Fourier-Jacobi expansion of cusp forms on $Sp(2;\mathbb{R})$

Hiro-aki Narita
Department of Mathematics
Faculty of Science and Engineering
Waseda University

Abstract

This note announces the recent result by the author about a general theory of the Fourier-Jacobi expansion of cusp forms on $Sp(2;\mathbb{R})$. It is also viewed as the write-up of author's talk at the RIMS workshop in January 2020. The theory covers the case of generic cusp forms. Explicit descriptions of such expansion are available for cusp forms generating large discrete series representations, generalized principal series representations induced from a Jacobi parabolic subgroup and principal series representations (induced from the minimal parabolic subgroup), which are known to be generic.

As the archimedean local ingredients we need the notion of Fourier-Jacobi type spherical functions and Whittaker functions, whose explicit formulas are obtained by Hirano and by Oda, Miyazaki-Oda, Niwa and Ishii et al. To realize these spherical functions in the Fourier-Jacobi expansion we use the spectral theory for the Jacobi group by Berndt-Böcherer and Berndt-Schmidt, which can be referred to as the global ingredient of our study. Based on the theory by Berndt-Böcherer we generalize the classical Eichler-Zagier correspondence in the representation theoretic context.

This note includes the correction to author's presentation at the workshop. The Fourier-Jacobi expansion has some contribution by Eisenstein-Poincaré series with the test functions given by the Whittaker functions, for which the author had completely no idea when he gave the talk.

1 Basic Notation

1.1 Real Lie groups

Let $G = Sp(2;\mathbb{R})$ be the real symplectic group of degree two defined by

$$G := \{g \in GL_4(\mathbb{R}) \mid {}^t g J_2 g = J_2\}$$

where $J_2 = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}$. This has a maximal parabolic subgroup P_J of G given by the Levi decomposition $N_J \rtimes L_J$. Here N_J is the nilpotent Lie group defined by

$$N_J := \left\{ n(u_0, u_1, u_2) := \begin{pmatrix} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u_0 & 1 \end{pmatrix} \;\middle|\; u_0,\ u_1,\ u_2 \in \mathbb{R} \right\}$$

and the Levi part L_J is the subgroup of G given by

$$L_J := \left\{ \begin{pmatrix} \alpha & & & \\ & a & & b \\ & & \alpha^{-1} & \\ & c & & d \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R}^\times,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}) \right\}.$$

We call this the Jacobi parabolic subgroup (also called the Klingen parabolic subgroup). This parabolic subgroup P_J has the Langlands decomposition $P_J = N_J A_J M_J$ with

$$A_J := \left\{ a_J = \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a_1^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; a \in \mathbb{R}_+^\times \right\},\ M_J := \left\{ \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & \epsilon & 0 \\ 0 & c & 0 & d \end{pmatrix} \;\middle|\; \begin{matrix} ad - bc = 1 \\ \epsilon \in \{\pm 1\} \end{matrix} \right\}.$$

The unipotent radical N_J of P_J is the Heisenberg group with the center Z_J, which is given by

$$Z_J := \{n(0, u_1, 0) \mid u_1 \in \mathbb{R}\}.$$

We introduce the non-reductive Lie group G_J called the Jacobi group by the semi-direct product $N_J \rtimes SL_2(\mathbb{R})$, which is given by replacing L_J with $SL_2(\mathbb{R})$ in P_J. More precisely, $SL_2(\mathbb{R})$ is viewed as a subgroup of G_J (or L_J) by putting $\alpha = 1$ in L_J. The Jacobi group G_J is the centralizer of Z_J in P_J.

We also need the minimal parabolic subgroup P_0 of G with the unipotent radical N_0, where N_0 is defined by

$$N_0 = \left\{ n(u_0, u_1, u_2, u_3) \in \begin{pmatrix} 1 & 0 & u_1 & u_2 \\ 0 & 1 & u_2 & u_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u_0 & 1 \end{pmatrix} \;\middle|\; u_i \in \mathbb{R}\ (0 \le i \le 3) \right\}.$$

We also review the Langlands decomposition $P_0 = N_0 A_0 M_0$ of the minimal parabolic subgroup P_0, where

$$A_0 := \{a_0 = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1,\ a_2 \in \mathbb{R}_{>0}\},$$
$$M_0 := \{\mathrm{diag}(\epsilon_1, \epsilon_2, \epsilon_1, \epsilon_2) \mid \epsilon_1,\ \epsilon_2 \in \{\pm 1\}\}.$$

The group N_0 admits the semi-direct product decomposition $N_0 = N_S \rtimes N_L$ with the subgroups N_S and N_L defined by

$$N_S = \{n(u_0, u_1, u_2, u_3) \in N_0 \mid u_0 = 0\}, \quad N_L = \{n(u_0, 0, 0, 0) \mid u_0 \in \mathbb{R}\}.$$

We remark that N_S is well known as the unipotent radical of the Siegel parabolic subgroup.

Let us introduce the Cartan involution θ of G defined by $\theta(g) := {}^t g^{-1}$ for $g \in G$. Then

$$K := \{g \in G \mid \theta(g) = g\} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in G \;\middle|\; A, B \in M_2(\mathbb{R}) \right\}$$

is a maximal parabolic subgroup of G. This is isomorphic to the unitary group $U(2)$ of degree two by the map

$$K \ni \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mapsto A + \sqrt{-1}B \in U(2).$$

We should remark that G has the Iwasawa decomposition $G = N_0 A_0 K$ with the notation above.

We furthermore introduce the real group $\widetilde{SL_2}(\mathbb{R})$ characterized by the non-split exact sequence

$$1 \to \{\pm 1\} \to \widetilde{SL_2}(\mathbb{R}) \to SL_2(\mathbb{R}) \to 1,$$

namely, the non-split double cover of the special linear group $SL_2(\mathbb{R})$ of degree two. This is realized by means of the unique non-trivial element of the second cohomology $H^2(SL_2(\mathbb{R}), \{\pm 1\})$ called the Kubota cocycle (cf. [10]). The group $SL_2(\mathbb{R})$ is known to have the special orthogonal group $SO_2(\mathbb{R})$ as a maximal compact subgroup and $\widetilde{SL_2}(\mathbb{R})$ has a maximal compact subgroup $\widetilde{SO_2}(\mathbb{R})$, non-split two fold cover of $SO_2(\mathbb{R})$.

1.2 Discrete subgroups

We explain the notation for the discrete subgroups of the real groups above necessary for the forthcoming argument. As the most fundamental notation we introduce the Siegel modular group $Sp(2;\mathbb{Z}) := G \cap GL_4(\mathbb{Z})$. In addition to this we will need $G_J(\mathbb{Z}) := G_J \cap Sp(2;\mathbb{Z})$, $N_J(\mathbb{Z}) := N_J \cap Sp(2;\mathbb{Z})$, $Z_J(\mathbb{Z}) := Z_J \cap Sp(2;\mathbb{Z})$, $N_S(\mathbb{Z}) := N_S \cap Sp(2;\mathbb{Z})$ and $N_0(\mathbb{Z}) := N_0 \cap Sp(2;\mathbb{Z})$ etc. We remark that $G_J(\mathbb{Z}) = N_J(\mathbb{Z}) \rtimes SL_2(\mathbb{Z})$, where $SL_2(\mathbb{Z})$ is viewed as a subgroup of $G_J(\mathbb{Z})$.

2 Eichler-Zagier correspondence in representation theoretic formulation

Hereafter $\mathbf{e}(z) := \exp(2\pi\sqrt{-1}z)$ for a complex number z.

2.1 Irreducible unitary representations of the Heisenberg group N_J and the Jacobi group G_J

It is well known that irreducible unitary representations of the Heisenberg group N_J are classified in terms of the central characters $\chi_m : Z_J \ni n(0,u_1,0) \mapsto \mathbf{e}(mu_1) \in \mathbb{C}^\times$ with $m \in \mathbb{R}$, where recall that Z_J denotes the center of N_J (cf. Section 1.1). In fact, irreducible unitary representations of N_J with the trivial central character are unitary characters of N_J. On the other hand, irreducible unitary representations of N_J other than unitary characters are infinite dimensional representations and are classified in terms of the central characters, up to unitary equivalence. This fact is verified easily, e.g. by means of the orbit method of nilpotent Lie groups (cf. Corwin-Greenleaf [5]).

We can extend a character χ_m to a character of the polarization subgroup $M := \{n(0,u_1,u_2) \mid u_1,\ u_2 \in \mathbb{R}\}$ of N_J, which is also denoted by χ_m. The group M is the image of the exponential map for the polarization subalgebra of $\mathfrak{n}$ (for a definition see [5, pp. 27–28]) with respect to a non-zero linear form of $\mathfrak{n}$ whose restriction to the center is non-trivial. We introduce a unitary representation $(\nu_m, \mathcal{U}_m)$ of N_J by the L^2-induction $L^2\text{-Ind}_M^{N_J}(\chi_m)$ from the character χ_m of M with the representation space $\mathcal{U}_m$. Here N_J acts on $\mathcal{U}_m$ by the right translation, denoted by ν_m. The classification of the unitary dual of N_J is stated as follows:

Proposition 2.1. *(1) Up to unitary equivalence, irreducible unitary representations of N_J are exhausted by*

1. *unitary characters $\eta_{m_0,m_2} : N_J \ni n(u_0,u_1,u_2) \mapsto \mathbf{e}(m_0u_0 + m_2u_2) \in \mathbb{C}^\times$ with $(m_0,m_2) \in \mathbb{R}^2$,*

2. *Infinite dimensional representations $(\nu_m, \mathcal{U}_m)$ with $m \in \mathbb{R} \setminus \{0\}$.*

(2) For $m,\ m' \in \mathbb{R} \setminus \{0\}$, $\nu_m \simeq \nu_{m'}$ if and only if $m = m'$.

We next describe irreducible unitary representations of G_J. We need to recall that, for a non-zero $m \in \mathbb{R}$, $(\nu_m, \mathcal{U}_m)$ is extended to the unitary representation $(\tilde{\nu}_m, \mathcal{U}_m)$ of $N_J \times \widetilde{SL}_2(\mathbb{R})$ by $\tilde{\nu}_m(n \cdot \tilde{g}) = \omega_m(\tilde{g})\nu_m(n)$ for $(n,\tilde{g}) \in N_J \times \widetilde{SL}_2(\mathbb{R})$, where ω_m denotes the Weil representation of $\widetilde{SL}_2(\mathbb{R})$. For this representation we remark that G_J has Z_J as the center and it is possible to consider the notion of the central character for an irreducible representation of G_J.

Proposition 2.2. *(1) Let $(\tilde{\nu}_m, \mathcal{U}_m)$ be as above. For an irreducible genuine representation $(\pi_1, \mathcal{W}_{\pi_1})$ of $\widetilde{SL}_2(\mathbb{R})$ (i.e. representation of $\widetilde{SL}_2(\mathbb{R})$ not factoring through $SL_2(\mathbb{R})$) we put the representation ρ_{m,π_1} of G_J by $\rho_{m,\pi_1}(n \cdot g) := \pi_1(g) \otimes \tilde{\nu}_m(n \cdot g)$ for $(n,g) \in N_J \times SL_2(\mathbb{R})$. Then $(\rho_{m,\pi_1}, \mathcal{U}_m)$ is an irreducible unitary representation of G_J. All irreducible unitary representations of G_J with non-trivial central characters are obtained in this manner.*
(2) Let $(\rho, \mathcal{F}_\rho)$ be an irreducible unitary representation of G_J with the trivial central character. Then ρ is unitarily equivalent to one of the following representations:

1. $\rho_{\pi_1,0}(n \cdot g) = \pi_1(g)\tilde{\nu}_0(n \cdot g)$ *for* $(n, g) \in N_J \times SL_2(\mathbb{R})$*, where* π_1 *(respectively* $\tilde{\nu}_0$*) is an irreducible unitary representation of* $SL_2(\mathbb{R})$ *(respectively the trivial representation of* G_J*),*

2. representations of G_J *induced from unitary characters of* N_0*.*

Proof. For this proposition see [6, Propositions 2.5, 2.6]. We cite Berndt-Schmidt [2, Theorem 2.6.1] and Satake [17, Appendix 1, Proposition 2] for the first assertion. As is remarked in [6, Section 2.3] (just before Proposition 2.6) the second assertion is deduced from Mackey's method for representations of semi-direct product groups, so called Mackey machine. □

Note that irreducible genuine representations $\sigma \in \widehat{\widetilde{SL}_2(\mathbb{R})}$ has the same multiplier with $\tilde{\nu}_m$, which is due to the uniqueness of the non-trivial element of $H^2(SL_2(\mathbb{R}), \{\pm 1\})$ (cf. Section 1.1). Thus ρ_{m,π_1}s above are well-defined representations of G_J.

2.2 Eichler-Zagier correspondence

We recall that we have introduced the discrete subgroup $G_J(\mathbb{Z})$ of G_J given by $G_J(\mathbb{Z}) = N_J(\mathbb{Z}) \rtimes SL_2(\mathbb{Z})$ (cf. Section 1.1). Following [2, Section 4.2] we introduce the space of cusp forms on G_J as follows:

Definition 2.3. (1) A subgroup N_J^* of G_J is called horo-spherical if N_J^* is G_J-conjugate to $V_J := \left\{ \begin{pmatrix} 1 & 0 & 0 & u_2 \\ 0 & 1 & u_2 & u_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; u_2,\ u_3 \in \mathbb{R} \right\}$. A horo-spherical subgroup N_J^* is defined to be cuspidal for $G_J(\mathbb{Z})$ if $N_J^*/N_J^* \cap G_J(\mathbb{Z})$ is compact.
(2) For a cuspidal subgroup N_J^*, which is in bijection with $\mathbb{R}^2$, we denote by dn the measure of N_J^* induced by the Euclidean measure of $\mathbb{R}^2$. The cuspidal space $\mathcal{H}^0$ of $L^2(G_J(\mathbb{Z})\backslash G_J)$ is defined as

$$\mathcal{H}^0 := \left\{ \phi \in L^2(G_J(\mathbb{Z})\backslash G_J) \;\middle|\; \int_{N_J^* \cap G_J(\mathbb{Z})\backslash N_J^*} \phi(ng)dn = 0 \quad \begin{array}{c} \text{for almost all } g \in G_J \text{ and} \\ \text{any cuspidal subgroup } N_J^* \end{array} \right\}.$$

For each $m \in \mathbb{Z}$ we introduce

$$\mathcal{H}_m := \{\phi \in L^2(G_J(\mathbb{Z})\backslash G_J) \mid \phi(n(0, z, 0)g) = \mathbf{e}(mz)\phi(g)\ \forall (z, g) \in \mathbb{R} \times G_J\},\ \mathcal{H}_m^0 := \mathcal{H}_m \cap \mathcal{H}^0.$$

On the cuspidal space we cite the following fact:

Proposition 2.4. *(1) (cf. [2, Theorem 4.3.1]) The representation of* G_J *on* $\mathcal{H}_m^0$ *defined by the right translation is completely reducible, and each irreducible component occurs in* $\mathcal{H}_m^0$ *with a finite multiplicity. The same assertion holds for* $\mathcal{H}^0$ *since* $\mathcal{H}^0 = \oplus_{m \in \mathbb{Z}} \mathcal{H}_m^0$*.*

(2) (cf. [2, Proposition 4.6 (i)]) Let V_J be as in Definition 2.3 and for $m_2,\ m_3 \in \mathbb{R}$, we put $\psi^{m_2,m_3}(\begin{pmatrix}1&0&0&u_2\\0&1&u_2&u_3\\0&0&1&0\\0&0&0&1\end{pmatrix}) := \mathbf{e}(m_2u_2+m_3u_3)$ *for* $(u_2,u_3)\in\mathbb{R}^2$. *Then* $\phi \in \mathcal{H}_m$ *belongs to* $\mathcal{H}_m^0$ *if and only if, for almost all* $g_0 \in G_J$, *we have that* $\int_{V_J\cap G_J(\mathbb{Z})\backslash V_J} \phi(ng_0)\overline{\psi^{n,s}(n)}$ $dn = 0$ *for* $(s,r)\in\mathbb{Z}$ *such that* $4ms-r^2=0$.

Now let us recall that, for $m \in \mathbb{R}$ and irreducible genuine representations π_1 of $\widetilde{SL}_2(\mathbb{R})$, we have introduced unitary representations $\rho_{m.\pi_1}$ of G_J, which exhaust the unitary dual of G_J (cf. Proposition 2.2) except for the case of Proposition 2.2 (2) part 2.

Definition 2.5. Let $m \in \mathbb{Z}$ and π_1 be an irreducible genuine representation of $\widetilde{SL}_2(\mathbb{R})$. Furthermore let ρ_{m,π_1} be as above. We define the space of Jacobi forms of index m and type π_1 as

$$\mathrm{Hom}_{\mathrm{G_J}}(\rho_{\mathrm{m},\pi_1},\mathcal{H}_{\mathrm{m}}^0).$$

As an immediate consequence from Proposition 2.4 we have the following:

Proposition 2.6. *For a non-zero* $m \in \mathbb{Z}$ *the space* $\mathrm{Hom}_{\mathrm{G_J}}(\rho_{\mathrm{m},\pi_1},\mathcal{H}_{\mathrm{m}}^0)$ *is finite dimensional.*

We next consider the space of the following intertwining operators

$$\Phi_m := \mathrm{Hom}_{\mathrm{G_J}}(\nu_{\mathrm{m}}, \mathrm{L}^2(\mathrm{N_J}(\mathbb{Z})\backslash\mathrm{N_J}))$$

for a non-zero integer $m \in \mathbb{Z}$. We recall that the representation space $\mathcal{U}_m$ is identified with $L^2(\mathbb{R})$. From a general theory by Corwin-Greenleaf [4] we can provide an explicit description of a basis of Φ_m as follows:

Proposition 2.7. *The space* Φ_m *has* $\{\theta_\alpha\}_{\alpha\in\mathbb{Z}/2m\mathbb{Z}}$ *as a basis, where*

$$\theta_\alpha(h)(n(u_0,u_1,u_2)) := \sum_{k\in\mathbb{Z}} \mathbf{e}(mu_1+(2km+\alpha)u_2)h(u_0+k+\frac{\alpha}{2m}) \quad (h\in L^2(\mathbb{R})\simeq\mathcal{U}_m).$$

Namely we have $\dim\Phi_m = 2|m|$.

Now we recall that $\widetilde{SL}_2(\mathbb{R})$ is realized as the group consisting of pairs (M,ϕ) with $M = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2(\mathbb{R})$ and a holomorphic function ϕ on the complex upper half plane $\mathfrak{h}$ satisfying $\phi^2(\tau) = c\tau+d$ $(\tau\in\mathfrak{h})$. For $M = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in SL_2(\mathbb{R})$ we put $\tilde{M} = (\begin{pmatrix}a&b\\c&d\end{pmatrix}, \sqrt{c\tau+d})$, where we choose the principal branch to define the square root $\sqrt{z}$ of $z \in \mathbb{C}$. We define $\widetilde{SL}_2(\mathbb{Z})$ as the double cover of $SL_2(\mathbb{Z})$ given by the inverse image of $SL_2(\mathbb{Z})$ for the covering map $\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$.

We now review that the Weil representation ω_m induces the $\widetilde{SL}_2(\mathbb{Z})$-module structure of Φ_m defined by

$$(\Omega_m(\delta) \cdot \theta_d)(h) := \theta_d(\omega_m(\delta)h) \quad (\delta \in \widetilde{SL}_2(\mathbb{Z}))$$

for $h \in L^2(\mathbb{R})$. From Borcherds [3, p.505] or Shintani [18, Proposition 1.6] we know such module structure of Φ_m explicitly. Given a fixed $m \in \mathbb{Z} \setminus \{0\}$, let $\mathbb{Z}$ be equipped with the bilinear form $\mathbb{Z} \times \mathbb{Z} \ni (x, y) \mapsto 2mxy \in \mathbb{Z}$. The dual lattice of $\mathbb{Z}$ with respect to this bilinear form is $\frac{1}{2m}\mathbb{Z}$. We can explicitly describe the $\widetilde{SL}_2(\mathbb{Z})$-module structure of Φ_m by the Weil representation of $\widetilde{SL}_2(\mathbb{Z})$ on the group algebra $\mathbb{C}[\frac{1}{2m}\mathbb{Z}/\mathbb{Z}]$ (cf. [3, p.505]).

Proposition 2.8. *For* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *let* $\tilde{\gamma}$ *be as above. We have* $\Omega_m(\tilde{\gamma})\theta_\alpha = \sum_{\beta \in \mathbb{Z}/2m\mathbb{Z}} c(\alpha, \beta)_\gamma \theta_\beta$*, where*

$$c(\alpha,\beta)_\gamma := \begin{cases} \sqrt{i}^{\,1-\mathrm{sgn}(d)} \delta_{\alpha, a\beta} \mathbf{e}(\dfrac{ab}{2}(\dfrac{\alpha^2}{2m})) \\ (c = 0), \\ \dfrac{1}{\sqrt{2c|m|}} \sqrt{i}^{\,\mathrm{sgn}(c)} \displaystyle\sum_{r \in \mathbb{Z}/c\mathbb{Z}} \mathbf{e}(\dfrac{a(2m)(\frac{\alpha}{2m} + r)^2 - 2(2m)\frac{\alpha}{2m}(\frac{\beta}{2m} + r) + d\frac{\beta^2}{2m}}{2c}) \\ (c \neq 0), \end{cases}$$

where $\delta_{*,*}$ *denotes the Kronecker delta.*

This is essentially the transformation formula of theta functions by Shintani [18, Proposition 1.6]. We follow the formulation by Borcherds [3, p.505].

We introduce the notion of Φ_m-valued cusp forms with respect to $\widetilde{SL}_2(\mathbb{Z})$.

Definition 2.9. For an irreducible genuine unitary representation π_1 of $\widetilde{SL_2}(\mathbb{R})$ we define $S_{\pi_1}(\widetilde{SL}_2(\mathbb{Z}); \Phi_m)$ to be the space of Φ_m-valued smooth functions f on $\widetilde{SL}_2(\mathbb{R})$ satisfying the followings:

1. $f(\delta b u) = \tau_{\min}(u)^{-1}\Omega_m(\delta)^{-1}f(b)$ for $(\delta, b, u) \in \widetilde{SL}_2(\mathbb{Z}) \times \widetilde{SL_2}(\mathbb{R}) \times \widetilde{SO_2}(\mathbb{R})$, where recall that $\tau_{\min}$ denotes the minimal K-type of π_1.

2. each coefficient of f is a cusp form with respect to $\widetilde{\Gamma(4m)}$, and generates π_1 as a $(\mathfrak{g}_1, \widetilde{SO}_2(\mathbb{R}))$-module, where $\mathfrak{g}_1$ denotes the Lie algebra of $\widetilde{SL}_2(\mathbb{R})$ (=the Lie algebra of $SL_2(\mathbb{R})$).

Here $\widetilde{\Gamma(4m)}$ denotes the pull-back of the principal congruence subgroup $\Gamma(4m)$ of $SL_2(\mathbb{Z})$ of level $4m$ by the covering map $\widetilde{SL}_2(\mathbb{R}) \to SL_2(\mathbb{R})$.

For this definition we note that $N = 4|m|$ is the minimal positive integer such that $\frac{N\alpha^2}{4m} \in \mathbb{Z}$ for any $\alpha \in \mathbb{Z}$ and that the representation Ω_m factors through $\widetilde{SL_2(\mathbb{Z})}/\widetilde{\Gamma(4m)}$.

Theorem 2.10. *We have an isomorphism*

$$S_{\pi_1}(\widetilde{SL}_2(\mathbb{Z}); \Phi_m) \simeq \mathrm{Hom}_{\mathrm{G_J}}(\rho_{\mathrm{m},\pi_1}, \mathcal{H}^0_{\mathrm{m}}).$$

When π_1 is an anti-holomorphic discrete series representation this isomorphism is nothing but the classical Eichler-Zagier correspondence. When π_1 is a principal series representation $\mathrm{Hom}_{\mathrm{G_J}}(\rho_{\mathrm{m},\pi_1}, \mathcal{H}^0_{\mathrm{m}})$ is an equivarent notion of Maass Jacobi cusp forms.

3 A review on Fourier-Jacobi type spherical functions and Whittaker functions for $Sp(2;\mathbb{R})$

In what follows, $\mathfrak{g}$ denotes the Lie algebra of $G = Sp(2;\mathbb{R})$.

3.1 Review on Whittaker functions

For an admissible representation π of G with K-type τ and a unitary character ψ of the maximal unipotent subgroup N_0 of G the Whittaker functions on G are defined as the restriction map ι_τ of the intertwining operators in $\mathrm{Hom}_{(\mathfrak{g},\mathrm{K})}(\pi, \mathrm{C}^\infty_\psi(\mathrm{N}_0\backslash \mathrm{G}))$ to the K-type τ, where $C^\infty_\psi(N_0\backslash G) := \{\phi \in C^\infty(G) \mid \phi(ng) = \eta(n)\phi(g) \quad \forall (n,g) \in N_0 \times G\}$. The image $\mathrm{Im}(\iota_\tau)$ of the restriction map is contained in $C^\infty_{\psi,\tau^*}(N_0\backslash G/K) := \{C^\infty\text{-function } W : G \to V^* \mid W(ngk) = \psi(n)\tau^*(k)^{-1}W(g) \ \forall (n,g,k) \in N_0 \times G \times K\}$, where (τ^*, V^*) denotes the contragredient representation of (τ, V) with the representation spaces V and V^*. By $W_{\psi,\pi}(\tau^*)$ we denote the image of the restriction map. We will need

$$W_{\psi,\pi}(\tau^*)^0 := \{w \in \mathrm{Im}(\iota_\tau) \mid w \text{ is rapidly decreasing}\}$$

which is motivated by the Fourier expansion of cusp forms.

Let $\dim V^* = d+1$ and $\{v^*_k\}_{k=0}^d$ be a basis of V^* consisting of weight vectors with highest weight vector v^*_d. When the highest weight of τ is (Λ_1, Λ_2), we have $d = \Lambda_1 - \Lambda_2$. We can express each Whittaker function $W \in W_{\psi,\pi}(\tau^*)^0$ as $W(g) = \sum_{k=0}^d c_k(g) v^*_k$ with coefficient functions $c_k(g)$. Now recall that G admits the Iwasawa decomposition of $G = N_0A_0K$, where $A_0 := \left\{a_0 = \mathrm{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_1,\ a_2 \in \mathbb{R}_{>0}\right\}$. We then see that the Whittaker function W is determined by the restriction to A_0.

As the references on explicit formulas for the Whittaker functions we cite Oda [16], Miyazaki-Oda [11], [12], Niwa [15] and Ishii [9].

3.2 Fourier-Jacobi type spherical functions

We review the notion of the Fourier-Jacobi type spherical functions after Hirano [6], [7] and [8].

For an admissible representation π of $G = Sp(2;\mathbb{R})$ with K-type τ and an irreducible unitary representation ρ of G_J the Fourier-Jacobi type spherical functions are defined as the restriction of the intertwining operators in $\mathrm{Hom}_{(\mathfrak{g},\mathrm{K})}(\pi, \mathrm{C}^{\infty}\mathrm{Ind}_{\mathrm{G_J}}^{\mathrm{G}}\rho)$ to the K-type τ. Such restricted intertwining operators are contained in $C^{\infty}_{\rho,\tau^*}(G_J\backslash G/K) := \{C^{\infty}\text{-function } W : G \to H_\rho \boxtimes V^* \mid W(rgk) = \rho(r)\boxtimes\tau^*(k)^{-1}W(g)\ \forall(r,g,k) \in G_J \times G \times K\}$, where H_ρ and V^* denote the representation spaces of ρ and the contragredient τ^* of τ respectively. Following Hirano [6], [7] and [8] we denote the image of the restriction map by $\mathcal{J}_{\rho,\pi}(\tau^*)$. We call this the space of the Fourier-Jacobi type spherical functions of type $(\rho,\pi;\tau^*)$. In terms of the theory of the Fourier expansion of cusp forms we are interested in

$$\mathcal{J}_{\rho,\pi}(\tau^*)^0 := \{W \in \mathcal{J}_{\rho,\pi}(\tau^*) \mid W \text{ is rapidly decreasing with respect to } A_J\}.$$

Now recall that irreducible unitary representations of G_J with the central character parametrized by $m \neq 0$ is of the form $\rho_{m,\pi_1} := \pi_1 \boxtimes \tilde{\nu}_m$ (cf. Proposition 2.2). Let us introduce the notation $\{w_l\}_{l\in L}$ and $\{u^m_j\}_{j\in J}$ for a basis of W_{π_1} and $\mathcal{U}_m$ respectively, where W_{π_1} denotes the representation space of π_1. We retain the notation $\{v^*_k\}_{0\le k\le d}$ for the representation space V^* of τ^* (cf. Section 3.1). As is pointed out in Hirano's papers (e.g. [6, Lemma 4.4]), the restriction of $W \in \mathcal{J}_{\rho,\pi_\lambda}(\tau^*)^0$ to A_J is written as

$$W(a_J) = \sum_{k=0}^{d} \sum_{\substack{j\in J \\ l=l(j,k)\in L}} c_{j,k}(a_J) w_l \otimes u^m_j \otimes v^*_k \quad (a_J \in A_J)$$

with coefficient functions $c_{j,k}(a_J)$. Here $l(j,k) = -j + k + \Lambda_2$, for which note that the highest weight of τ^* is $(-\Lambda_2, -\Lambda_1)$ when that of τ is (Λ_1, Λ_2). The index $l(j,k)$ is due to the compatibility of τ^* and the $SO(2)$-types of ρ_{m,π_1} with respect to the $K\cap SL_2(\mathbb{R}) = SO(2)$-action. We should note that Hirano obtained an explicit formulas for the Fourier-Jacobi type spherical functions (cf. [6], [7] and [8]) in terms of the Mejer G-function $G^{m,n}_{p,q}$. For the detail on the Mejer G-function $G^{m,n}_{p,q}$ we cite [6, Appendix] and the references by Meijer cited therein.

4 Fourier-Jacobi expansion

4.1 Cusp forms and the working assumption

We first discuss cusp forms on $G = Sp(2;\mathbb{R})$ with respect to the Siegel modular group $Sp(2;\mathbb{Z})$ in a general context by representation theory.

Definition 4.1. Let π be an irreducibel admissible representation of G and (τ, V) be a K-type of π with the representation space V. Recall that we have denoted the contragredient representation of (τ, V) by (τ^*, V^*) with the representation space V^*. Let $F : G \to V^*$ be a cusp form of weight τ^* with respect to $Sp(2;\mathbb{Z})$, namely a cusp form F satisfying

$$F(\gamma g k) = \tau^*(k)^{-1} F(g) \quad \forall (\gamma, g, k) \in Sp(2;\mathbb{Z}) \times G \times K.$$

A cusp form F of weight τ^* with respect to $Sp(2;\mathbb{Z})$ is said to be generating π if the G-module generated by the right G-translations of coefficient functions $\{\langle F(g), v\rangle \mid v \in V\}$ of F is isomorphic to π as $(\mathfrak{g}, K)$-modules, where $V^* \times V \ni (v^*, v) \mapsto \langle v^*, v\rangle \in \mathbb{C}$ denotes a K-invariant pairing.

It is well known that the total space of cusp forms on a reductive group decomposes discretely into irreducible pieces with finite multiplicities. This is the reason why we assume π to be irreducible.

When π is a holomorphic (respectively anti-holomorphic) discrete series representation of G and τ is the minimal K-type of π, this notion is nothing but holomorphic (respectively anti-holomorphic) Siegel cusp forms.

We should note that he Fourier-Jacobi expansion which we are going to discuss is applicable when an irreducible admissible representation π admits the Whittaker model, i.e. for a non-degenerate character ψ of N_0 we have

$$\mathrm{Hom}_{(\mathfrak{g},\mathrm{K})}(\pi, \mathcal{A}_\psi(\mathrm{N}_0\backslash\mathrm{G})) \neq 0,$$

where

$$\mathcal{A}_\psi(N_0\backslash G) := \{W \in C^\infty_\psi(N_0\backslash G) \mid W \text{ is of moderate growth}\}.$$

As is well known, an irreducible admissible representation π with the above property is called generic and we thus call cusp forms generating such a representation generic cusp forms. We note that holomorphic or anti-holomorphic discrete series representation are not generic, which ca be also said to be well known.

To discuss the Fourier-Jacobi expansion of cusp forms generating an irreducible admissible representations π we make the following working assumption on the multiplicity free property of the Whittaker functions and the Fourier-Jacobi type spherical functions for π:

Working Assumption. There is a multiplicity one K-type τ of π such that

- $\dim W_{\psi,\pi}(\tau^*)^0 \leq 1$ holds for any non-degenerate $\psi \in \hat{N}_0$ and there is no rapidly decreasing element in $W_{\psi,\pi}(\tau^*)^0$ for any degenerate $\psi \in \hat{N}_0$,
- $\dim \mathcal{J}_{\rho,\pi_1}(\tau^*)^0 \leq 1$ holds for any irreducible unitary representations ρ of G_J with the non-trivial central character.

For large discrete series representations, irreducible P_J-principal series representations and irreducible principal series representations (including non-spherical principal series representations), the working assumption totally holds.

4.2 Main result

The Fourier Jacobi expansion of a cusp form F is written as

$$F(g) = \sum_{m\in\mathbb{Z}} F_m(g)(g \in G), \quad F_m(g) := \int_{\mathbb{R}/\mathbb{Z}} F(n(0,0,z)g)\mathbf{e}(-mz)dz.$$

To state the result on this we introduce a couple of ingredients.

1. We first let

$$F_{\xi_0,\xi_3}(g) := \int_{N_0(\mathbb{Z})\backslash N_0} F(n(u_0,u_1,u_2,u_3)g)\psi_{\xi_0,\xi_3}(n(u_0,u_1,u_2,u_3))^{-1}dn \quad (g \in G)$$

   for the unitary character $\psi_{\xi_0,\xi_3} : N \ni n(u_0,u_1,u_2,u_3) \mapsto \exp(2\pi\sqrt{-1}(\xi_0u_0+\xi_3u_3))$ of N_0 with $(\xi_0,\xi_3) \in \mathbb{Z}^2$, where dn denotes the invariant measure of $N_0(\mathbb{Z})\backslash N_0$ normalized so that $\mathrm{vol}(N_0(\mathbb{Z})\backslash N_0) = 1$. By the working assumption F_{ξ_0,ξ_3} is a constant multiple of a Whittaker function in $W_{\psi_{\xi_0,\xi_3},\pi}(\tau^*_\lambda)^0$ for a non-degenerate ψ_{ξ_0,ξ_3}. This is proved to contribute to all the F_m-terms.

2. We need more ingredients to describe the F_m-term for a non-zero $m \in \mathbb{Z}$. For this purpose we remark that singular semi-integral matrices of degree two with the fixed upper left entry m is of the forms

$$S_{\alpha,m}(k) = \begin{pmatrix} m & km+\frac{\alpha}{2} \\ km+\frac{\alpha}{2} & m(k+\frac{\alpha}{2m})^2 \end{pmatrix}$$

   with some integers $k \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}$ such that $\dfrac{\alpha^2}{4m} \in \mathbb{Z}$, where α is determined modulo $2m$. We put

$$S_{\alpha,m} := S_{\alpha,m}(0), \quad \gamma_{\alpha,m} := n(\frac{\alpha}{2m},0,0,0).$$

   and provide

$$F_{S_{\alpha,m}}(g) := \int_{N_S(\mathbb{Z})\backslash N_S} F(n(0,u_1,u_2,u_3)g)\mathbf{e}(-\operatorname{tr}(S_{\alpha,m}\begin{pmatrix} u_1 & u_2 \\ u_2 & u_3 \end{pmatrix}))du_1du_2du_3.$$

   We introduce

$$L_{\alpha,m} := d_{\alpha,m}\mathbb{Z} \quad \text{with } d_{\alpha,m} := 2m/(2m,\alpha),$$

   where we take α in $1 \le \alpha \le 2|m|$, which forms a complete set of representatives for $\mathbb{Z}/2m\mathbb{Z}$. The lattice $L_{\alpha,m}$ coincides with

$$\{u_0 \in \mathbb{R} \mid \gamma_{\alpha,m}^{-1}\,{}^t n(u_0,0,0,0)\gamma_{\alpha,m} \in Sp(2;\mathbb{Z})\}.$$

The dual lattice $\widehat{L_{\alpha,m}}$ of $L_{\alpha,m}$ is $\frac{1}{d_{\alpha,m}}\mathbb{Z}$. We then furthermore provide

$$F_{S_{\alpha,m},n}(g) := \int_{\mathbb{R}/L_{\alpha,m}} F_{S_{\alpha,m}}(\gamma_{\alpha,m}^{-1}\,{}^t n(u_0,0,0,0)\gamma_{\alpha,m}g)\mathbf{e}(-nu_0)du_0$$

for $n \in \widehat{L_{\alpha,m}} \setminus \{0\}$. This turns out to be the left translate of a Whittaker function by $(\xi^t\gamma_{\alpha,m})^{-1}$, where $\xi = .\begin{pmatrix} J_2 & 0_2 \\ 0_2 & J_2 \end{pmatrix}$ with $J_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For $\alpha \mod 2m \in \mathbb{Z}/2m\mathbb{Z}$ with $\dfrac{\alpha^2}{4m} \in \mathbb{Z}$ we introduce

$$E_\alpha(F_{S_{\alpha,m},n}(*g))(r) := \sum_{\gamma \in Z_J(\mathbb{Z})(G_J(\mathbb{Z})\cap\gamma_{\alpha,m}^{-1}V_J\gamma_{\alpha,m})\backslash G_J(\mathbb{Z})} F_{S_{\alpha,m},n}(\gamma \cdot rg).$$

This is a Jacobi Eisenstein-Poincaré series with the test function $F_{S_{\alpha,m},n}(g)$. Though this is called an "incomplete theta series" and should be denoted by θ_α as in [2, Section 4.4] we use the notation E_α to avoid the confusion with $\theta_\alpha \in \Phi_m$.

3. We further recall from section 3.2 that the Fourier-Jacobi type spherical function of type $(\rho,\pi;\tau^*)$ restricted to A_J has been written as $\sum_{k=0}^d \sum_{\substack{j\in J \\ l=l(j,k)\in L}} c_{j,k}(a_J)w_l \otimes u_j^m \otimes v_k^*$ $(a_J \in A_J)$ when the irreducible unitary representation ρ has the non-trivial central character indexed by $m \in \mathbb{Z}\setminus\{0\}$. In what follows, when ρ is specified as ρ_{m,π_1} with $\pi_1 \in \widehat{\widetilde{SL}_2(\mathbb{R})}$, we denote $c_{j,k}$ by $c_{j,k}^{(\pi_1)}$ in order to indicate the dependence of $c_{j,k}$ on π_1. We thus write

$$\sum_{k=0}^d \sum_{\substack{j\in J \\ l=l(j,k)\in L}} c_{j,k}^{(\pi_1)}(a_J)w_l \otimes u_j^m \otimes v_k^* \quad (a_J \in A_J)$$

for the Fourier-Jacobi type spherical function. The notation just explained is necessary to complete the description of the F_m-term with $m \neq 0$.

We remark that the Jacobi parabolic subgroup P_J coincides with $G_J A_J$, for which recall that $A_J = \{a_J = \begin{pmatrix} a_1 & & & \\ & 1 & & \\ & & a_1^{-1} & \\ & & & 1 \end{pmatrix} \mid \alpha \in \mathbb{R}_+^\times\}$. With the coordinate $(r, a_J) \in G_J \times A_J$ we have the theorem as follows:

Theorem 4.2. *Let π be an irreducible admissible representation of G with the multiplicity one K-type τ satisfying the working assumption and let F be a cusp form of weight τ^* with respect to $Sp(2;\mathbb{Z})$ generating π.*

Each term F_m of the Fourier-Jacobi expansion $\sum_{m\in\mathbb{Z}} F_m$ of F is expressed as

$$\sum_{(\xi_0,\xi_3)\in\mathbb{Z}^2,\ \xi_0\xi_3\neq 0}\ \sum_{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})_\infty\backslash SL_2(\mathbb{Z})} F_{\xi_0,\xi_3}(\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & a & 0 & b\\ 0 & 0 & 1 & 0\\ 0 & c & 0 & d\end{pmatrix} ra_J)$$

for $m=0$ and

$$\sum_{\substack{1\le\alpha\le 2|m|\\ s.t.\ \alpha^2/4m\in\mathbb{Z}}}\ \sum_{n\in\widehat{L_{\alpha,m}}\backslash\{0\}} E_\alpha(F_{S_{\alpha,m},n}(*a_J))(r)+$$

$$\sum_{\pi_1\in\widehat{\widetilde{SL}_2(\mathbb{R})},m(\pi_1)\neq 0}\ \sum_{i=1}^{m(\pi_1)}\sum_{k=0}^{d}\ \sum_{\substack{j\in J,\\ l=l(j,k)\in L}} c_{j,k}^{(\pi_1)}(a_J)\phi_{\pi_1}^{(i)}(w_l\otimes u_j^m)(r)\otimes v_k^*$$

for $m\neq 0$, where

- *recall that $\{v_k^*\}_{k=0}^d$ denotes a basis of V^* consisting of weight vectors (cf. Section 3.1),*
- $SL_2(\mathbb{Z})_\infty := \left\{\begin{pmatrix} 1 & n\\ 0 & 1\end{pmatrix}\ \middle|\ n\in\mathbb{Z}\right\}$,
- $\mathfrak{m}(\pi_1) := \dim \mathrm{Hom}_{\mathrm{G_J}}(\rho_{\mathrm{m},\pi_1}, \mathcal{H}_{\mathrm{m}}^0)$ *and* $\{\phi_{\pi_1}^{(i)}\}$ *is a basis of* $\mathrm{Hom}_{\mathrm{G_J}}(\rho_{\mathrm{m},\pi_1}, \mathcal{H}_{\mathrm{m}}^0)$.

The term F_0 is contributed by representations of G_J with the trivial central character, which are representations induced from characters of N_0 (cf. Proposition 2.2 (2) part 2). We then find it quite natural to see that F_0 is written as a sum of the Whitaker functions.

When m is a non-zero integer we have $F_m\in\mathcal{H}_m$. For this we should note that $\mathcal{H}_m = \mathcal{H}_m^c\oplus\mathcal{H}_m^0$ with the orthogonal complement $\mathcal{H}_m^c$ to $\mathcal{H}_m^0$ in $\mathcal{H}_m$. The space $\mathcal{H}_m^c$ is the continuous spectrum of $\mathcal{H}_m$. The summation $\sum_{\substack{1\le\alpha\le 2|m|\\ \text{s.t. } \alpha^2/4m\in\mathbb{Z}}}\sum_{n\in\widehat{L_{\alpha,m}}\backslash\{0\}} E_\alpha(F_{S_{\alpha,m},n}(*a_J))(r)$ is the $\mathcal{H}_m^c$-part of F_m, which the author did not notice in the presentation at the workshop.

Remark 4.3. (1) When π is holormophic or anti-holomorphic discrete series π is not generic as we have pointed out. We therefore see that $F_0\equiv 0$ and $E_\alpha(F_{S_{\alpha,m},n}(*a_J))\equiv 0$ for such π.
(2) The paper [14] in preparation includes more specific descriptions of the Fourier-Jacobi expansions for the cases of large discrete series representations, P_J-principal series representations and principal series representations. In addition, it also includes examples of Jacobi cusp forms $\phi_{\pi_1}^{(i)}(w_l\otimes u_j^m)(r)$ contributing to the Fourier-Jacobi expansions.

References

[1] R. Berndt and S. Böcherer, Jacobi forms and discrete series representations of the Jacobi group, Math. Z., 204 no.1 (1990), 13–44.

[2] R. Berndt and R. Schmidt, Elements of the Representation Theory of the Jacobi Group, Birkhäuser Verlag (1998).

[3] R. E. Borcherds, Automorphic forms with singularities on Grassmannians, Invent math. 132, (1998), 491–562.

[4] L. Corwin and F. P. Greenleaf, Intertwining operators for representations induced from uniform subgroups, Acta Math., 136 (1976), 275–301.

[5] L. Corwin and F. P. Greenleaf, Representations of nilpotent Lie groups and their applications, Part 1: Basic theory and examples, Cambridge university press, Cambridge, (1989).

[6] M. Hirano, Fourier-Jacobi Type Spherical Functions for Discrete Series Representations of $Sp(2,\mathbb{R})$, Compos. Math., 128 (2001), 177–216.

[7] M. Hirano, Fourier-Jacobi Type Spherical Functions for P_J-principal series representations of $Sp(2;\mathbb{R})$, Journal of London Math. Soc., 65 (2002), 524–546.

[8] M. Hirano, Fourier-Jacobi Type Spherical Functions for principal series representations of $Sp(2;\mathbb{R})$, Indagationes math., 15 (2004), 43–53.

[9] T. Ishii, On principal series Whittaker functions on $Sp(2,\mathbb{R})$, J. Funct. Anal., 225 (2005), 1–32.

[10] T. Kubota, Topological covering of $SL(2)$ over a local field, J. Math. Soc. Japan. 19 (1967), 114–121.

[11] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2;\mathbb{R})$, Explicit formulae of differential equations, Automorphic forms and related topics, The Pyungsan Inst. Math. Sci., 59–92 (Seoul, 1993).

[12] T. Miyazaki and T. Oda, Principal series Whittaker functions on $Sp(2;\mathbb{R})$, II, Tohoku Math. J., 50 (1998), 243–260.

[13] T. Miyazaki and T. Oda, Errata: Principal series Whittaker functions on $Sp(2;\mathbb{R})$, II, (Tohoku Math. J., 50 (1998), 243–260), Tohoku Math. J., 54 (2002), 243–260.

[14] H. Narita, Fourier-Jacobi expansion of cusp forms on $Sp(2;\mathbb{R})$, in preparation.

[15] S. Niwa, Commutation relations of differential operators and Whittaker functions on $Sp_2(\mathbb{R})$, Proc. Japan Acad. 71 Ser. A (1995) 189-191.

[16] T. Oda, An explicit integral representation of Whittaker functions on $Sp(2;\mathbb{R})$ for the large discrete series representations, Tohoku Math. J., 46 (1994), 261–279.

[17] I. Satake, Unitary representations of a semi-direct product of Lie groups on $\bar{\partial}$-cohomology spaces, Math. Ann., 190 (1971), 177–202.

[18] T. Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J., 58 (1971), 83–126.

Hiro-aki Narita
Department of Mathematics
Faculty of Science and Engineering
Waseda University
3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555
JAPAN
E-mail: hnarita@waseda.jp