular function such that $\partial_y^\alpha \Psi \in L^1(\mathbb{R}^d_y)$ for $|\alpha| \le 2d+1$. Then for all $f \in C_0^\infty(\mathbb{R})$, the operator $(\Psi f(H(\epsilon)))$ is trace class and the following asymptotics holds :*

$$\mathrm{tr}\,(\Psi f(H(\epsilon))) \sim \sum_{j=0}^{\infty} a_j \epsilon^{-d+j}, \tag{13}$$

with

$$a_0 = -\iint_{\mathbb{R}^d\times\mathbb{R}_t} \Psi(y) f'(t)\rho(t - V(0, y))dydt. \tag{14}$$

Let $N([a, b]; \epsilon)$ be the number of eigenvalues of $H(\epsilon)$ in $[a, b] \subset]-\infty, e_0(0)[$ counted with their multiplicities.

Corollary 3.2. *Assume that V tends to zero at infinity, and let $f \in C_0^\infty(]-\infty, e_0(0)[;\mathbb{R})$. We have*

$$\mathrm{tr}(f(H)) \sim \sum_{j=0}^{\infty} b_j \epsilon^{-d+j}, \tag{15}$$

with

$$b_0 = -\iint_{\mathbb{R}^d \times \mathbb{R}_t} f'(t)\rho(t - V(0,y))dydt. \tag{16}$$

In particular,

$$\lim_{\epsilon \searrow 0} \Big[\epsilon^d N([a,b];\epsilon)\Big] = \int_{\mathbb{R}^d} \Big[\rho(b - V(0,y)) - \rho(a - V(0,y))\Big]dy. \tag{17}$$

Theorem 3.3. *Assume* (12) *with $\delta > d$. For $f \in C_0^\infty(\mathbb{R})$ the operator $f(H(\epsilon)) - f(H)$ is trace class. Moreover, the following asymptotics holds*

$$\mathrm{tr}(f(H(\epsilon)) - f(H)) \sim \sum_{j=0}^{\infty} c_j \epsilon^{-d+j} \tag{18}$$

with

$$c_0 = \iint_{\mathbb{R}^d \times \mathbb{R}_t} f'(t)(\rho(t) - \rho(t - V(0,y)))\, dydt. \tag{19}$$

The above theorem, enables us to define the spectral shift function $\xi(\cdot,\epsilon) \in \mathcal{D}'(\mathbb{R})$, related to the operators $H(\epsilon)$ and H (see (1)). Theorem 3.3 tells us that $\xi(\cdot,\epsilon)$ converges to $\int(\rho(t) - \rho(t - V(0,y))dy$ in the sense of distribution. Under a non-trapping condition, the following result gives a pointwise asymptotic expansion in powers of ϵ of $\xi'(\cdot;\epsilon)$.

Theorem 3.4. *Fix $\lambda > e_0(0)$ with $\lambda \notin \{e_1(0), e_2(0), \ldots\}$, and assume that* [2]

$$k \cdot \nabla e_j(k) - y \cdot \nabla_y V(0,y) \geq c > 0 \ \text{in} \ \{(y,k) \in \mathbb{R}^{2d}; e_j(k) + V(0,y) = \lambda\}. \tag{20}$$

There exists $\eta > 0$ such that the following complete asymptotic expansion holds uniformly on $\lambda \in]\lambda - \eta, \lambda + \eta[$:

$$\xi'(t,\epsilon) \sim \sum_{j=0}^{\infty} \kappa_j(t)\epsilon^{-d+j}, \tag{21}$$

with

$$\kappa_0(t) = \int (\rho'(t) - \rho'(t - V(0,y)))dy.$$

4 Effective Hamiltonian

We need some basic result about pseudo-differential operators with operator-valued symbol (see [14] and the references cited therein). We shall consider a family of Hilbert space $\mathcal{A}_X$, $X = \mathbb{R}^{2d}$ satisfying :

$$\mathcal{A}_X = \mathcal{A}_Y, \forall X, Y \in \mathbb{R}^{2d}, \tag{22}$$

[2] By (4), this assumption is satisfied under the assumption : $-y \cdot \nabla_y V(0,y) \geq 0$ and $-y \cdot \nabla_y V(0,y) > 0$, on $\{y \in \mathbb{R}^d; V(0,y) = \lambda - e_j(0)\}$.

there exist $N \in \mathbb{N}$ and $C > 0$ such that for all $u \in \mathcal{A}_0$ and all $X, Y \in \mathbb{R}^{2d}$ we have

$$\|u\|_{\mathcal{A}_X} \le C\langle X - Y\rangle^N \|u\|_{\mathcal{A}_Y}. \tag{23}$$

Notice that (47) means that only the norm of $\mathcal{A}_X$ depends on X, not on the space itself. Let $\mathcal{B}_X$ be a second family with the same properties. We say that $p \in C^\infty(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_0, \mathcal{B}_0))$ belongs to the symbol class $S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$ if for every $\alpha \in \mathbb{N}^{2d}$ there exists C_α such that

$$\|\partial_X^\alpha p\|_{\mathcal{L}(\mathcal{A}_X, \mathcal{B}_X)} \le C_\alpha, \ \forall X \in \mathbb{R}^{2d}. \tag{24}$$

If p depends on a semiclassical parameter ϵ and possibly on other parameters as well, we require (24) to hold uniformly with respect to these parameters. For ϵ-dependent symbols, we say that $p(y, k; \epsilon)$ has an asymptotic expansion in powers of ϵ , and we write

$$p(y, k; \epsilon) \sim \sum_j p_j(y, k)\epsilon^j \text{ in } S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$$

if for every $N \in \mathbb{N}$, $\epsilon^{-N-1}\Big(p(y, k; \epsilon) - \sum_{j=0}^N p_j(y, k)\epsilon^j\Big) \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$.

We can then associate to p an ϵ-pseudodifferential operator

$$p^w(y, \epsilon D_y; \epsilon)u(y) = \iint e^{\frac{i}{\epsilon}(y-t)k} p(\frac{y+t}{2}, k; \epsilon)u(t)\frac{dtdk}{(2\pi\epsilon)^d}, \ u \in \mathcal{A}_0.$$

Here we use the Weyl quantization. Similarly to the scalar case, the following results hold.

Theorem 4.1. *Let $p \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$ where $\mathcal{A}_X, \mathcal{B}_X$ satisfy* (47) *and* (48) *then $p^w(y, \epsilon D_y, \epsilon)$ is uniformly continuous from $S(\mathbb{R}^d; \mathcal{A}_0)$ into $S(\mathbb{R}^d; \mathcal{B}_0)$.*

Theorem 4.2. *Assume $\mathcal{A}_X = \mathcal{A}_0$ and $\mathcal{B}_X = \mathcal{B}_0$ for all $X \in \mathbb{R}^{2d}$. If $p \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_0, \mathcal{B}_0))$ then $p^w(y, \epsilon D_y; \epsilon)$ is bounded from $L^2(\mathbb{R}^d, \mathcal{A}_0)$ into $L^2(\mathbb{R}^d, \mathcal{B}_0)$.*

Let $\mathcal{C}_X$ be a third Hilbert space which satisfies (47), (48).

Theorem 4.3. *Let $p \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{B}_X, \mathcal{C}_X))$, $q \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{A}_X, \mathcal{B}_X))$. Then*

$$p^w(y, \epsilon D_y) \circ q^w(y, \epsilon D_y) = r^w(y, \epsilon D_y; \epsilon),$$

where r is given by

$$r(y, k; \epsilon) \sim \sum_{j=0} \frac{1}{j!}(\frac{i\epsilon}{2}\sigma(D_y, D_k; D_x, D_\xi))^j p(y, k)q(x, \xi)\Big|_{x=y, k=\xi}. \tag{25}$$

4.1 Grushin problem: brief description

In this paragraph we recall the basic results about Grushin problem. Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be three Hilbert spaces, and let $P \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3)$ be self-adjoint. Assume that there exist $R_+ \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $R_- \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_3)$ such that the following operator

$$\mathcal{P}(z) = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_3 \times \mathcal{H}_2$$

is bijective for $z \in \Omega$. Here Ω is an open bounded set in $\mathbb{C}$. Let

$$\mathcal{E}(z) = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{\text{eff}}(z) \end{pmatrix}$$

be its inverse. We refer to the problem $\mathcal{P}(z)$ as a Grushin problem and the operator $E_{\text{eff}}(z)$ is called effective Hamiltonian. Notice that, an effective Hamiltonian is a Hamiltonian that acts in a reduced space and only describes a part of the eigenvalue spectrum of the true Hamiltonian P. Morally, effective Hamiltonians are much simpler than the true Hamiltonian and hence their eigensystems can often be determined analytically or with little effort numerically.

The following useful properties (relating the operator P and its effective Hamiltonian) are consequences of the identities $\mathcal{E} \circ \mathcal{P} = I$ and $\mathcal{P} \circ \mathcal{E} = I$:

$$(P - z) \text{ is invertible if and only if } \; E_{\text{eff}}(z) \; \text{ is invertible,} \tag{26}$$

$$\dim \ker(P - z) = \dim \ker(E_{\text{eff}}(z)), \tag{27}$$

$$(P - z)^{-1} = E(z) - E_+(z) E_{\text{eff}}^{-1}(z) E_-(z), \tag{28}$$

$$E_{\text{eff}}^{-1}(z) = -R_+(P - z)^{-1} R_-. \tag{29}$$

The last two equalities hold for all $\Im z \neq 0$. On the other hand, since $z \mapsto (P - z)$ is holomorphic, it follows that the operators $E(z), E_\pm(z)$ and $E_{\text{eff}}(z)$ are also holomorphic in $z \in \Omega$. Moreover, we have

$$\partial_z E_{\text{eff}}(z) = E_-(z) E_+(z). \tag{30}$$

This identity comes from the fact that $R_\pm$ are independent of z.

5 Spectral Reduction to an ϵ-pseudodifferential operator

Throughout this section we assume that V is independent on x. The proof of the general case is quite similar with minor modifications (see Remark 6.2). Fix an interval $I = [\alpha, \beta]$, and set

$$\mathbb{U} = \{J \in \mathbb{N}^d; e_J(k) \leq \beta + \|V\|_\infty\}.$$

According to Theorem 2.1 and Corollary 2.2, $e_J(0)$ (respectively $e_J(k)$) tends to infinity as $|J| \to \infty$ (respectively $|k| \to \infty$). Therefore $\mathbb{U}$ is finite. In what follows, $(\Psi_1(\cdot, k), \cdots, \Psi_N(\cdot, k))$ denotes the family $(\Psi_J(\cdot, k))_{J \in \mathbb{U}}$, where $N = \#\mathbb{U}$.

To shorten notation, we omit the index d in Ω_d and Λ_d. For $k \in \mathbb{R}^d$, let $\mathcal{H}_{\Lambda,k} = \mathcal{H}_\Lambda$ be the Hilbert space with k-dependent norm: $\|u\|^2_{\Lambda,k} = \|u\|^2_{H^2(\Lambda)} + |k|^4 \|u\|^2_{L^2(\Lambda)}$. We denote by $\mathbb{C}^N_k$ the space $\mathbb{C}^N$ equipped with norm $(1 + |k|^2)| \cdot |_{\mathbb{C}^N}$.

By the change of variable $y \mapsto y/\epsilon$, the operator $H(\epsilon)$ is unitarily equivalent to

$$H_1 := H_{1,0} + V(y), \tag{31}$$

where

$$H_{1,0} := \sum_{j=1}^{d} D^2_{x_j} + (\epsilon D_{y_j} + \mu_j x_j)^2.$$

Let $G(y,k) = H_0(k)+V(y)$ be the linear bounded operator from $\mathcal{H}_\Lambda$ into $L^2(\Lambda)$, where $H_0(k)$ is given by (3). Obviously, $G \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{H}_{\Lambda,k}, L^2(\Lambda))$. Thus, by quantizing G we have

$$G(y, \epsilon D_y) = H_1.$$

More precisely, H_1 can be viewed as an ϵ-pseudodifferential operator on y with operator valued symbol $G(y,k)$.

For $k \in \mathbb{R}^d$, and $N \in \mathbb{N}^*$, define $R_+(k) : L^2(\Lambda) \to \mathbb{C}^N$, $R_-(k) = R_+^*(k) : \mathbb{C}^N \to L^2(\Lambda)$ by

$$\begin{aligned} R_+(k)u &= (\langle u, \Psi_1(\cdot,k)\rangle, \cdots, \langle u, \Psi_N(\cdot,k)\rangle), \\ R_-(k)(c_1,\cdots,c_N) &= \sum_{j=1}^N c_j \Psi_j(\cdot,k). \end{aligned}$$

According to Corollary 2.2 the family $(\Psi_J(\cdot,k))_{J\in\mathbb{N}^d}$ is an orthonormal basis in $L^2(\Lambda)$. Hence, a simple computation yields

$$\begin{aligned} R_+(k)R_-(k) &= I_{\mathbb{C}^N}, \\ R_-(k)R_+(k)u &= \sum_{j=1}^N \langle u, \Psi_j(\cdot,k)\rangle \Psi_j(\cdot,k) =: \Pi_N u, \ \forall u \in L^2(\Lambda). \end{aligned} \tag{32}$$

The following proposition reduces the spectral study of the operator $G(y,k) : \mathcal{H}_{\Lambda,k} \to L^2(\Lambda)$ near the energy z, to the study of an $N \times N$-square matrix $E_{\text{eff}}(y,k,z)$.

Proposition 5.1. *Fix a bounded interval I. There exists $N \in \mathbb{N}^*$ such that for all $z \in I$ the operator*

$$\mathcal{P}(y,k) := \begin{pmatrix} G(y,k)-z & R_-(k) \\ R_+(k) & 0 \end{pmatrix} : \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N \to L^2(\Lambda) \times \mathbb{C}^N_k, \tag{33}$$

is bijective with bounded two-sided inverse

$$\mathcal{E}(y,k,z) := \begin{pmatrix} \widehat{G_N}(y,k,z) & R_-(k) \\ R_+(k) & E_{\text{eff}}(y,k,z) \end{pmatrix}. \tag{34}$$

Here $\widehat{G_N}(y,k,z) = (G(y,k)-z)^{-1}(1-\Pi_N)$ and $E_{\text{eff}}(y,k,z)$ is the square diagonal matrix $(z - e_j(k) - V(0,y))\delta_{ij})_{1\le i,j\le N}$. Moreover

$$\mathcal{P} \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{H}_{\Lambda,k} \times \mathbb{C}^N; L^2(\Lambda) \times \mathbb{C}^N_k)). \tag{35}$$

$$\mathcal{E} \in S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}^N_k; \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N)). \tag{36}$$

Proof. By construction, we have

$$e_J(k) + V(y) - z \ge c > 0,$$

uniformly for $(z,k,y) \in I \times \mathbb{R}^{2d}$ and $J \notin \mathbb{U}$. Thus, the operator

$$(G(y,k)-z)^{-1}(1-\Pi_N) : L^2(\Lambda) \to \mathcal{H}_{\Lambda,Y},$$

is well-defined and uniformly bounded on $(z, y, k) \in I \times \mathbb{R}^{2d}$. Using (32), an easy computation shows that $\mathcal{P}(y, k) \circ \mathcal{E}(y, k, z) = I$ and $\mathcal{E}(y, k, z) \circ \mathcal{P}(y, k) = I$. On the other hand, it follows from (7) that $(y, k) \to R_-(k) \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathbb{C}^N; L^2(\Lambda))$ and $(y, k) \to R_+(k) \in S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathcal{H}_{\Lambda,k}; \mathbb{C}^N))$

□

Proposition 5.2. *The operator*

$$\mathcal{P} := \begin{pmatrix} G(y, \epsilon D_y) - z & R_-(\epsilon D_y) \\ R_+(\epsilon D_y) & 0 \end{pmatrix} : \mathcal{H}_\Omega^D \times H^2(\mathbb{R}^d; \mathbb{C}^N) \to L^2(\Omega) \times L^2(\mathbb{R}^d; \mathbb{C}^N), \tag{37}$$

is bijective with an inverse

$$\mathcal{E}(z; \epsilon) := \mathcal{E}^w(z; \epsilon) = \begin{pmatrix} E^w(y, \epsilon D_y, z; \epsilon) & E_+^w(y, \epsilon D_y, z; \epsilon) \\ E_-^w(y, \epsilon D_y, z; \epsilon) & E_{\text{eff}}^w(y, \epsilon D_y, z; \epsilon) \end{pmatrix},$$

uniformly bounded with respect to $z \in I$ and ϵ small enough. Moreover, $\mathcal{E}(z; \epsilon)$ depend holomorphically on z, and $\mathcal{E}(y, k, z; \epsilon)$ has an asymptotic expansion in $S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}_k^N; \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N))$, i.e.,

$$\mathcal{E}(y, k, z; \epsilon) = \begin{pmatrix} E(y, k, z; \epsilon) & E_+(y, k, z; \epsilon) \\ E_-(y, k, z; \epsilon) & E_{\text{eff}}(y, k, z; \epsilon) \end{pmatrix} \sim \sum_{j=0}^{\infty} \mathcal{E}_j(y, k, z)\epsilon^j. \tag{38}$$

In particular $E_{\text{eff}}(y, k, z; \epsilon) \sim \sum_{j=0}^{\infty} E_{\text{eff},j}(y, k, z)\epsilon^j$ in $S^0(\mathbb{R}^{2d}; \mathcal{L}(\mathbb{C}_k^N; \mathbb{C}^N))$. The leading terms $\mathcal{E}_0(y, k, z)$ and $E_{\text{eff},0}(y, k, z)$ are given by Proposition 5.1, i.e.,

$$\mathcal{E}_0(y, k, z) = \mathcal{E}(y, k, z; 0) \text{ and } E_{\text{eff},0}(y, k, z) = E_{\text{eff}}(y, k, z; 0).$$

Proof. The fact that $\mathcal{P}$ can be viewed as an ϵ-pseudodifferential operator valued symbol $\mathcal{P}(y, k)$ and Theorem 4.3 show that

$$\mathcal{P}^w(y, \epsilon D_y) \circ \mathcal{E}^w(y, \epsilon D_y, z) = I + \epsilon \mathcal{R}^w(y, \epsilon D_y, z; \epsilon) \tag{39}$$

where $\mathcal{R}(y, k, z; \epsilon) \sim \sum_{j=0}^{\infty} \mathcal{R}_j(y, k, z)\epsilon^j$ in $S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}^N; L^2(\Lambda) \times \mathbb{C}^N))$. It follows from Theorem 4.2 that $\mathcal{R}^w(y, \epsilon D_y, z; \epsilon)$ is uniformly bounded for $z \in I$ and $|\epsilon| \leq 1$. Thus, for ϵ small enough the right hand side of (39) is invertible. On the other hand we know that if $P = p^w(y, k, \epsilon)$ is an invertible ϵ-pseudodifferential with $p(y, k; \epsilon) \sim \sum_{j=0}^{\infty} p_j(y, k)\epsilon^j$ then its inverse q^w is also an ϵ-pseudodifferential operator with $q(y, k; \epsilon) \sim \sum_{j=0}^{\infty} q_j(y, k)\epsilon^j$. Consequently, $\mathcal{E}^w(y, \epsilon D_y, z; \epsilon) := \mathcal{E}^w(y, \epsilon D_y, z) \circ (I + \epsilon \mathcal{R}^w(y, \epsilon D_y, z; \epsilon))^{-1}$ satisfies all the desired properties. □

Remark 5.3. *Let $\mathcal{E}_0(z)$ be the operator given by Proposition 5.2 corresponding to the non-perturbed operator H_0 (i.e., $V = 0$). Since $\mathcal{P}(y, k) = \mathcal{P}(k)$ is y-independent, we have*

$$\mathcal{E}_0(z) = \begin{pmatrix} \widehat{G}_N(\epsilon D_y, z) & E_+^0(\epsilon D_y) \\ E_-^0(\epsilon D_y) & E_{\text{eff}}^0(\epsilon D_y, z) \end{pmatrix},$$

where $E_+^0(k) = R_-(k), E_-^0(k) = R_+(k)$ and $E_{\text{eff}}^0(k, z) = (z - e_j(k))\delta_{ij})_{1 \leq i,j \leq N}$

6 Proof of the main results

6.1 Proof of Theorem 3.1

In the following we fix a bounded interval I containing $\mathrm{supp}(f)$, and we apply Proposition 5.1 and Proposition 5.2 on I. For the simplicity of the notation we ignore the dependence of $E, E_+, E_-, E_{\text{eff}}$ on (y, k, z, ϵ). We denote by $E^0, E_+^0, E_-^0, E_{\text{eff}}^0$ the operators given by Proposition corresponding to We shall sometimes use the same symbol for an ϵ-pseudododifferential operator and for its Weyl symbol.

Applying formulas (28) and (29) to Proposition 5.2 we obtain

$$(H_1 - z)^{-1} = E - E_+ E_{\text{eff}}^{-1} E_-, \tag{40}$$

$$\partial_z E_{\text{eff}} = E_- E_+. \tag{41}$$

Assume that $f \in C_0^\infty(\mathbb{R})$ is real-valued, we can construct an almost analytic extension $\tilde{f} \in C_0^\infty(\mathbb{C})$ of f satisfying the following properties (see [14]) :

$$\tilde{f}(z) = f(z), \forall z \in \mathbb{R}, \tag{42}$$

for all $N \in \mathbb{N}$ there exists C_N such that

$$|\frac{\partial \tilde{f}}{\partial \overline{z}}(z)| \leq C_N |\Im z|^N. \tag{43}$$

Let H be any self-adjoint operator, the Dynkin-Helffer-Sjöstrand formula reads [14]:

$$f(H) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(z - H)^{-1} L(dz), \text{ with } z = x + iy, \tag{44}$$

which yields

$$f(H_1) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \overline{z}}(z)(z - H_1)^{-1} L(dz). \tag{45}$$

Here $L(dz)$ is the Lebesgue measure on the complex plane $\mathbb{C} \sim \mathbb{R}^2_{x,y}$.

Inserting (40) in the right hand side of (45) and using the fact that $z \to E^w(y, \epsilon D_y, z; \epsilon)$ is holomorphic, we get

$$f(H_1) = -\frac{1}{\pi} \int \frac{\partial \tilde{f}}{\partial \overline{z}}(z) E_+ E_{\text{eff}}^{-1} E_- L(dz). \tag{46}$$

Here and in what follows we use the fact that $\int \overline{\partial}_z \tilde{f}(z) K(z) L(dz) = 0$ provided that $K(z)$ is holomorphic in a neighborhood of $\mathrm{supp}(\tilde{f})$. We recall that the principal symbol of E_{eff} is given by

$$E_{\text{eff},0}(y, k, z) = ((z - V(y) - e_j(k))\delta_{i,j})_{1 \leq i,j \leq N},$$

and that $e_j(k) \sim |k|^2$ at infinity from (6) in Theorem 2.1. For $j = 1, \cdots, N$, let $\tilde{e}_j(k)$ be a regular function such that $\tilde{e}_j(k) = e_j(k)$ for $|k|$ large enough and

$$|z - V(y) - \tilde{e}_j(k)| \geq c_0(1 + |k|^2), \ \forall \ (z, y, k) \in \mathrm{supp}\tilde{f} \times \mathbb{R}^d \times \mathbb{R}^d. \tag{47}$$

Put

$$\tilde{E}_{\text{eff}}(y,k,z;\epsilon) = E_{\text{eff}}(y,k,z;\epsilon) + \tilde{E}_{\text{eff}}(y,k,z) - E_{\text{eff}}(y,k,z),$$

where $\tilde{E}_{\text{eff}}(y,k,z) = ((z - V(y) - \tilde{e}_j(k))\delta_{i,j})_{1\le i,j\le N}$. We conclude from (47) that $\tilde{E}_{\text{eff}}(y,k,z;\epsilon)$ is elliptic for ϵ small enough, hence that $\tilde{\tilde{E}}_{\text{eff}} := \tilde{E}^w_{\text{eff}}(y,\epsilon D_y,z;\epsilon)$ is invertible and holomorphic for $z \in \text{supp}(\tilde{f})$, and finally that

$$\int \frac{\partial \tilde{f}}{\partial \overline{z}}(z) E_+ \tilde{E}_{\text{eff}}^{-1} E_- L(dz) = 0.$$

Combining the above equality with (46), we obtain

$$f(H_1) = -\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \overline{z}}(z) E_+ (E_{\text{eff}}^{-1} - \tilde{E}_{\text{eff}}^{-1}) E_- L(dz). \tag{48}$$

Let Ψ be as in Theorem 3.1. Writing $E_{\text{eff}}^{-1} - \tilde{E}_{\text{eff}}^{-1} = \tilde{E}_{\text{eff}}^{-1}(\tilde{E}_{\text{eff}} - E_{\text{eff}})E_{\text{eff}}^{-1}$ and using the fact that $\tilde{E}_{\text{eff}} - E_{\text{eff}} = ((E_j(k) - \tilde{E}_j(k))_{1\le i,j\le N}$ has a compact support, we deduce that the operator $\Psi\Big(E_+\tilde{E}_{\text{eff}}^{-1}(\tilde{E}_{\text{eff}} - E_{\text{eff}})E_{\text{eff}}^{-1}E_-\Big)$ is trace class. Thus, by using the cyclicity of the trace we get

$$\begin{aligned} \text{tr}\Big(\Psi f(H_1)\Big) &= -\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \overline{z}}(z)\text{tr}\Big(E_{\text{eff}}^{-1} - \tilde{E}_{\text{eff}}^{-1})E_-\Psi E_+\Big)L(dz), \\ &= \text{tr}\Big(-\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \overline{z}}(z)E_{\text{eff}}^{-1}E_-\Psi E_+ L(dz)\Big). \end{aligned} \tag{49}$$

In the last equality we have used the fact the operator $\tilde{E}_{\text{eff}}^{-1}E_-\Psi E_+$ is holomorphic on $z \in \text{supp}(\tilde{f})$.

According to Proposition 5.2 and Theorem 4.3 the operator $A = E_-\Psi E_+$ is an ϵ-pseudodifferential operator on $L^2(\mathbb{R}^d;\mathbb{C}^N)$ with $A = A^w(y,\epsilon D_y,z;\epsilon)$ where $A(y,k,z;\epsilon) \sim \sum_{j=0}^\infty A_j(y,k,z)\epsilon^j$ in $S^0(\mathbb{R}^{2d};\mathcal{L}(\mathbb{C}^N;\mathbb{C}^N))$. Moreover, from Proposition 5.1 we have $A_0(y,k,z) = \Psi(y)$.

The proof of the following lemma is similar to the one in [10].

Lemma 6.1. *Fix $\delta \in]0,1/2[$. There exists $r \in S^0(\mathbb{R}^{2d};\mathcal{L}(\mathbb{C}^N,\mathbb{C}^N)$ such that $r(y,k;\epsilon) \sim \sum_{j=0}^\infty r_j(y,k)\epsilon^j$ and*

$$r^w(y,\epsilon D_y;\epsilon) = -\frac{1}{\pi}\int_{|\Im z|\ge \epsilon^\delta} \frac{\partial \tilde{f}}{\partial \overline{z}}(z)E_{\text{eff}}^{-1}E_-\Psi E_+ L(dz),$$

with

$$r_0(y,k) = -\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \overline{z}}(z)\Big((z - E_j(k) - V(y))\delta_{i,j}\Big)_{1\le i,j\le N} L(dz)\Psi(y).$$

We now turn to the proof of Theorem 3.1. If we restrict the integral in the right hand side of (49) to the domain $|\Im z| \le \epsilon^\delta$ then we get a term $\mathcal{O}(\epsilon^\infty)$ in trace norm. Here we have used the fact that $|\frac{\partial \tilde{f}}{\partial \overline{z}}(z)| = \mathcal{O}(|\Im z|^M)$ for all $M \in \mathbb{N}$ (see (43)). If we restrict our attention to the domain $|\Im z| \geq \epsilon^\delta$ then by Lemma 6.1 we get a complete asymptotic expansion in powers of ϵ, which yields (14). To finish the proof let us compute a_0. We have

$$a_0 = \iint \widehat{\mathrm{tr}}(r_0(y,k))\frac{dydk}{(2\pi)^d} = \sum_{j=1}^{N} \iint \left(-\frac{1}{\pi}\int \frac{\partial \tilde f}{\partial \overline{z}}(z)(z - e_j(k) - V(y))^{-1} L(dz) \right) \Psi(y)\frac{dydk}{(2\pi)^d}.$$

Here $\widehat{\mathrm{tr}}$ stands for the trace of square matrices. Since $\frac{1}{\pi}\overline{\partial}_z \frac{1}{z-z_0} = \delta(\cdot - z_0)$, it follows that $-\frac{1}{\pi}\int \frac{\partial \tilde f}{\partial \overline{z}}(z)(z-e_j(k)-V(y))^{-1}L(dz) = f(e_j(k)+V(y))$. Consequently,

$$a_0 = \sum_{j=1}^{N} \iint f(e_j(k)+V(y))\frac{dydk}{(2\pi)^d} = \sum_j \iint f(e_j(k)+V(y))\frac{dydk}{(2\pi)^d}.$$[3]

Combining this with the obvious equality

$$\sum_j \int f(e_j(k)+V(y))\frac{dk}{(2\pi)^d} = -\sum_j \int f'(t)\int_{e_j(k)\leq t-V(y)} dkdt = -\int f'(t)\rho(t-V(y))dt,$$

we get (14).

6.2 Proof of Corollary 3.2

Let f be as in Corollary 3.2, and fix $\eta > 0$ small enough such that $\mathrm{supp}(f) \subset]-\infty, E_1(0)-\eta]$. Put $\omega_\eta := \{y \in \mathbb{R}^d; \exists (j,k) \in \mathbb{N}^* \times \mathbb{R}^d \text{ s.t. } e_j(k)+V(y) \in \mathrm{supp}(f)\}$. Since V tends to zero at infinity and $e_j(k) \geq e_j(0)$ for all j,k, it follows that ω_η is a compact set.

Let $\tilde V$ be a regular function such that $\tilde V(y) \in [-\eta/2, \eta/2]$ for all $y \in \mathbb{R}^d$ and $\tilde V(y) = V(y)$ for $|y|$ large enough. Put

$$\tilde E_{\mathrm{eff}}(y,k,z;\epsilon) = E_{\mathrm{eff}}(y,k,z;\epsilon) + (\tilde V(y) - V(y))I_N.$$

By construction of $\tilde V$, we have

$$|z - e_j(k) - \tilde V(y)| \geq C(1+|k|^2),$$

uniformly on $(j,y,k) \in \mathbb{N}^* \times \mathbb{R}^{2d}$ and z in small complex neighborhood of $\mathrm{supp}(\tilde f)$.

Hence, the principal symbol $\tilde E_{\mathrm{eff}}(y,k,z) = ((z - \tilde V(y) - e_j(k))\delta_{i,j})_{1\leq i,j\leq N}$ of $\tilde E_{\mathrm{eff}}$ is elliptic. We can now proceed analogously to the proof of (48), and obtain

$$f(H_1) = -\frac{1}{\pi}\int \frac{\partial \tilde f}{\partial \overline{z}}(z)E_+(E_{\mathrm{eff}}^{-1} - \tilde E_{\mathrm{eff}}^{-1})E_- L(dz). \tag{50}$$

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ be equal to one in a neighborhood of $\mathrm{supp}(\tilde V - V = \tilde E_{\mathrm{eff}} - E_{\mathrm{eff}})$. Writing $E_+(E_{\mathrm{eff}}^{-1} - \tilde E_{\mathrm{eff}}^{-1})E_- = E_+\tilde E_{\mathrm{eff}}^{-1}(\tilde E_{\mathrm{eff}} - E_{\mathrm{eff}})E_{\mathrm{eff}}^{-1}E_-$ and using the fact that $\mathrm{supp}(1-\psi) \cap \mathrm{supp}(\tilde V - V) = \emptyset$, we deduce from (50) and (25) that $\|(1-\psi)f(H_1)\|_{\mathrm{tr}} = \mathcal{O}(\epsilon^\infty)$. Consequently,

$$\mathrm{tr}(f(H_1)) = \mathrm{tr}(\psi f(H_1)) + \mathcal{O}(\epsilon^\infty), \tag{51}$$

[3] We recall that for $j \notin \{1,\dots,N\}$ $e_j(k)+V(y) \notin \mathrm{supp}(f)$ for $(y,k) \in \mathbb{R}^d \times \mathbb{R}^d$.

which together with Theorem 3.1 yields (3.2) and (16).[4]

It remains to prove (17). For every small $\eta > 0$, choose $\overline{f_\eta}, \underline{f_\eta} \in C_0^\infty(\mathbb{R};[0,1])$ with

$$1_{[a+\eta,b-\eta]} \leq \underline{f_\eta} \leq 1_{[a,b]} \leq \overline{f_\eta} \leq 1_{[a-\eta,b+\eta]}.$$

It then suffices to observe that

$$\mathrm{tr}\left[\underline{f_\eta}(H(\epsilon))\right] \leq N([a,b];\epsilon) \leq \mathrm{tr}\left[\overline{f_\eta}(H(\epsilon))\right],$$

which yields

$$\lim_{\eta\searrow 0}\lim_{h\searrow 0}\left((2\pi\epsilon)^d\mathrm{tr}\left[\underline{f_\eta}(H(\epsilon))\right]\right) \leq \lim_{\epsilon\searrow 0}(2\pi\epsilon)^d N([a,b];\eta) \leq \lim_{\eta\searrow 0}\lim_{\epsilon\searrow 0}\left((2\pi\epsilon)^d\mathrm{tr}\left[\overline{f_\eta}(H(\epsilon))\right]\right),$$

and to apply Theorem 3.1.

6.3 Proof of Theorem 3.3

We only mention the steps in the proof of Theorem 3.3 which are the same as in the proof of Theorem 3.1. Fix $z_0 < \inf(\sigma(H_j))$ $(j=0,1)$, and let $m > d/2+1$. From the assumption (12) the operator $(H_1 - z_0)^{-m} - (H_0 - z_0)^{-m}$ is trace class. Therefore, $f(H_1) - f(H_0)$ is trace class for all $f \in C_0^\infty(\mathbb{R})$. In contrast to the proof of Theorem 3.1, we don't need to introduce the function Ψ, since $f(H_1) - f(H_0)$ is trace class.

As in the proof of (46), Proposition 5.2 and Remark 5.3 yield

$$f(H_0) = -\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \bar{z}}(z) E_+^0 (E_{\mathrm{eff}}^0)^{-1} E_-^0 L(dz),$$

which together with (46) gives

$$\mathrm{tr}\,(f(H_1) - f(H_0)) = \mathrm{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \bar{z}}(z)\left[E_+^1 {E_{\mathrm{eff}}}^{-1} E_-^1 - E_+^0 (E_{\mathrm{eff}}^0)^{-1} E_-^0\right] L(dz)\right), \quad (52)$$

Next, analysis similar to that in the proof of (49) shows that

$$\mathrm{tr}\,(f(H_1) - f(H_0)) = \mathrm{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \bar{z}}(z)\left[{E_{\mathrm{eff}}}^{-1} E_-^1 E_+^1 - (E_{\mathrm{eff}}^0)^{-1} E_-^0 E_+^0\right] L(dz)\right). \quad (53)$$

According to (30), Proposition 5.2 and Remark 5.3, we have

$$\partial_z E_{\mathrm{eff}} = E_-^1 E_+^1, \quad \partial_z E_{\mathrm{eff}}^0 = E_-^0 E_+^0.$$

Combining this with (54), we obtain

$$\mathrm{tr}\,(f(H_1) - f(H_0)) = \mathrm{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde{f}}{\partial \bar{z}}(z)\left[{E_{\mathrm{eff}}}^{-1}\partial_z E_{\mathrm{eff}} - (E_{\mathrm{eff}}^0)^{-1}\partial_z E_{\mathrm{eff}}^0\right] L(dz)\right). \quad (54)$$

We now apply the same arguments after Lemma 6.1, with (49) replaced by (54), to obtain Theorem 3.3.

[4]Notice that the right hand side of (51) is independent modulo $\mathcal{O}(\epsilon^\infty)$ of the choice of ψ, since $\psi = 1$ near the characteristic set Σ_η of E_{eff}.

6.4 Proof of Theorem 3.4

The starting point is formula (54). Let θ and g be C^∞-functions with compact support such that $\theta = 1$ near zero, $g = 1$ on $]\lambda - \eta, \lambda + \eta[$ and $\text{supp}(g) \subset]\lambda - 2\eta, \lambda + 2\eta[$. We choose $\eta > 0$ small enough so that (20) holds on $]\lambda - 2\eta, \lambda + 2\eta[$. Applying (1) and (54) to the function $f(x) = g(x)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - x)$, we obtain

$$\begin{aligned} &- \langle \xi'(\cdot;\epsilon), g(\cdot)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - \cdot)\rangle = \text{tr}\left(g(H_1)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - H_1) - g(H_0)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - H_0)\right) \\ &= \text{tr}\left(-\frac{1}{\pi}\int \frac{\partial \tilde{g}}{\partial \bar{z}}(z)(\mathcal{F}_\epsilon^{-1}\theta)(\lambda - z)\left[{E_{\text{eff}}}^{-1}\partial_z E_{\text{eff}} - (E^0_{\text{eff}})^{-1}\partial_z E^0_{\text{eff}}\right]L(dz)\right). \end{aligned} \tag{55}$$

Here $\tilde{g}$ is an almost analytic extension of g, and $\mathcal{F}_\epsilon^{-1}$ is the semiclassical Fourier transform of θ :

$$(\mathcal{F}_\epsilon^{-1}\theta)(\tau) = \frac{1}{(2\pi\epsilon)}\int_{\mathbb{R}} e^{\frac{i}{\epsilon}t\tau}\theta(t)dt.$$

According to Proposition 5.2, E_{eff} is an ϵ-pseudodifferential operator. On the other hand, the assumption (20) means that the classical symbol corresponding to E_{eff} is non-trapping[5]. The asymptotic expansion with respect to ϵ of an integral similar to the right-hand side of the second equality in (55) have been studied by many authors (see [1, 12, 13, 14, 28] and the references given therein). In particular, under the assumption (20), it follows from the arguments in the proofs of Theorems 2.5 and 2.6 in [12] (see also [1]) that the left-hand side of has a complete asymptotic expansion in powers of ϵ, and

$$\xi'(\tau,\epsilon)g(\tau) = \langle \xi'(\cdot;\epsilon), g(\cdot)(\mathcal{F}_\epsilon^{-1}\theta)(\tau - \cdot)\rangle + \mathcal{O}(\epsilon^\infty),$$

uniformly for $\tau \in]\lambda - 2\eta, \lambda + 2\eta[$. This implies (21). The explicit formula of $\kappa_0(t)$ follows from (19).

Remark 6.2. *We will now show how to treat the case when V depends on x. The only modification to be made is the proof of Proposition 5.1. Fix $m \in \mathbb{N}^*$. By Taylor's formula we have*

$$V(\epsilon x, y) = V(0,y) + \sum_{|\alpha|=1}^{m} \frac{\epsilon^{|\alpha|}}{\alpha!}x^\alpha \frac{\partial^\alpha}{\partial x^\alpha}V(0,y) + \epsilon^{m+1}\mathcal{O}(1) =: V(0,y) + \epsilon W(x,y;\epsilon), \tag{56}$$

uniformly for $(x,y) \in \Omega_d$. Let $\mathcal{P}(y,k)$ and $\mathcal{E}(y,k,z)$ be the operators given in Proposition 5.1 corresponding to the operator $V(y) = V(0,y)$. Now, consider the Grushin problem related to $\tilde{G}(y,k,\epsilon) = G(y,k) + \epsilon W(x,y,\epsilon)$:

$$\tilde{\mathcal{P}}(y,k,,\epsilon) = \begin{pmatrix} \tilde{G}(y,k,\epsilon) - z & R_-(k) \\ R_+(k) & 0 \end{pmatrix} = \mathcal{P}(y,k) + \epsilon \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N \to L^2(\Lambda) \times \mathbb{C}^N_k,$$

[5] A symbol $(y,k) \to A(y,k,z) \in \mathcal{L}(\mathbb{C}^N;\mathbb{C}^N)$ is non-trapping at the energy $z = z_0$ if and only if there exists a scalar escape function $G \in C^\infty(\mathbb{R}^{2d};\mathbb{R})$ such that

$$\exists C > 0,\ \frac{\partial G}{\partial y}\cdot\frac{\partial A}{\partial k} - \frac{\partial G}{\partial k}\cdot\frac{\partial A}{\partial y} \geq C,\ \ \forall (y,k) \text{ with } \det A(y,k,z_0) = 0$$

Since $W(\cdot, y, \epsilon) : \mathcal{H}_{\Lambda,k} \to L^2(\Lambda)$ is uniformly bounded with respect to $y \in \mathbb{R}^d$ and $\epsilon \in [0,1]$, it follows from Proposition 5.1 that, for ϵ small enough the operator $\tilde{\mathcal{P}}(y,k,,\epsilon)$ is bijective with bounded two-sided inverse

$$\tilde{\mathcal{E}}(y,k,z;\epsilon) := \begin{pmatrix} \widehat{G_N}(y,k,z;\epsilon) & E_+(k,z,\epsilon) \\ E_-(k,z,\epsilon) & E_{\text{eff}}(y,k,z;\epsilon) \end{pmatrix} = \left[\mathcal{E}(y,k,z) \left(I + \epsilon \mathcal{E}(y,k,z) \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} \right) \right]^{-1}. \tag{57}$$

From (56) and the above equality it follows that, modulo $\mathcal{O}(\epsilon^{m+1})$, $\tilde{\mathcal{E}}(y,k,z;\epsilon)$ has an asymptotic expansion in powers of ϵ in $S^0(\mathbb{R}^{2d}; \mathcal{L}(L^2(\Lambda) \times \mathbb{C}_k^N; \mathcal{H}_{\Lambda,k} \times \mathbb{C}^N))$. This gives Proposition 5.1 when V depends on (x,y).

We can now proceed analogously to the proof of the case $V = V(y)$.

7 Appendix : Proof of Theorem 2.3

Fix $J = (j_1, j_2, \cdots, j_d) \in \mathbb{N}^{*d}$, and let $e_J(k) = e_{j_1}(k_1) + \cdots + e_{j_d}(k_d)$ be one eigenvalue of the operator $H_0(k)$. Set

$$\kappa(t) = \int_{\{k \in \mathbb{R}^d; e_J(k) \leq t\}} dk.$$

Lemma 7.1. *The function κ is analytic in a neighborhood of $\mathbb{R} \setminus \{e_J(0)\}$.*

Proof. Fix $t_0 \neq e_J(0)$, and let ε be a small positive constant such that $\nabla e_J(k) \neq 0$ when $k \in \Sigma_\epsilon(t_0) := e_J^{-1}(]t_0 - \varepsilon, t_0 + \varepsilon[)$. Without any loss of generality we may assume that $\partial_{k_1} e_J(k) \neq 0$ for all $k \in \Sigma_\epsilon(t_0)$. By the change of variable $U : k \mapsto \tilde{k} = (e_J(k), k_2, ..., k_d)$, we have

$$\int_{\{k \in \Sigma_\epsilon(t_0)\,;\, e_J(k) \leq t\}} dk = \int_{\{\tilde{k} \in U(\Sigma_\epsilon(t_0))\,;\, \tilde{k}_1 \leq t\}} \text{Jac}(U^{-1}(\tilde{k})) d\tilde{k},$$

where $\text{Jac}(U^{-1}(\tilde{k}))$ denotes the Jacobian determinant of U^{-1}. Clearly the right-hand side of the above equality is analytic. Combining this with the fact that $\int_{\{k \in \mathbb{R}^d \setminus \Sigma_\epsilon(t_0)\,;\, e_J(k) \leq t\}} dk$ is constant for t near t_0, we get the lemma. □

Thus, the function ρ is analytic in a neighborhood of $\Sigma = \mathbb{R} \setminus \sigma(H_0(0))$. The remainder of the proof of Theorem 2.3 is a simple consequence of the following lemma.

Lemma 7.2. *There exists an analytic function g with $g(s) \sim_{s \to 0} \frac{\text{vol}(S^{d-1})}{d\sqrt{\det(\frac{\nabla^2 e_J(0)}{2})}} s^d$ such that*

$$\kappa(t) = Y(t - e_J(0)) g(\sqrt{t - e_J(0)}),$$

for $|t - e_J(0)|$ small enough. Here $Y(t)$ is the Heaviside function, and S^{d-1} stands for the unit sphere in $\mathbb{R}^d$.

Proof. By Morse Lemma there exist a neighborhood $\mathcal{V}$ of $k = 0$, $\varepsilon > 0$ and a local analytic diffeomorphism $\mathcal{D} : \mathcal{V} \to B(0, \epsilon)$ satisfying $\mathcal{D}(k) = k + \mathcal{O}(k^2)$ such that

$$e_J \circ \mathcal{D}^{-1}(k) = e_J(0) + \frac{1}{2} \langle \nabla^2 e_J(0) k, k \rangle.$$

On the other hand, for $|t - e_J(0)|$ small enough we have

$$\{k \in \mathbb{R}^d; e_J(k) \le t\} = \{k \in \mathcal{V}\,;\, e_J(k) \le t\}.$$

Thus making the change of variable $k = \mathcal{D}^{-1}(\xi)$ and using polar coordinates, we obtain

$$\begin{aligned}\kappa(t) &= \int_{\{k\in\mathcal{V}\,;\,e_J(k)\le t\}} dk = \left(\det\left(\frac{\nabla^2 e_J(0)}{2}\right)\right)^{-1/2} \int_{\{\xi\in B(0,\epsilon)\,;\,|\xi|^2\le t-e_J(0)\}} \mathrm{Jac}(\mathcal{D}^{-1}(\xi))d\xi \\ &= \left(\det\left(\frac{\nabla^2 e_J(0)}{2}\right)\right)^{-1/2} \int_0^{\sqrt{\max(t-e_J(0),0)}} \int_{S^{d-1}} \mathrm{Jac}(\mathcal{D}^{-1}(r\omega))\, r^{d-1} dr d\omega,\end{aligned}$$

which yields the lemma since $\mathrm{Jac}(\mathcal{D}^{-1}(r\omega)) = 1 + \mathcal{O}(r)$. □

We now turn to the proof of Theorem 2.3. For $t_0 \in \Sigma$, we let $\mathcal{S}_{t_0} := \{J \in \mathbb{N}^d; e_J(0) = t_0\}$ and $m_{t_0} := \#\mathcal{S}_{t_0}$ be its multiplicity. Writing

$$\rho(t) = \underbrace{\sum_{(j_1,\cdots,j_d)\notin\Sigma_{t_0}} \int_{\{k\in\mathbb{R}^d; e_{j_1}(k_1)+\cdots+e_{j_d}(k_d)\le t\}} dk}_{(1)} + \underbrace{\sum_{(j_1,\cdots,j_d)\in\Sigma_{t_0}} \int_{\{k\in\mathbb{R}^d; e_{j_1}(k_1)+\cdots+e_{j_d}(k_d)\le e\}} dk}_{(2)}.$$

It follows from Theorem 2.1 that $\nabla_k e_J(k) = \nabla_k(e_{j_1}(k_1) + \cdots + e_{j_d}(k_d)) \ne 0$ on $\Sigma_\eta(t_0)$ for η small enough and $(j_1, \cdots, j_d) \notin \mathcal{S}_{t_0}$. Combining this with Lemma, we deduce that (1) is analytic for $|t - t_0|$ small enough. Thus applying Lemma 7.2 to each term of (2) we get Theorem 2.3.

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