ARCHIMEDEAN NON-VANISHING AND COHOMOLOGICAL TEST VECTOR

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ABSTRACT. The standard L-functions of GL_{2n} expressed in terms of the Friedberg-Jacquet global zeta integrals have better structure for arithmetic applications, due to the relation of the linear periods with the modular symbols. In this paper, we just give an overview for our recent work on the archimedean local integrals of Friedberg-Jacquet([CJLT19],[LT20]). We will focus on the complex case, explicitly construct a uniform cohomological test vector v for a new twisted linear functional $\Lambda_{s,\chi}$ and establish the non-vanishing property for the archimedean local Friedberg-Jacquet integral when evaluating at v.

1. INTRODUCTION

Let k be a number field, and $\mathbb{A}$ be the ring of adeles of k. Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$. The standard L-function $L(s,\pi\otimes\chi)$ of π, twisted by an idèle-class character χ of $k^\times$, was first studied by R. Godement and H. Jacquet in 1972 ([GJ72]), and then by the Rankin-Selberg convolution method of Jacquet, I. Piatetski-Shapiro and J. Shalika in 1983 ([JPSS83]). In 1993, S. Friedberg and Jacquet found in [FJ93] a new global zeta integral for $L(s,\pi\otimes\chi)$, assuming that π has a non-zero Shalika period.

Let ω_π be the central character of π and take an idele-class character ω such that $\omega^n=\omega_\pi$. The global zeta integral $Z(\varphi_\pi,\chi,\omega,s)$ of Friedberg-Jacquet is given by

$$\int_{[\mathrm{GL}_n\times\mathrm{GL}_n]}\varphi_\pi(\begin{pmatrix} g_1 & 0\\ 0 & g_2\end{pmatrix})|\frac{\det g_1}{\det g_2}|^{s-\frac{1}{2}}\chi(\frac{\det g_1}{\det g_2})\omega(\det g_2)dg_1dg_2, \tag{1.1}$$

where $\varphi_\pi\in\pi$, $[\mathrm{GL}_n\times\mathrm{GL}_n]:=Z_{2n}(\mathbb{A})(\mathrm{GL}_n(k)\times\mathrm{GL}_n(k))\backslash(\mathrm{GL}_n(\mathbb{A})\times\mathrm{GL}_n(\mathbb{A}))$ with Z_{2n} the center of GL_{2n}. In [FJ93, Proposition 2.3], it is proved that $Z(\varphi_\pi,\chi,\omega,s)$ converges absolutely for all $s\in\mathbb{C}$. Particularly, for $\mathrm{Re}(s)$ sufficiently large, it is equal to the absolutely convergent integral

$$Z(\varphi_\pi,\chi,s):=\int_{\mathrm{GL}_n(\mathbb{A})}\mathcal{S}^\omega_\psi(\varphi_\pi)\begin{pmatrix} g & 0\\ 0 & \mathrm{I}_n\end{pmatrix}\chi(\det g)|\det g|^{s-\frac{1}{2}}dg, \tag{1.2}$$

where ψ is a non-trivial additive character of $k\backslash\mathbb{A}$ and $\mathcal{S}^\omega_\psi$ is the global Shalika period of φ_π that is defined as follows. Let S be the Shalika subgroup of GL_{2n} consisting of matrices of the form

$$s(x,g)=\begin{pmatrix} g & 0\\ 0 & g\end{pmatrix}\begin{pmatrix} \mathrm{I}_n & x\\ 0 & \mathrm{I}_n\end{pmatrix},$$

where $x\in\mathrm{M}_n$ and $g\in\mathrm{GL}_n$. Define $\theta(s(x,g)):=\omega(\det g)\psi(\mathrm{Tr}(x))$. The Shalika period is defined by

$$\mathcal{S}^\omega_\psi(\varphi_\pi)(h):=\int_{Z_{2n}(\mathbb{A})S(k)\backslash S(\mathbb{A})}\varphi_\pi(s(x,g)h)\theta^{-1}(s(x,g))ds.$$

By the local uniqueness of the Shalika model ([Ni09], [AGJ09], and [CS19]), for the factorizable $\varphi_\pi=\otimes'_v\varphi_v$, one has that $(\mathcal{S}^\omega_\psi(\varphi_\pi))(h)=\prod_v(\mathcal{S}^{\omega_v}_{\psi_v}(\varphi_v))(h_v)$ with $\mathcal{S}^{\omega_v}_{\psi_v}(\varphi_v)$ being the local

2010 *Mathematics Subject Classification.* Primary 22E45; Secondary 11F67.

Key words and phrases. Non-Vanishing of Archimedean Local Integral, Linear Model, Shalika Model, Friedberg-Jacquet Integral, Cohomological Test Vector, Standard L-functions for General Linear Groups.

Shalika function associated to the local Shalika model at each place v, and an euler product decomposition:

$$Z(\varphi_\pi, \chi, s) = \prod_v Z_v(\mathcal{S}^{\omega_v}_{\psi_v}(\varphi_v), \chi_v, s),$$

where the local zeta integrals are defined by

$$Z_v(\varphi_v, \chi_v, s) := \int_{\mathrm{GL}_n(k_v)} \mathcal{S}^{\omega_v}_{\psi_v}(\varphi_v)\begin{pmatrix} g & 0 \\ 0 & \mathrm{I}_n \end{pmatrix} \chi_v(\det g)|\det g|_v^{s-\frac{1}{2}} dg. \tag{1.3}$$

Furthermore, it is proved that the local zeta integral $Z_v(\varphi_v, \chi_v, s)$ is a holomorphic multiple of the local L-function $L(s, \pi_v \otimes \chi_v)$ ([FJ93] and [AGJ09]). It is clear that the Friedberg-Jacquet global zeta integral for $L(s, \pi\otimes\chi)$ is a natural generalization of the Hecke zeta integral for GL_2 ([JL70]).

In the three constructions of different global zeta integrals for $L(s, \pi \otimes \chi)$, it seems that the Friedberg-Jacquet global zeta integral for $L(s, \pi\otimes\chi)$ is better for arithmetic applications with π being cohomological, since the construction is closely related to the generalized modular symbols ([AB89]). For instance, the work of A. Ash and D. Ginzburg ([AG94]) that constructs p-adic L-functions for GL_{2n}; the work of Grobner and A. Raghuram ([GR14]) that studies arithmetic properties of the critical values of $L(s, \pi\otimes\chi)$ and the recent work of D. Jiang, B. Sun and F. Tian ([JST19]) that establishes the period relations of the critical values at different critical places for the automorphic L-functions $L(s, \pi \otimes \chi)$ are all focus on the Friedberg-Jacquet integrals for cohomological representations. Among these applications, there are two basic assumptions:

(1) **Non-vanishing Assumption**: $Z_v(\varphi_v, \chi_v, \frac{1}{2}) \neq 0$ for suitable cohomological vector φ_v in cohomological representation π_v when $k_v = \mathbb{R}$ and $\mathbb{C}$.

(2) **Uniform Cohomological Test Vector**: The archimedean local zeta integral $Z_v(\varphi_v, \chi_v, s)$ admits a uniform cohomological test vector φ_v in the sense that

$$\frac{1}{L(s, \pi_v \otimes \chi_v)} Z_v(\varphi_v, \chi_v, s) = 1$$

holds for all complex values $s \in \mathbb{C}$.

For (1), A. Ash and D. Ginzburg only show that such assumption is satisfied for $n = 2$ in [AG94]. For (2), the best result to the date is Sun's existence of cohomological test vector in [Sun19, Theorem 5.1], which shows that for any irreducible essentially tempered cohomological Casselman-Wallach representation π_v of $\mathrm{GL}_{2n}(\mathbb{R})$ and every $s \in \mathbb{C}$, there exists a cohomological vector $\varphi_{v,s}$, depending on s, such that the normalized Friedberg-Jacquet integral

$$\frac{1}{L(s, \pi_v \otimes \chi_v)} Z_v(\varphi_{v,s}, \chi_v, s) = 1.$$

As explained [JST19], this is not enough to obtain the global period relation of the critical values of the twisted standard L-functions $L(s, \pi \otimes \chi)$ at different critical places.

The objective of our recent work [CJLT19],[LT20] is to develop a constructive approach towards Problems (1) and (2) for archimedean case, which is complementary to the approach taken by Sun in [Sun19]. In this paper, we will sketch out the explicit construction of the cohomological test vectors. The strategies between real and complex cases are similar. However, the latter case has extra complications. We will focus on the complex case in this paper.

2. Cohomological Representations, Shalika models and linear models

For further arithmetic application, we often consider the cohomological representations. Hence, as in [JST19], we assume that the cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A})$ is regular and algebraic in the sense of Clozel([Cl88]). Under this condition, Clozel shows that the archimedean local representation is essentially tempered and cohomological. In this section, we will give the description for the cohomological representations of $\mathrm{GL}_{2n}(\mathbb{K})$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) with Shalika models and construct a new linear model for explicit computation.

2.1. **Cohomological Representations of $\mathrm{GL}_{2n}(\mathbb{K})$.** Let $G = \mathrm{GL}_{2n}(\mathbb{K})$ with center Z, and K be the maximal compact subgroup of G. Set

$$H = \left\{ \begin{pmatrix} g_1 & \\ & g_2 \end{pmatrix} \middle| g_1, g_2 \in \mathrm{GL}_n(\mathbb{K}) \right\} \simeq \mathrm{GL}_n(\mathbb{K}) \times \mathrm{GL}_n(\mathbb{K}).$$

Let B be the standard Borel subgroup of G consisting of all upper-triangular matrices in G. We fix the usual root system of G so that B contains all simple root vectors. Then the half sum of all positive roots, denoted by ρ, is

$$\rho = (\frac{2n-1}{2}, \frac{2n-3}{2}, \cdots, \frac{3-2n}{2}, \frac{1-2n}{2}). \tag{2.1}$$

To fix notation, we will use capital letters G, H etc. for certain Lie groups, G^0, H^0 etc. for their identity components, German letters $\mathfrak{g}, \mathfrak{h}$ etc. for their Lie algebras, and $\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}$ for the complexifications of the Lie algebras.

Real Case: Let F_ν be a highest weight representation of $\mathrm{GL}_{2n}(\mathbb{C})$ with highest weight ν, which can be written as a vector: $\nu = (\nu_1, \nu_2, \cdots, \nu_{2n}) \in \mathbb{Z}^{2n}$, with $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{2n}$. Let π be an irreducible essentially tempered Casselman-Wallach representations of $\mathrm{GL}_{2n}(\mathbb{R})$ with property that the total relative Lie algebra cohomology

$$\mathrm{H}^*(\mathfrak{g}, K^0 Z^0, \pi \otimes F_\nu^\vee) \neq 0.$$

Here we also use π for its underlying $(\mathfrak{g}, K)$-module when no confusion arises. By [Cl88, Section 3], the highest weight ν satisfies the following purity condition:

$$\nu_1 + \nu_{2n} = \nu_2 + \nu_{2n-1} = \cdots = \nu_{2n} + \nu_1 := m \in \mathbb{Z}. \tag{2.2}$$

Meanwhile, we must also have that

$$n^2 \leq j \leq n^2 + n - 1.$$

One should observe that the top non-vanishing degree $n^2 + n - 1$ is exactly the dimension d of the quotient space $\mathrm{Lie}(H)/\mathrm{Lie}((K \cap H)Z)$, where $\mathrm{Lie}(H)$ is the Lie algebra of H. It is the same as the dimension of the modular symbol generated by H(ref.[AG96]). Now set

$$l_i = \nu_i - \nu_{2n+1-i} + (2n + 1 - 2i) \quad \text{for all } 1 \leq i \leq 2n. \tag{2.3}$$

We note that all l_i share the same parity, which is different from the parity of m. For each positive integer k, we write D_k for the relative discrete series of $\mathrm{GL}_2(\mathbb{R})$ with quadratic central character whose minimal K-type has highest weight $k+1$. Then we have the langlands parameter for π(see [Ma05, Section 3.1] and [GR14, Section 3.4])

Proposition 2.1. *Let (π, V_π) be an irreducible essentially tempered Casselman-Wallach representation of $GL_{2n}(\mathbb{R})$ with property that*

$$H^*(\mathfrak{g}, K^0 Z^0, \pi \otimes F_\nu^\vee) \neq 0.$$

Then π is equivalent to the normalized induced representation

$$Ind_P^{GL_{2n}(\mathbb{R})} D_{l_1} |\det|^{\frac{m}{2}} \otimes D_{l_2} |\det|^{\frac{m}{2}} \otimes \cdots \otimes D_{l_n} |\det|^{\frac{m}{2}},$$

where P is the standard parabolic subgroup of $GL_{2n}(\mathbb{R})$ associated with the partition $[2^n]$. Moreover, the minimal K-type τ of π has the highest weight $(l_1 + 1, l_2 + 1, \cdots, l_n + 1)$.

In this situation, the central character of π takes the form

$$\omega_\pi(a\mathrm{I}_{2n}) = \begin{cases} |a|^{mn} & m \text{ is even;} \\ |a|^{mn}(\mathrm{sgn}(a))^n & m \text{ is odd,} \end{cases}$$

We defines a character of $\mathbb{R}^\times$ as follows:

$$\omega(a) = \begin{cases} |a|^{m} & m \text{ is even;} \\ |a|^{m}\mathrm{sgn}(a) & m \text{ is odd,} \end{cases}$$

which is exactly the cental character of $D_{l_j}|\det|^{\frac{m}{2}}$ for all j and $\omega_\pi = \omega^n$.

Complex Case: Let ρ_1, ρ_2 be two highest weight representations of $\mathrm{GL}_{2n}(\mathbb{C})$ with highest weights $(\nu_1, \nu_2, \cdots, \nu_{2n})$ and $(\nu_{2n+1}, \nu_{2n+2}, \cdots, \nu_{4n})$ respectively, where each $\nu_j \in \mathbb{Z}$ and

$$\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{2n}; \qquad \nu_{2n+1} \geq \nu_{2n+2} \geq \cdots \geq \nu_{4n}.$$

We consider a complex finite dimensional representation (ρ, F_ν) of the real algebraic group $\mathrm{GL}_{2n}(\mathbb{C})$ defined by $\rho(g) := \rho_1(g) \otimes \rho_2(\bar{g})$. For consistency, we denote ν_{2n+j} by $\bar{\nu}_j$ for all $j = 1, 2, \cdots, 2n$. The $4n$-integers

$$\nu = (\nu_1, \nu_2, \cdots, \nu_{2n}; \bar{\nu}_1, \bar{\nu}_2, \cdots, \bar{\nu}_{2n}) \tag{2.4}$$

is called the highest weight of F_ν. Similar to the real case, we assume that ν satisfies the purity condition as in [Cl88]. Namely, there exists an integer $m \in \mathbb{Z}$ such that for all $j = 1, 2, \cdots, 2n$,

$$\bar{\nu}_j + \nu_{2n-j+1} = m. \tag{2.5}$$

Let (π, V_π) be an irreducible essentially tempered Casselman-Wallach representation of $\mathrm{GL}_{2n}(\mathbb{C})$ such that the total relative Lie algebra cohomology

$$H^*(\mathfrak{g}, K, \pi \otimes F_\nu^\vee) \neq 0.$$

By [Cl88, Lemma 3.14], the j^{th}-cohomology group

$$H^j(\mathfrak{g}, K, \pi \otimes F_\nu^\vee) \neq 0 \Longleftrightarrow 2n^2 - n \leq j \leq 2n^2 + n - 1.$$

In particular, when j is taken to be the dimension $d = 2n^2 - 1$ of the modular symbol generated by H, the cohomology is nonzero. Now we recall the Langlands parameter for π, which is discussed in [Ra16, Section 2.4.2].

Proposition 2.2. *Let (π, V_π) be an irreducible essentially tempered Casselman-Wallach representation of $GL_{2n}(\mathbb{C})$ such that the relative Lie algebra cohomology*

$$H^*(\mathfrak{g}, K, \pi \otimes F_\nu^\vee) \neq 0.$$

Then π is equivalent to the principal series representation

$$Ind_B^{GL_{2n}(\mathbb{C})} z^{\nu_1 + \frac{2n-1}{2}} \bar{z}^{m-\nu_1-\frac{2n-1}{2}} \otimes z^{\nu_2+\frac{2n-3}{2}} \bar{z}^{m-\nu_2-\frac{2n-3}{2}} \otimes \cdots \otimes z^{\nu_{2n}+\frac{1-2n}{2}} \bar{z}^{m-\nu_{2n}-\frac{1-2n}{2}}, \tag{2.6}$$

where ν and m are described in (2.4) and (2.5) resp.. B is the standard Borel subgroup of G.

For any integer L, denote by χ_L the unitary character of $\mathbb{C}^\times$ sending z to $(\frac{z}{|z|})^L$. We set $l_j = 2\nu_j + (2n+1-2j) - m$, then $(l_1, l_2, \cdots, l_{2n})$ is a sequence of integers in a strictly decreasing order such that $l_j + l_{2n+1-j} = 2\nu_j + 2\nu_{2n+1-j} - 2m$ is an even integer. We then rewrite the cohomological representation π given in (2.6) as

$$\pi \simeq \mathrm{Ind}_B^G | \quad |_{\mathbb{C}}^{\frac{m}{2}} \chi_{l_1} \otimes | \quad |_{\mathbb{C}}^{\frac{m}{2}} \chi_{l_2} \otimes \cdots \otimes | \quad |_{\mathbb{C}}^{\frac{m}{2}} \chi_{l_{2n}}. \tag{2.7}$$

2.2. **Shalika model and linear model.** Let us fix a non-trivial unitary character ψ of $\mathbb{K}$ and a multiplicative character ω of $\mathbb{K}^\times$. We say a Casselman-Wallach representation π of G has a non-zero $(\omega, \psi)-$Shalika model if there exists a non-zero continuous linear functional λ on the Fréchet space V_π, which is called a *Shalika functional*, such that

$$\langle \lambda, \pi(\begin{pmatrix} \mathrm{I}_n & Y \\ & \mathrm{I}_n \end{pmatrix} \begin{pmatrix} g & \\ & g \end{pmatrix})v \rangle = \omega(\det g)\psi(\mathrm{tr}(Y))\langle \lambda, v \rangle, \tag{2.8}$$

for any $v \in V_\pi$, $g \in \mathrm{GL}_n(\mathbb{K})$ and any $n \times n$ matrix $Y \in \mathrm{M}_n(\mathbb{K})$. For a character χ of $\mathbb{K}^\times$, the local archimedean integral of Friedberg-Jacquet as in (1.3) can be re-written as

$$Z(v, s, \chi) = \int_{\mathrm{GL}_n(\mathbb{K})} \langle \lambda, \pi \begin{pmatrix} g & \\ & \mathrm{I}_n \end{pmatrix} v \rangle |\det g|_{\mathbb{K}}^{s-\frac{1}{2}} \chi(\det g) dg. \tag{2.9}$$

By [AGJ09, Theorem 3.1], the integral (2.9) converges absolutely when $\mathrm{Re}(s)$ is sufficiently large. $Z(v, s, \chi)$ is a homomorphic multiple of $L(s, \pi \otimes \chi)$ in the sense of meromorphic continuation and there exists a smooth vector $v \in V_\pi$ such that $Z(v, s, \chi) = L(s, \pi \otimes \chi)$. Thus whenever

$s = s_0$ is not a pole of $L(s, \pi\otimes\chi)$, $Z(v, s, \chi)$ has no pole at $s = s_0$. This implies that the map: $v \mapsto Z(v, s_0, \chi)$ defines a nonzero element in

$$\mathrm{Hom}_H(V_\pi, |\det|_{\mathbb{K}}^{-s_0+\frac{1}{2}}\chi^{-1}(\det)\otimes|\det|_{\mathbb{K}}^{s_0-\frac{1}{2}}(\chi\omega)(\det)),$$

which is called the space of *twisted linear functionals* of π. The uniqueness of the twisted linear model is proved in [CS19]. In our scenario, we apply [CS19, Theorem B] and conclude that for all but countably many characters χ,

$$\dim \mathrm{Hom}_H(\pi, |\det|_{\mathbb{K}}^{-s_0+\frac{1}{2}}\chi^{-1}(\det)\otimes|\det|_{\mathbb{K}}^{s_0-\frac{1}{2}}(\chi\omega)(\det)) \le 1. \tag{2.10}$$

In fact, if $|\det|^{-s_0+\frac{1}{2}}\chi^{-1}(\det)\otimes|\det|^{s_0-\frac{1}{2}}(\chi\omega)(\det)$ is a good character of H, then (2.10) holds. For the precise definition of a good character of H, we refer to [CS19]. Here we remark that when χ and ω are both trivial and $s_0=\frac{1}{2}$, the uniqueness theorem is proved in [AG09].

2.3. **Cohomological Representations with Shalika Models.** The goal of this subection is to prove a hereditary property of Shalika models with respect to normalized parabolic induction. Let us first start from the case of $\mathrm{GL}_2(\mathbb{K})$. Let σ be a generic Casselman-Wallach representation of $\mathrm{GL}_2(\mathbb{K})$ with a central character ω_σ. We fix a nontrivial unitary character ψ of $\mathbb{K}$. Then σ admits a non-zero Whittaker model $\mathcal{W}(\sigma,\psi)$, i.e. there exists a continuous linear functional λ_σ on the Fréchet space V_σ such that

$$\langle\lambda_\sigma, \sigma(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}a & \\ & a\end{pmatrix})v\rangle = \omega_\sigma(a)\psi(x)\langle\lambda_\sigma, v\rangle,$$

which exactly coincides with (2.8) for $n=1$. Hence any such σ has a non-zero $(\omega_\sigma,\psi)-$Shalika model. Generally, suppose that we have l Frechét spaces $V_1, V_2, \cdots, V_l$. Let λ_j be a continuous linear functional on V_j $(j=1,2,\cdots,l)$. Then $\bigotimes_{j=1}^l \lambda_j$ is a continuous linear functional on the projective tensor space $\widehat{\bigotimes}_{j=1}^l V_j$, which is also a Fréchet space.

Theorem 2.3. *Let ω be a character of $\mathbb{K}^\times$ and ψ be a nontrivial unitary character of $\mathbb{K}$. For two positive even integers $n_1 = 2m_1$ and $n_2 = 2m_2$, take two Casselman-Wallach representations π_1 and π_2 of $GL_{n_1}(\mathbb{K})$ and $GL_{n_2}(\mathbb{K})$, respectively, and assume that both π_1 and π_2 have $(\omega,\psi)-$Shalika models. Then the normalized parabolic induction $\pi := Ind_{P_{n_1,n_2}(\mathbb{K})}^{GL_{n_1+n_2}(\mathbb{K})}\pi_1\otimes\pi_2$ also has a non-zero $(\omega,\psi)-$Shalika model. Here P_{n_1,n_2} is a standard parabolic subgroup of $GL_{n_1+n_2}$ with its Levi part isomorphic to $GL_{n_1}\times GL_{n_2}$.*

Proof. We can show this theorem by constructing a new linear function, which satisies the equivariant property (2.8). Please refer to [CJLT19, Theorem 2.1] for more details. □

Then combining Theorem 2.3 and the example on $\mathrm{GL}_2(\mathbb{K})$ discussed at the beginning of this section, we have

Corollary 2.4. *Any irreducible essentially tempered Casselman-Wallach representation (π, V_π) of $GL_{2n}(\mathbb{R})$ with*

$$H^*(\mathfrak{g}, K^0Z^0, \pi\otimes F_\nu^\vee)\neq 0$$

has a non-zero Shalika model defined by the Shalika functional as in (2.8).

However, not all ireducible essentially tempered cohomological Casselman-Wallach representations of $\mathrm{GL}_{2n}(\mathbb{C})$ has Shallika modules automatically. Actually, we have the following description for such kind of representations with Shalika models.

Theorem 2.5. *Let π be the cohomological representation given in (2.7) with a central character ω_π. Given a non-trivial additive character ψ of $\mathbb{C}$ and a multiplicative character ω of $\mathbb{C}^\times$ such that $\omega^n = \omega_\pi$, we have the following equivalent statements:*

(1) *π has a non-zero (ω,ψ)-Shalika model defined at the beginning of the Introduction;*

(2) *There exists a discrete, countable subset $S \subset \mathbb{C}$ such that for every complex number $s \notin S$, π has a nonzero twisted linear model for arbitrary character χ of $\mathbb{C}^\times$, i.e.*

$$\mathrm{Hom}_H(V_\pi, |\det|_{\mathbb{C}}^{-s+\frac{1}{2}}\chi^{-1}(\det) \otimes |\det|_{\mathbb{C}}^{s-\frac{1}{2}}(\chi\omega)(\det)) \neq 0.$$

(3) *There exists an integer L such that $l_j + l_{2n+1-j} = 2L$.*

Proof. **(3)⇒(1):** If $l_j+l_{2n+1-j} = 2L$ holds, then we can rewrite π as the normalized parabolically induced representation

$$\pi \simeq \mathrm{Ind}_P^{\mathrm{GL}_{2n}(\mathbb{C})}\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n, \tag{2.11}$$

where P is the standard parabolic subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ associated with the partition $[2^n]$, and each σ_j is the principal series of $\mathrm{GL}_2(\mathbb{C})$:

$$\sigma_j := \mathrm{Ind}_{B_2}^{\mathrm{GL}_2(\mathbb{C})}|\det|_{\mathbb{C}}^{\frac{m}{2}}\chi_{l_j} \otimes |\det|_{\mathbb{C}}^{\frac{m}{2}}\chi_{l_{2n+1-j}}. \tag{2.12}$$

All the principal series σ_j share the same central character $\omega = |\cdot|_{\mathbb{C}}^m\chi_{2L}$. Hence all σ_j have nonzero Shalika models associated with the same characters ω and ψ. Then by Theorem 2.3(see [CJLT19, Thoerem 2.1]), π automatically has a nonzero (ω,ψ)-Shalika model.

(1)⇒(2): We assume that π has a nonzero (ω,ψ)-Shalika model. Then the local integral $Z(v,s,\chi)$ defines a nonzero element in

$$\mathrm{Hom}_H(V_\pi, |\det|_{\mathbb{C}}^{-s+\frac{1}{2}}\chi^{-1}(\det) \otimes |\det|_{\mathbb{C}}^{s-\frac{1}{2}}(\chi\omega)(\det)),$$

whenever s is not a pole of the standard local L-function $L(s,\pi\otimes\chi)$. Thus, Statement **(2)** follows from the fact that the poles of the local L-function $L(s,\pi\otimes\chi)$ form a countable discrete subset of $\mathbb{C}$.

(2)⇒(3): This is the most difficult part of the proof. Please refer [LT20, Theorem 2.1] for more details. □

2.4. **A New construction of Linear Model.** In this subsection, we will construct another linear period so that we do the computation more easily. Let π be an irreducible essentially tempered cohomological representation of $G = \mathrm{GL}_{2n}(\mathbb{K})$ with a nonzero (ω,ψ)-Shalika model. According to the discussion in previous subsections, we see that π must be isomorphic to the normalized parabolically induced representation

$$\mathrm{Ind}_P^G\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_n, \tag{2.13}$$

where P is the standard parabolic subgroup of G associated with the partition $[2^n]$, and $\sigma_j = D_{l_j}|\det|^{\frac{m}{2}}$ or $\mathrm{Ind}_{B_{\mathrm{GL}_2}}^{\mathrm{GL}_2(\mathbb{C})}|\cdot|_{\mathbb{C}}^{\frac{m}{2}}\chi_{l_j} \otimes |\cdot|_{\mathbb{C}}^{\frac{m}{2}}\chi_{l_{2n+1-j}}$ when $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ respectively. It is clear that all σ_j are generic and share the same central character ω. Fix a character χ of $\mathbb{K}^\times$. For each σ_j, which admits a nonzero Whittaker model $\mathcal{W}(\sigma_j,\psi)$, the archimedean local integral

$$\lambda_{s,j}(v) := \int_{\mathbb{K}^\times} W_v(\begin{pmatrix} a & \\ & 1\end{pmatrix}) |a|_{\mathbb{K}}^{s-\frac{1}{2}}\chi(a)d^\times a \tag{2.14}$$

has a meromorphic continuation to the whole complex plane, and it is a holomorphic multiple of the L-function $L(s,\sigma_j\otimes\chi)$. Whenever $s = s_0$ is not a pole of the $L-$function $L(s,\sigma_j\otimes\chi)$, $\lambda_{s_0,j}$ defines a nonzero continuous linear functional in

$$\mathrm{Hom}_{\mathrm{GL}_1(\mathbb{K})\times\mathrm{GL}_1(\mathbb{K})}(\sigma_j, |\quad|_{\mathbb{K}}^{\frac{1}{2}-s_0}\chi^{-1} \otimes |\quad|_{\mathbb{K}}^{s_0-\frac{1}{2}}\omega\chi).$$

Then we consider a continuous linear functional $\bigotimes_{j=1}^n \lambda_{s,j}$ on the projective tensor product $\hat{\bigotimes}_{j=1}^n V_{\sigma_j}$ of Fréchet spaces. Take $\varphi \in V_\pi$, we can define a function on H by

$$F_s(h;\varphi) = \langle \bigotimes_{j=1}^n \lambda_{s,j}, \varphi(wh)\rangle, \tag{2.15}$$

where w is inverse of the Weyl element which changes the sequence $(1,2,3,\cdots,2n)$ to $(1,3,\cdots,2n-1,2,4,\cdots,2n)$. Let B_1 and B_2 be the standard Borel subgroups of $\mathrm{GL}_n(\mathbb{K})$. By the equivariance of $\lambda_{s,j}$, it is easy to check that $F_s(h;\varphi)$ satisfies a $B_1\times B_2$ equivariant property: for $(b_1,b_2)\in B_1\times B_2$

$$F_s(\begin{pmatrix} b_1 & \\ & b_2\end{pmatrix} h;\varphi)=\delta_{B_1}(b_1)\delta_{B_2}(b_2)\chi_{1,s}(\det b_1)\cdot\chi_{2,s}(\det b_2)\cdot F_s(h;\varphi), \tag{2.16}$$

where δ_{B_i} is the modular character of $B_i(i=1,2)$, $\chi_{1,s}=|\cdot|_{\mathbb{K}}^{\frac{1}{2}-s}\chi^{-1}$, and $\chi_{2,s}=|\cdot|_{\mathbb{K}}^{s-\frac{1}{2}}\chi\omega$. Now we define a continuous linear functional $\Lambda_{s,\chi}$ on V_π by the following convergent integral

$$\Lambda_{s,\chi}(\varphi):=\int_{K\cap H}F_s(\begin{pmatrix} k_1 & \\ & k_2\end{pmatrix};\varphi)\chi_{1,s}^{-1}(\det k_1)\chi_{2,s}^{-1}(\det k_2))dk_1dk_2. \tag{2.17}$$

In terms of φ, $\Lambda_{s,\chi}$ can be rewritten as:
(2.18)

$$\Lambda_{s,\chi}(\varphi)=\int_{K\cap H}\langle\bigotimes_{j=1}^{n}\lambda_{s,j},\varphi(w\begin{pmatrix} k_1 & \\ & k_2\end{pmatrix})\rangle\chi_{1,s}^{-1}(\det k_1)\chi_{2,s}^{-1}(\det k_2))dk_1dk_2=\langle\bigotimes_{j=1}^{n}\lambda_{s,j},\tilde{\varphi}(w)\rangle,$$

where $\tilde{\varphi}$ is obtained by averaging φ against the character $\chi_{1,s}^{-1}(\det k_1)\chi_{2,s}^{-1}(\det k_2)$ over the compact group $K\cap H$. In particular, if φ satisfies the right $K\cap H$-equivariant property:

$$\varphi(g\begin{pmatrix} k_1 & \\ & k_2\end{pmatrix})=\chi_{1,s}(\det k_1)\cdot\chi_{2,s}(\det k_2)\cdot\varphi(g)=\chi^{-1}(\det k_1)\cdot(\omega\chi)(\det k_2)\cdot\varphi(g), \tag{2.19}$$

then

$$\Lambda_{s,\chi}(\varphi)=\langle\bigotimes_{i=1}^{n}\lambda_{s,i},\tilde{\varphi}(w)\rangle=\langle\bigotimes_{i=1}^{n}\lambda_{s,i},\varphi(w)\rangle. \tag{2.20}$$

We can show that the linear functional $\Lambda_{s,\chi}$ has the following property.

Proposition 2.6. *For every $\varphi\in V_\pi$, $\Lambda_{s,\chi}(\varphi)$ defined by (2.17) has a meromorphic continuation in s to the whole complex plane. It is a holomorphic multiple of $L(s,\pi\otimes\chi)$. $\Lambda_{s,\chi}$ defines a nonzero element in*

$$Hom_H(\pi,\chi_{1,s}(\det)\otimes\chi_{2,s}(\det))=Hom_H(\pi,|\det|_{\mathbb{K}}^{-s+\frac{1}{2}}\chi^{-1}(\det)\otimes|\det|_{\mathbb{K}}^{s-\frac{1}{2}}(\chi\omega)(\det)),$$

whenever s is not a pole of the L-function $L(s,\pi\otimes\chi)$. In particular, one can choose φ such that $\Lambda_{s,\chi}(\varphi)=L(s,\pi\otimes\chi)$.

3. Cohomological Vectors in the Induced Representation

Although Proposition 2.6 tells us that we can find a test vector φ for $\Lambda_{s,\chi}$, it does not show such φ is cohomological, namely, φ lies in the minimal K-type of π. Thus, The goal of this Section is to explicitly construct a cohomological vector of π for $\Lambda_{s,\chi}$. Since the construction of real and complex cases is similar, here we only focus on the case $G=\mathrm{GL}_{2n}(\mathbb{C})$. One can also refer to [CJLT19] for the real case, where we use a conceptual method. Now we will start with some reductions and then outline our strategy on the construction of the function in the minimal K-type. Throughout this Section, we use the bold letter $\boldsymbol{i}$ for $\sqrt{-1}$.

3.1. **Some reductions.** First, we briefly recall the notations in Section 2. Let $K=\mathrm{U}_{2n}$ be the standard maximal compact subgroup of $G=\mathrm{GL}_{2n}(\mathbb{C})$, B be the standard Borel subgroup of G, and T be the split torus contained in B. Then T is a product of $2n$ copies of $\mathbb{C}^\times$, and $T\cap K$ is a product of $2n$ copies of U_1.

Let π be the irreducible generic cohomological representation given in (2.7) with $l_j+l_{2n+1-j}=2L(j=1,\cdots,n)$. Set $l_j=N_j+L$. Then $(N_1,N_2,\cdots,N_{2n})$ is a sequence of integers in the strictly decreasing order satisfying $N_j + N_{2n+1-j} = 0$ $(j = 1, 2, \cdots, n)$. As in subsection 2.4, π can be rewritten as

$$\mathrm{Ind}_B^G | \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{N_1+L} \otimes | \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{-N_1+L} \otimes \cdots \otimes | \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{N_n+L} \otimes | \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{-N_n+L}. \tag{3.1}$$

Let τ be the minimal K-type of π. By Frobenius reciprocity law, we see that the highest weight of τ is $\Lambda = (N_1 + L, \cdots, N_n + L, -N_n + L, \cdots - N_1 + L)$. For simplicity, we set

$$\chi_{\vec{N}} := \chi_{N_1+L} \otimes \chi_{-N_1+L} \otimes \cdots \otimes \chi_{N_n+L} \otimes \chi_{-N_n+L},$$

where $\vec{N} := (N_1 + L, -N_1 + L, \cdots, N_n + L, -N_n + L)$. Then by [Vo86, Proposition 8.1], τ is also the minimal K-type of the representation $\pi_K = \mathrm{Ind}_{T\cap K}^K \chi_{\vec{N}}$. Every function in π_K belongs to the space $C^\infty(K)$ of smooth functions on K satisfying the left equivariant property given by $\chi_{\vec{N}}$. Inspired by this, we may regard $C^\infty(K)$ as a left $T\cap K$- and right K-module under the left and right regular actions. As a $(T \cap K) \times K$-module, the space $C^\infty(K)_{\mathrm{fin}}$ of K-finite vectors of $C^\infty(K)$ is completely reducible and decomposed into a direct sum:

$$C^\infty(K)_{\mathrm{fin}} = \bigoplus_{(\chi,\eta)\in\widehat{T\cap K}\times\widehat{K}} m_{\chi,\eta}\chi \otimes \eta = \bigoplus_{\chi\in\widehat{T\cap K}} \Big(\bigoplus_{\eta\in\widehat{K}} m_{\chi,\eta}\chi \otimes \eta\Big), \tag{3.2}$$

where $m_{\chi,\eta}$ is the multiplicity of $(T \cap K) \times K$-submodule $\chi \otimes \eta$ occurring in $C^\infty(K)_{\mathrm{fin}}$.

Thus $\bigoplus_{\eta\in\widehat{K}} m_{\chi_{\vec{N}},\eta}\chi_{\vec{N}} \otimes \eta$ is the space of K-finite vectors of π_K and the minimal K-type τ of π_K is an irreducible summand in this space with highest weight $(-\vec{N}) \times \Lambda$. Here the left irreducible $T \cap K$-module is given by the tensor product of $2n$ characters on U_1, and we write its highest weight as the form $-\vec{N}$.

Now we consider the restriction map

$$\iota : C^\infty(\mathrm{Mat}_{2n}^{\mathbb{R}}) \longrightarrow C^\infty(K), \qquad f \longmapsto f|_K,$$

where $\mathrm{Mat}_{2n}^{\mathbb{R}}$ is the realification of the complex vector space Mat_{2n} of $2n \times 2n$ complex matrices. Once we equip $C^\infty(\mathrm{Mat}_{2n}^{\mathbb{R}})$ a left $T \cap K$- and right K-module structures by the left and right regular actions, ι becomes a $T \cap K \times K$-module homomorphism. To explicitly construct a cohomological vector in the minimal K-type τ, we only need to produce a polynomial $F_{\vec{N},\chi_{-l}\otimes\chi_{l+2L}}$ in $C^\infty(\mathrm{Mat}_{2n}^{\mathbb{R}})_{\mathrm{fin}} = \mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$ such that

(1) its restriction $\iota(F_{\vec{N},\chi_{-l}\otimes\chi_{l+2L}})$ lies in a $T \cap K \times K$-submodule of $C^\infty(K)_{\mathrm{fin}}$ with highest weight $(-\vec{N}) \times \Lambda$;
(2) its restriction $\iota(F_{\vec{N},\chi_{-l}\otimes\chi_{l+2L}})$ satisfies the right $K \cap H$-equivariant property (2.19).

Such a polynomial is called $(\vec{N}, \chi_{-l} \otimes \chi_{l+2L})$-equivariant. Here $\chi_{-l} \otimes \chi_{l+2L}$ is a character of $\mathrm{U}_n \times \mathrm{U}_n$.

3.2. **Strategy of the construction.** As is known to all, every highest weight representation of a connected compact Lie group can be realized as the Cartan component of a tensor product of fundamental representations. Following the principle of this realization, we state our construction as follows.

(i) First, we consider the case that the right $K \cap H$-equivariance (2.19) is given by the trivial characters $\chi = \omega = id$. In this situation, $l = L = 0$. We write Λ_0 and $\vec{N}_0$ for the corresponding Λ and $\vec{N}$, that is,

$$\Lambda_0 = (N_1, \cdots, N_n, -N_n, \cdots, -N_1) = \sum_{j=1}^{n-1}(N_j - N_{j+1})\Lambda_j + N_n\Lambda_n,$$

$$\vec{N}_0 = (N_1, -N_1, \cdots, N_n, -N_n) = \sum_{j=1}^{n-1}(N_j - N_{j+1})\vec{N}_j + N_n\vec{N}_n,$$

where $\Lambda_j = (\underbrace{1, \cdots, 1}_{j}, 0, \cdots, 0, \underbrace{-1, \cdots, -1}_{j})$ and $\vec{N}_j = (\underbrace{1, -1, \cdots, 1, -1}_{j \text{ pairs}}, 0, \cdots, 0)$. Thus, we only need to construct some right $K \cap H = \mathrm{U}_n \times \mathrm{U}_n$-invariant polynomials $F_j (1 \leq j \leq n)$ in $\mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$ such that $\iota(F_j)$ lies in the minimal K-type τ_j (with the highest weight Λ_j) of the right K-module π_j of $C^\infty(K)$, where

$$\pi_j := \mathrm{Ind}_{T \cap K}^{K} \underbrace{\chi_1 \otimes \chi_{-1} \otimes \cdots \otimes \chi_1 \otimes \chi_{-1}}_{\text{j pairs}} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}. \tag{3.3}$$

(ii) For each integer k, denote by S_k the symmetric group that permutes $\{1, \ldots, k\}$. Then for $j = 1, 2, \cdots, n$, we define

$$S^{(j)} := \{s \in S_{2j} | s(2i-1) \text{ is an odd number}, s(2i) = s(2i-1) + 1, 1 \leq i \leq j\} \simeq S_j.$$

From the definition of π_j in (3.3), it is clear that π_j is a left $S^{(j)}$-module. Here the left $S^{(j)}$-action is given by

$$(s \cdot F)(Z) := F(s^{-1} Z).$$

Moreover, we can show the highest weight function in the minimal K-type τ_j of π_j is $S^{(j)}$-invariant by a direct matrix computation. Thus any function in τ_j is $S^{(j)}$-invariant. For this reason, we require that every $\iota(F_j)$ which will be constructed in **(i)** is left $S^{(j)}$-invariant.

(iii) Finally, we consider the case that the characters $\chi = \chi_l$ and ω which define the right $K \cap H$-equivariance in (2.19) are non-trivial. In this situation, except for constructing above functions F_j, we also need to produce some extra functions $\Delta_{1,\pm}$ and $\Delta_{2,\pm}$ in $\mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$ such that

(1) they contribute to the right $K \cap H$-equivariant property;
(2) they lie in some suitable irreducible K-modules whose highest weights match with the data $\vec{N}$ and Λ combining with the highest weights Λ_j of τ_j and $\vec{N}_j$ for $1 \leq j \leq n$.

In order to show that $\iota(F_j)$ belongs to the minimal K-type τ_j of π_j, we need to check $\iota(F_j)$ is an eigenfunction for the Casimir operator Ω corresponding the correct eigenvalue.

3.3. **Explicit Construction of a Cohomological Test Vector.** To simplify notations, for any matrix $Z = X + \boldsymbol{i}Y = (x_{i,j})_{1 \leq i,j \leq 2n} + \boldsymbol{i}(y_{i,j})_{1 \leq i,j \leq 2n}$ in Mat_{2n}, we write it as

$$Z = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ \vdots & \vdots \\ u_{2n} & v_{2n} \end{pmatrix},$$

where $u_j := (z_{j,1}, \cdots, z_{j,n})$ and $v_j := (z_{j,n+1}, \cdots, z_{j,2n})$. Every polynomial $f \in \mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$ is a polynomial in x_{ij} and y_{ij} in prior. Yet for convenience of the future calculations, we will regard f as a polynomial in z_{ij} and $\bar{z}_{ij}$ by a change of variables: $z_{ij} = x_{ij} + \boldsymbol{i}y_{ij}$; $\bar{z}_{i,j} = x_{ij} - \boldsymbol{i}y_{ij}$. For any $1 \leq k \neq l \leq 2n$, we set

$$\Phi_{kl} := \langle u_k, u_l \rangle_c = u_k \cdot \bar{u}_l^t. \tag{3.4}$$

Then $\Phi_{kl} \in \mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$. Now we will give the construction of certain Casimir eigen-polynomials. For each integer $1 \leq j \leq n$, we define a polynomial $F_j(Z, \bar{Z}) \in \mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$ as follows:

$$F_j := \sum_{s \in S_j} \mathrm{sgn}(s) \cdot \Phi_{1,s(1)+1} \Phi_{3,s(3)+1} \cdots \Phi_{2j-1,s(2j-1)+1}, \tag{3.5}$$

where S_j is the permutation group of the set $\{1, 3, \cdots, 2j-1\}$ and $\mathrm{sgn}(s)$ is the sign of the permutation s, i.e. it is 1 if s is an even permutation, and it is -1 if s is an odd permutation. From the definition (3.5), it is obvious that $\iota(F_j)$ is left $S^{(j)}$-invariant. By direct computation, we show that

Theorem 3.1. *$\iota(F_j)$ lies in π_j and it generates an irreducible submodule (τ_j, V_j) of K with highest weight Λ_j.*

Proof. Please refer to [LT20, Theorem 3.9, Corollary 3.11]. □

Then we define the $\Delta_{1,\pm}$ and $\Delta_{2,\pm}$ as follows, which will contribute to the right $K \cap H$-equivariance in the cohomological vector.

$$\begin{aligned} &\Delta_{1,+}(Z,\bar{Z}) := \det\begin{pmatrix} u_1 \\ u_3 \\ \cdots \\ u_{2n-1} \end{pmatrix}, \qquad \Delta_{1,-}(Z,\bar{Z}) := \det\begin{pmatrix} \bar{u}_2 \\ \bar{u}_4 \\ \cdots \\ \bar{u}_{2n}, \end{pmatrix}, \\ &\Delta_{2,+}(Z,\bar{Z}) := \det(Z), \qquad \Delta_{2,-}(Z,\bar{Z}) := \det(\bar{Z}), \end{aligned} \tag{3.6}$$

where $\bar{u}_j := (\bar{z}_{j,1}, \cdots, \bar{z}_{j,n})$ for each $1 \le j \le 2n$. Similarly, we have

Proposition 3.2. *$\Delta_{1,+}$(resp. $\Delta_{1,-}$) generates an irreducible K-submodule W_1(resp. W_{-1}) of $\mathbb{C}^n[Mat_{2n}^{\mathbb{R}}]$ with highest weight $(\underbrace{1,\cdots,1}_{n},0,\cdots,0)$(resp. highest weight $(0,\cdots,0,\underbrace{-1,\cdots,-1}_{n})$). $\Delta_{2,+}$(resp. $\Delta_{2,-}$) generates an irreducible K-submodule W_1'(resp. W_{-1}') of $\mathbb{C}^{2n}[Mat_{2n}^{\mathbb{R}}]$ with highest weight $(1,\ldots,1)$(resp. $(-1,\ldots,-1)$).*

Finally, combining above polynomials, we will give the explicit construction of a smooth function in the minimal K-type τ of π satisfying the right $K \cap H$-equivariant property (2.19). Fix two integers $l, L \in \mathbb{Z}$. Assume that $\vec{N} := (N_1+L, -N_1+L, \cdots, N_n+L, -N_n+L)$ is a sequence of integers with the property that $N_1 \ge \cdots \ge N_n \ge |l+L|$. We define a polynomial function $F_{\vec{N},\chi_{-l}\otimes\chi_{l+2L}}$ in $\mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$ as follows,

$$F_{\vec{N},\chi_{-l}\otimes\chi_{l+2L}} := \begin{cases} (\prod_{i=1}^{n-1} F_i^{N_i-N_{i+1}}) \cdot F_n^{N_n-l-L}\Delta_{1,-}^{2(l+L)}\Delta_{2,+}^{l+2L}, & \text{if } l+L \ge 0,\ l+2L \ge 0; \\ (\prod_{i=1}^{n-1} F_i^{N_i-N_{i+1}}) \cdot F_n^{N_n-l-L}\Delta_{1,-}^{2(l+L)}\Delta_{2,-}^{-(l+2L)}, & \text{if } l+L \ge 0,\ l+2L \le 0; \\ (\prod_{i=1}^{n-1} F_i^{N_i-N_{i+1}}) \cdot F_n^{N_n+l+L}\Delta_{1,+}^{-2(l+L)}\Delta_{2,+}^{l+2L}, & \text{if } l+L \le 0,\ l+2L \ge 0; \\ (\prod_{i=1}^{n-1} F_i^{N_i-N_{i+1}}) \cdot F_n^{N_n+l+L}\Delta_{1,+}^{-2(l+L)}\Delta_{2,-}^{-(l+2L)}, & \text{if } l+L \le 0,\ l+2L \le 0. \end{cases} \tag{3.7}$$

Theorem 3.3. *Let π be the cohomological representation of G given in (3.1), i.e.*

$$Ind_B^G |\ |_{\mathbb{C}}^{\frac{m}{2}}\chi_{N_1+L} \otimes |\ |_{\mathbb{C}}^{\frac{m}{2}}\chi_{-N_1+L} \otimes \cdots \otimes |\ |_{\mathbb{C}}^{\frac{m}{2}}\chi_{N_n+L} \otimes |\ |_{\mathbb{C}}^{\frac{m}{2}}\chi_{-N_n+L}$$

with $L \in \mathbb{Z}$ and $(N_1, N_2, \cdots, N_n)$ being a sequence of positive integers in the strictly decreasing order. For any integer l satisfying $|l+L| \ge N_n$, $\varphi = \iota(F_{\vec{N},\chi_{-l}\otimes\chi_{l+2L}})$ lives in the minimal K-type τ of π.

Proof. Please refer to [LT20, Theorem 3.13 and Corollary 3.14]. □

4. Non-vanishing of Archimedean Local Integrals

In this Section, we will establish the non-vanishing property for the archimedean local integrals of Friedberg-Jacquet. For convenience, we begin with the $\mathrm{GL}_2(\mathbb{C})$ case and then reduce the $\mathrm{GL}_{2n}(\mathbb{C})$ case to $\mathrm{GL}_2(\mathbb{C})$ blocks in the sense of linear periods.

4.1. **$\mathrm{GL}_2(\mathbb{C})$ Case.** Given a non-negative integer N, two integers m and L, we consider the principal series

$$\sigma = \mathrm{Ind}_{B_{\mathrm{GL}_2}}^{\mathrm{GL}_2(\mathbb{C})} |\ \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{N+L} \otimes |\ \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{-N+L}.$$

The minimal K-type τ_σ has highest weight $(N+L, -N+L)$. As a representation of SU_2, τ_σ is a $2N+1$ dimensional vector space and it has a weight space decomposition:

$$\tau_\sigma = \bigoplus_{k=-N}^{N} \tau_{\sigma,k},$$

where $\tau_{\sigma,k}$ is the one dimensional weight $2k$ space.

Given the character χ of $\mathbb{C}^\times$, we consider the continuous linear functional $\lambda_{s,\sigma,\chi}$ defined as in (2.14) for $\mathbb{K} = \mathbb{C}$.

Proposition 4.1. [Po08, Proposition 1, Theorem 1] *If $|l+L| \leq N$, then for all $k \neq -l-L$, $\lambda_{s,\sigma,\chi}$ vanishes on the weight $2k$ space $\tau_{\sigma,k}$; for $k = -l-L$, there exists a vector $v \in \tau_{\sigma,k}$ such that $\lambda_{s,\sigma,\chi}(v) = L(s, \sigma \otimes \chi)$. If $|l+L| > N$, then $\lambda_{s,\sigma,\chi}$ vanishes identically on the minimal K-type τ_σ.*

In Section 3, we provide a strategy to construct a function in the minimal K-type by restricting a polynomial in $\mathbb{C}[\mathrm{Mat}_{2n}^{\mathbb{R}}]$ on U_{2n}. In the special case of $n=1$, we have the following Corollary:

Corollary 4.2. *Assume that $|l+L| \leq N$. Set $\vec{N} = (N+L, N-L)$. We parameterize a 2×2 complex matrix as $Z = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$. Then the polynomial φ_σ defined below*

$$\varphi_\sigma = \begin{cases} (x_1\bar{x}_2)^{N-l-L}(\bar{x}_2)^{2(l+L)}(x_1y_2 - x_2y_1)^{l+2L}, & \text{if } l+L \geq 0,\ l+2L \geq 0; \\ (x_1\bar{x}_2)^{N-l-L}(\bar{x}_2)^{2(l+L)}(\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1)^{-(l+2L)}, & \text{if } l+L \geq 0,\ l+2L \leq 0; \\ (x_1\bar{x}_2)^{N+l+L}(x_1)^{-2(l+L)}(x_1y_2 - x_{2i}y_1)^{l+2L}, & \text{if } l+L \leq 0,\ l+2L \geq 0; \\ (x_1\bar{x}_2)^{N+l+L}(x_1)^{-2(l+L)}(\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1)^{-(l+2L)}, & \text{if } l+L \leq 0,\ l+2L \leq 0 \end{cases} \tag{4.1}$$

is a $(\vec{N}, \chi_{-l} \otimes \chi_{l+2L})$-equivariant polynomial function in $\mathbb{C}[Mat_{2n}^{\mathbb{R}}]$. The restriction map $\iota_2 : \mathbb{C}[Mat_{2n}^{\mathbb{R}}] \to C^\infty(\mathrm{U}_2)$ sends the above φ_σ to a smooth function $\iota_2(\varphi_\sigma)$ living in the minimal K-type τ_σ of σ.

Since $\tau_{\sigma,k}$ is one dimensional for all k, $\iota_2(\varphi_\sigma)$ is a nonzero multiple of the vector v in Proposition 4.1. Thus, we can define a continuous linear functional $\tilde{\lambda}_{s,\sigma,\chi}$ on V_σ, which is a nonzero scalar multiple of $\lambda_{s,\sigma,\chi}$ such that

$$\tilde{\lambda}_{s,\sigma,\chi}(\iota_2(\varphi_\sigma)) = \begin{cases} 0 & \text{if } |l+L| > N; \\ L(s, \sigma \otimes \chi) & \text{if } |l+L| \leq N. \end{cases} \tag{4.2}$$

4.2. **$\mathrm{GL}_{2n}(\mathbb{C})$ Case.** Generally, we consider an irreducible essentially tempered cohomological representation π given in (2.13), namely

$$\pi \simeq \mathrm{Ind}_P^G \sigma_1 \otimes \sigma_2 \cdots \otimes \sigma_n, \quad \text{each } \sigma_j \simeq \mathrm{Ind}_{B_{\mathrm{GL}_2}}^{\mathrm{GL}_2(\mathbb{C})} |\ \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{N_j+L} \otimes |\ \ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{-N_j+L}.$$

For each σ_j, we can define a nonzero continuous linear functional $\tilde{\lambda}_{s,j} := \tilde{\lambda}_{s,\sigma_j,\chi}$ such that the equation (4.2) holds. Then we can define a nonzero continuous linear function $\tilde{\Lambda}_{s,\chi}$ on π by replacing $\lambda_{s,j}$ with $\tilde{\lambda}_{s,j}$ in (2.18), which is a nonzero scalar multiple of $\Lambda_{s,\chi}$ constructed in (2.17). Similarly, if φ satisfies the right $K \cap H$-equivariant property (2.19), we have

$$\tilde{\Lambda}_{s,\chi}(\varphi) = \langle \bigotimes_{j=1}^{n} \tilde{\lambda}_{s,j}, \varphi(w) \rangle. \tag{4.3}$$

Here w is the same Weyl element as in (2.15). Thus to evaluate the twisted linear functional $\tilde{\Lambda}_{s,\chi}$ at the cohomological vector, we only need to track the isomorphism in the double induction formula:

(4.4)

$$\eta : \mathrm{Ind}_P^G \sigma_1 \otimes \cdots \otimes \sigma_n \simeq \mathrm{Ind}_B^G |\ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{N_1+L} \otimes |\ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{-N_1+L} \otimes \cdots \otimes |\ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{N_n+L} \otimes |\ |_{\mathbb{C}}^{\frac{m}{2}} \chi_{-N_n+L}.$$

If $\varphi = \iota(f)$ is the cohomological vector constructed in Theorem 3.3 using the model of π on the RHS of (4.4), then $\eta^{-1}(\varphi)(w) = \varphi(xw)$ is a smooth function (defined on M) valued in $V_{\sigma_1} \otimes \cdots \otimes V_{\sigma_n}$. By direct computation, we have

Theorem 4.3. *If the integer l satisfies $|l + L| > N_n$, then $\tilde{\Lambda}_{s,\chi}$ vanishes identically on the minimal K-type τ of π. If l satisfies $|l + L| \le N_n$, we take $\varphi = \iota(F_{\vec{N},\chi_{-l}\otimes\chi_{l+2L}})$ to be the cohomological vector constructed in Theorem 3.3 using the model for π on the RHS of (4.4), then*

$$\tilde{\Lambda}_{s,\chi}(\eta^{-1}(\varphi)) = L(s, \pi \otimes \chi).$$

Proof. Please refer to [LT20, Theorem 4.3]. □

Finally, by the uniqueness of linear period(see [CS19, Theorem B]), we show that

Corollary 4.4. *There exists a holomorphic function $G(s, \chi)$ such that*

$$Z(v, s, \chi) = e^{G(s,\chi)} \tilde{\Lambda}_{s,\chi}(v).$$

Remark 4.5. *(1) If we take $v = \eta^{-1}(\varphi)$ as in Theorem 4.3, then $Z(v, s, \chi)$ does not vanish for all s and χ. This shows the* **Non-vanishing Assumption** *mentioned in Section 1.*
(2) In the recent work [JST19], the authors actually show that $Z(v, s, \chi) = \tilde{\Lambda}_{s,\chi}(v)$. It implies that the function constructed in Theorem 3.3 is just the cohomological test vector for archimedean Friedberg-Jacquet zeta integral and it is independent of s. This gives the answer for the **Uniform Cohomological Test Vector** *problem, which is also mentioned in Section 1.*

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