On linear relations for special L-values over certain totally real number fields

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1 Introduction

Let F, $\mathfrak{o}_F$, and $\mathfrak{d}_F$ be a totally real number field over $\mathbb{Q}$ with degree m, the ring of integers of F, and the different of F over $\mathbb{Q}$. We denote the m embeddings of F to $\mathbb{R}$ by $\iota_1, \iota_2, \dots, \iota_m$. We define a congruence subgroup $\mathbf{\Gamma}$ by

$$\mathbf{\Gamma} = \mathbf{\Gamma}[{\mathfrak{d}_F}^{-1}, 4\mathfrak{d}_F] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \,\middle|\, a, d \in \mathfrak{o}_F,\ b \in {\mathfrak{d}_F}^{-1},\ c \in 4\mathfrak{d}_F \right\}.$$

Let $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper-half plane. As usual, we embed $SL_2(F)$ into $SL_2(\mathbb{R})^m$ by $\gamma \mapsto (\iota_1(\gamma), \iota_2(\gamma), \dots, \iota_m(\gamma))$ and consider the Möbius transformation of $SL_2(F)$ on $\mathfrak{h}^m$ by this embedding.
For $\xi \in F$ and $\boldsymbol{z} = (z_1, z_2, \dots, z_m) \in \mathfrak{h}^m$, we set $q^\xi = e^{2\pi\sqrt{-1}\sum_{i=1}^m \iota_i(\xi) z_i}$. The standard theta series θ_F is given by

$$\theta_F(\boldsymbol{z}) = \sum_{\xi \in \mathfrak{o}_F} q^{\xi^2}.$$

We define the factor of automorphy $j_F(\gamma, \boldsymbol{z})$ by

$$j_F(\gamma, \boldsymbol{z}) = \frac{\theta_F(\gamma \boldsymbol{z})}{\theta_F(\boldsymbol{z})},$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{\Gamma}$ and $\boldsymbol{z} = (z_1, z_2, \dots, z_m) \in \mathfrak{h}^m$. When $F = \mathbb{Q}$, we write $\theta(z) = \theta_{\mathbb{Q}}(z)$ and $j(\gamma, z) = j_{\mathbb{Q}}(\gamma, z)$ briefly. It is known that

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz + d)^{\frac{1}{2}}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),\ z \in \mathfrak{h}$$

where $(\frac{\cdot}{\cdot})$ is the Shimura's quadratic reciprocity symbol [5], and ε_d is 1 or $\sqrt{-1}$ according as $d \equiv 1 \pmod 4$ or $d \equiv 3 \pmod 4$.
Let k be in $\frac{1}{2}\mathbb{Z}$, a holomorphic function f on $\mathfrak{h}^m$ is a Hilbert modular form on $\mathbf{\Gamma}$ of parallel weight k if f satisfies

$$f(\gamma \boldsymbol{z}) = j_F(\gamma, \boldsymbol{z})^{2k} f(\boldsymbol{z}) \text{ for any } \gamma \in \mathbf{\Gamma}, \boldsymbol{z} \in \mathfrak{h}^m$$

and that has the q-expansions of the forms

$$f(g\boldsymbol{z}) \prod_{i=1}^{m} (\iota_i(c) + \iota_i(d) z_i)^{-k} = c_g(0) + \sum_{\substack{\xi \in \mathfrak{o}_F \\ \xi \succ 0}} c_g(\xi) q^{\frac{\xi}{h_g}}$$

for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F)$ where $h_g > 0$ is the constant which depends only on g and $\xi \succ 0$ means that $\iota_i(\xi) > 0$ for any $i = 1, 2, \ldots, m$. We denote the space of Hilbert modular forms on $\mathbf{\Gamma}$ of parallel weight k by $M_k(\mathbf{\Gamma})$. We also denote the space of cusp forms by $S_k(\mathbf{\Gamma})$.
In 1975, Cohen [1] constructed a special modular form $\mathscr{H}_r \in M_{r+\frac{1}{2}}(\Gamma_0(4))$ for all positive integers r, called the Cohen Eisenstein Series of weight $r + \frac{1}{2}$ which has the q-expansion of the form

$$\mathscr{H}_r(z) = \zeta(1-2r) + \sum_{\substack{N \geq 1 \\ (-1)^r N \equiv 0,1 \pmod 4}} L(1-r, \mathfrak{X}_{(-1)^r N}) \times \sum_{d | f_{(-1)^r N}} \mu(d) \mathfrak{X}_{(-1)^r N}(d) d^{r-1} \sigma_{2r-1}\left(\frac{f_{(-1)^r N}}{d}\right) q^N$$

where $\mathfrak{X}_N$ is the quadratic character corresponding to $\mathbb{Q}(\sqrt{N})/\mathbb{Q}$, f_N is the natural number such that $N = D_N f_N^2$, and D_N is the discriminant of $\mathbb{Q}(\sqrt{N})/\mathbb{Q}$. The space of modular forms on $\Gamma_0(4)$ of weight $r + \frac{1}{2}$ whose nth Fourier coefficient vanishes unless $(-1)^r n$ is congruent to 0 or 1 modulo 4 is called the Kohnen plus space introduced and investigated by Kohnen in 1980 [3].
In 2013, Hiraga and Ikeda gave a generalization of the Kohnen plus space for Hilbert modular forms of half-integral weight [2].
Let κ be a positive integer, the Kohnen plus space $M^+_{\kappa+\frac{1}{2}}(\mathbf{\Gamma})$ with respect to $M_{\kappa+\frac{1}{2}}(\mathbf{\Gamma})$ is defined by the subspace of $M_{\kappa+\frac{1}{2}}(\mathbf{\Gamma})$ which consists of all $h \in M_{\kappa+\frac{1}{2}}(\mathbf{\Gamma})$ with Fourier coefficient of the form

$$h(z) = c(0) + \sum_{\substack{\xi \in \mathfrak{o}_F, \xi \succ 0 \\ (-1)^\kappa \xi \equiv \square \ (4)}} c(\xi) q^{\xi}.$$

Here, we define $\xi \equiv \square\ (4)$ if there exists $x \in \mathfrak{o}_F$ such that $\xi - x^2 \in 4\mathfrak{o}_F$. We also define $S^+_{\kappa+\frac{1}{2}}(\mathbf{\Gamma}) = M^+_{\kappa+\frac{1}{2}}(\mathbf{\Gamma}) \cap S_{\kappa+\frac{1}{2}}(\mathbf{\Gamma})$.

In 2016, Su constructed the Eisenstein series $G_{\kappa+\frac{1}{2},\chi'} \in M^+_{\kappa+\frac{1}{2}}(\mathbf{\Gamma})$ which is a generalization of the Cohen Eisenstein series [6]. Let χ' be a character of the class group of F, then $G_{\kappa+\frac{1}{2},\chi'}$ is defined by

$$G_{\kappa+\frac{1}{2},\chi'}(\boldsymbol{z}) = L_F(1-2\kappa, \overline{\chi'}^2) + \sum_{\substack{(-1)^\kappa \xi \equiv \square\ (4) \\ \xi \succ 0}} \mathcal{H}_\kappa(\xi, \chi') q^\xi$$

where $L_F(s, \chi)$ is the L-function over F with respect to the character χ defined by

$$L_F(s, \chi) = \sum_{\substack{0 \neq \mathfrak{A} \subset \mathfrak{o}_F \\ \text{ideal}}} \frac{\chi(\mathfrak{A})}{N_{F/\mathbb{Q}}(\mathfrak{A})^s}$$

for $\mathrm{Re}(s) > 1$ and

$$\begin{aligned} \mathcal{H}_\kappa(\xi, \chi') = \chi'(\mathfrak{D}_{(-1)^\kappa \xi}) & L_F(1-\kappa, \overline{\mathfrak{X}_{(-1)^\kappa \xi}\chi'}) \\ & \times \sum_{\mathfrak{a} | \mathfrak{F}_\xi} \mu(\mathfrak{a}) \mathfrak{X}_\xi(\mathfrak{a}) \chi'(\mathfrak{a}) N_{F/\mathbb{Q}}(\mathfrak{a})^{\kappa-1} \sigma_{F, 2\kappa-1, \chi'^2}(\mathfrak{F}_\xi \mathfrak{a}^{-1}). \quad (1) \end{aligned}$$

Here, $\mathfrak{D}_\xi$ and $\mathfrak{X}_\xi$ are the relative discriminant and the quadratic character corresponding to $F(\sqrt{\xi})/F$ respectively, $\mathfrak{F}_\xi$ is the integral ideal such that $\mathfrak{F}_\xi^2 \mathfrak{D}_\xi = (\xi)$, μ is the Möbius function for ideals, and $\sigma_{F,k,\chi}$ is defined by

$$\sigma_{F,k,\chi}(\mathfrak{A}) = \sum_{\mathfrak{b} | \mathfrak{A}} N_{F/\mathbb{Q}}(\mathfrak{b})^k \chi(\mathfrak{b}).$$

When F is a real quadratic field such that $\mathfrak{d}_F = (\delta)$ with a totally real positive element δ, Su gave linear relations between special L-values over F and some arithmetic functions [7].

In this paper, we give generalization of these linear relations.

We define arithmetic functions $\alpha_k(n)$ and $\beta_k(n)$ by

$$\alpha_k(n) := \begin{cases} -\frac{2k}{(2^k-1)B_k} \tilde{\sigma}_{k-1}(\frac{n}{2}) & \text{if } k \in 2\mathbb{Z}, \\ \frac{2}{L(-k+1, \chi_{-4})} \sigma_{k-1, \chi_{-4}}(n) & \text{if } k \in \mathbb{Z} \setminus 2\mathbb{Z}, \\ -\frac{2k-1}{(2^{k-\frac{1}{2}}-1)B_{k-\frac{1}{2}}} \sum_{s^2<n} \tilde{\sigma}_{k-\frac{3}{2}}(\frac{n-s^2}{2}) + 2\lambda(n) & \text{if } k \notin \mathbb{Z}, k - \frac{1}{2} \in 2\mathbb{Z}, \\ \frac{2}{L(-k+\frac{3}{2}, \chi_{-4})} \sum_{s^2<n} \sigma_{k-\frac{3}{2}, \chi_{-4}}(n-s^2) + 2\lambda(n) & \text{if } k \notin \mathbb{Z}, k - \frac{1}{2} \in \mathbb{Z} \setminus 2\mathbb{Z}, \end{cases}$$

$$\beta_k(n) := \begin{cases} \frac{(-1)^{\frac{k}{2}+1}2k}{(2^k-1)B_k}\sigma'_{k-1,\chi_4}(n) & \text{if } k \in 2\mathbb{Z}, \\ \frac{2^k(-1)^{\frac{k-1}{2}}}{L(-k+1,\chi_{-4})}\sigma'_{k-1,\chi_{-4}}(n) & \text{if } k \in \mathbb{Z}\setminus 2\mathbb{Z}, \\ \frac{(-1)^{\frac{k}{2}+\frac{3}{4}}(2k-1)}{(2^{k-\frac{1}{2}}-1)B_{k-\frac{1}{2}}}\sum_{s^2<n}\sigma'_{k-\frac{3}{2},\chi_4}(n-s^2) & \text{if } k \notin \mathbb{Z}, k-\frac{1}{2} \in 2\mathbb{Z}, \\ \frac{2^{k-\frac{1}{2}}(-1)^{\frac{k}{2}-\frac{3}{4}}}{L(-k+\frac{3}{2},\chi_{-4})}\sum_{s^2<n}\sigma'_{k-\frac{3}{2},\chi_{-4}}(n-s^2) & \text{if } k \notin \mathbb{Z}, k-\frac{1}{2} \in \mathbb{Z}\setminus 2\mathbb{Z}, \end{cases}$$

where B_k is the k-th Bernoulli number,

$$\tilde{\sigma}_k(n) = \sum_{d|n} d^k(-1)^d, \quad \lambda(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{otherwise} \end{cases},$$

and we set $a(x) = 0$ for an arithmetic function $a(n)$ and $x \notin \mathbb{N} \cup \{0\}$.
Our main result is the following.

Theorem 1.1. *Let F be a totally real number field such that $8 \nmid m$ and $\mathfrak{d}_F = (\delta)$ with totally real positive element δ, we have*

$$\mathcal{R}G_{\kappa+\frac{1}{2},\chi'} = L_F(1-2\kappa, \overline{\chi'}^2)\left\{E_{m(\kappa+\frac{1}{2})} + 2^{-m\kappa}(-1)^{\frac{m\kappa(\kappa+1)}{2}}E^*_{m(\kappa+\frac{1}{2})}\right\} + Q$$

where Q is some cusp form in $S_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$.

By comparing the Fourier coefficients of both sides, we deduce the following corollary.

Corollary 1.1. *With the above notation, if $\mathcal{H}_\kappa(\xi, \chi')$ is as in (1), we have*

$$\sum_{\substack{\xi \in \mathfrak{o}_F, \xi \succ 0 \\ (-1)^\kappa \xi \equiv \square\ (4) \\ \mathrm{Tr}(\frac{\xi}{\delta}) = n}} \mathcal{H}_\kappa(\xi, \chi') = L_F(1-2\kappa, \overline{\chi'}^2)$$

$$\times \left\{\alpha_{m(\kappa+\frac{1}{2})}(n) + 2^{-m\kappa}(-1)^{\frac{m\kappa(\kappa+1)}{2}}\beta_{m(\kappa+\frac{1}{2})}(n)\right\} + c(n)$$

where $c(n)$ is the q-coefficient of some cusp form in $S_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$.

2 Outline of the proof

From here until the end, we assume that F is a totally real number field such that $\mathfrak{d}_F = (\delta)$ with totally real positive element δ.
Let $\mathbb{A}_F$, $\psi = \prod \psi_v$ be the adele ring of F, the additive character on $\mathbb{A}_F/F$ with

$\psi_v(x) = e^{(-1)^\kappa 2\pi\sqrt{-1}x}$ for archimedian places v. Let f be a complex valued function on $\mathfrak{h}^m$, we define a complex valued function $\mathcal{R}f$ on $\mathfrak{h}$ as follows.

$$(\mathcal{R}f)(z) = f\left(\frac{z}{\iota_1(\delta)}, \frac{z}{\iota_2(\delta)}, \dots, \frac{z}{\iota_m(\delta)}\right).$$

Lemma 2.1. [7, Theorem 2.1] *For $f \in M_{k+\frac{1}{2}}(\mathbf{\Gamma})$, we have*

$$\mathcal{R}f \in M_{m(\kappa+\frac{1}{2})}(\Gamma_0(4)).$$

Moreover, if we write $f(\boldsymbol{z}) = \sum_{\xi\in\mathfrak{o}_F} c(\xi)q^\xi$, $\mathcal{R}f(z)$ has the q-expansion of the form

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty}\left(\sum_{\mathrm{Tr}_{F/\mathbb{Q}}(\frac{\xi}{\delta})=n} c(\xi)\right) q^n. \tag{2}$$

The most important part of the proof of Theorem 1.1 is the calculations of the constant terms of $\mathcal{R}G_{\kappa+\frac{1}{2},\chi'}$ at each cusp. For a complex valued function $f : \mathfrak{h}^m \longrightarrow \mathbb{C}$, we put

$$(\mathcal{W}_F f)(\boldsymbol{z}) = \prod_{i=1}^{m}(-2\sqrt{-1}\iota_i(\delta)z_i)^{-\kappa-\frac{1}{2}} f\left(-(4\iota_1(\delta)^2 z_1)^{-1}, \dots, -(4\iota_m(\delta)^2 z_m)^{-1}\right)$$

and

$$(\mathcal{U}_F f)(\boldsymbol{z}) = \prod_{i=1}^{m}(2\iota_i(\delta)z_i + 1)^{-\kappa-\frac{1}{2}} f\left(\frac{z_1}{2\iota_1(\delta)z_1+1}, \dots, \frac{z_m}{2\iota_m(\delta)z_m+1}\right).$$

The following lemmas give the constant terms of $\mathcal{W}_F G_{\kappa+\frac{1}{2},\chi'}$ and $\mathcal{U}_F G_{\kappa+\frac{1}{2},\chi'}$.

Lemma 2.2. [7] *The constant term of $\mathcal{W}_F G_{\kappa+\frac{1}{2},\chi'}$ is equal to*

$$2^{-m\kappa}(-1)^{\frac{m\kappa(\kappa+1)}{2}} L_F(1-2\kappa, \overline{\chi'}^2).$$

Lemma 2.3 (Kuga)**.** *The constant term of $\mathcal{U}_F G_{\kappa+\frac{1}{2},\chi'}$ is equal to*

$$2^{-m\kappa} L_F(1-2\kappa, \overline{\chi'}^2) \prod_{v|2}\int_{\mathfrak{o}_v} \psi_v\left(\frac{x^2}{2\delta}\right) dx.$$

Especially when $8 \nmid m$, this value is equal to 0.

Sketch of proof of the Theorem 1.1.

When $4 \nmid m$, the proof is similar to that of [7].
When $4 \mid m$ and $8 \nmid m$, we define operators $\mathcal{U}$ and $\mathcal{W}$ on $M_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$ as follows.

For $h \in M_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$,

$$(\mathcal{W}h)(z) = \left(\frac{2z}{\sqrt{-1}}\right)^{-m(\kappa+\frac{1}{2})} h\left(-\frac{1}{4z}\right).$$

$$(\mathcal{U}h)(z) = (2z+1)^{-m(\kappa+\frac{1}{2})} h\left(\frac{z}{2z+1}\right).$$

Then, by a simple caluculation, we can check that

$$\mathcal{W}\mathcal{R}G_{\kappa+\frac{1}{2},\chi'} = \mathcal{R}\mathcal{W}_F G_{\kappa+\frac{1}{2},\chi'}$$

and

$$\mathcal{U}\mathcal{R}G_{\kappa+\frac{1}{2},\chi'} = \mathcal{R}\mathcal{U}_F G_{\kappa+\frac{1}{2},\chi'}.$$

By Lemmas 2.1, 2.2, and 2.3, we have

$$\begin{aligned}\mathcal{R}G_{\kappa+\frac{1}{2},\chi'} =& L_F(1-2\kappa, \overline{\chi'}^2) E_{m(\kappa+\frac{1}{2})} \\ &+ 2^{-m\kappa}(-1)^{\frac{m\kappa(\kappa+1)}{2}} L_F(1-2\kappa, \overline{\chi'}^2) E^*_{m(\kappa+\frac{1}{2})} + Q\end{aligned}$$

where

$$E_k(z) = \begin{cases} \sum_{\gamma\in\Gamma_0(4)_\infty\backslash\Gamma_0(4)} j(\gamma,z)^{-2k} & \text{if } k \in \mathbb{Z} \\ \theta(z)\sum_{\gamma\in\Gamma_0(4)_\infty\backslash\Gamma_0(4)} j(\gamma,z)^{-2k+1} & \text{if } k \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}, \end{cases}$$

$$E^*_k(z) = \left(\frac{2z}{\sqrt{-1}}\right)^{-k} E_k\left(-\frac{1}{4z}\right).$$

and $Q \in S_{m(\kappa+\frac{1}{2})}(\Gamma_0(4))$. By comparing the Fourier coefficients of both side, we complete the proof. Indeed, $\alpha_k(n)$ and $\beta_k(n)$ represent the nth Fourier coefficients of E_k and E^*_k respectively. □

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