Vector bundle over a GKM graph and combinatorial Borel-Hirzebruch formula and Leray-Hirsh theorem

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1. Introduction

This article is the research announcement of the paper [**Ku21**]. The aim of the paper [**Ku21**] is to study an equivariant vector bundle over GKM manifolds from combinatorial point of view by using the notion of legs which are introduced in [**KU**] (also see [**LS17**] for a non-compat edge). In the paper [**Ku21**], we will use the notations from [**LS17**] to define an *equivariant vector bundle over a GKM graph*; however, in this article, we will use the notations which used in [**KU**].

1.1. GKM manifold and GKM graph. We first briefly recall the GKM manifold and the (abstract) GKM graph (see [**GZ01**] and [**Ku19**] also). Let T^n be the n-dimensional torus and M^{2m} be a $2m$-dimensional, compact, connected, manifold with almost effective T^n-action. We denote such manifold as (M^{2m}, T^n), or M^{2m}, M, (M, T) (if its torus action or dimensions of a manifold and a torus are obviously known from the context). We call (M^{2m}, T^n) a *GKM manifold* if it satisfies the following properties:

(1) the set of fixed points is not empty and isolated, i.e., M^T is 0-dimensional;
(2) the closure of each connected component of 1-dimensional orbits is equivariantly diffeomorphic to the 2-dimensional sphere, called an *invariant* 2*-sphere*.

Regarding fixed points as vertices and invariant 2-spheres as edges, this condition is equivalent to that the one-skeleton of (M^{2m}, T^n) has the structure of a graph, where a *one-skeleton* of (M^{2m}, T^n) is the orbit space of the set of 0- and 1-dimensional orbits. By attaching the tangential representations around the fixed points, we can define the labels on edges. This labeled graph is called a *GKM graph* of a GKM manifold (M, T).

Abstractly, the GKM graph can be defined as follows. Let Γ be an m-valent graph with the set of vertices $V(\Gamma)$ and the set of edges $E(\Gamma)$. We put a label $\alpha : E(\Gamma) \to \mathrm{Hom}(T, S^1) \simeq H^2(BT) \simeq \mathbb{Z}^n$ on Γ, where BT^n (often denoted by BT) is a classifying space of an n-dimensional torus T. Note that the cohomology ring (over $\mathbb{Z}$-coefficient) of BT^n is isomorphic to the polynomial ring

$$H^*(BT) \simeq \mathbb{Z}[a_1, \ldots, a_n],$$

where a_i is a variable with $\deg a_i = 2$ for $i = 1, \ldots, n$. Set

$$\alpha_{(p)} = \{\alpha(e) \mid e \in E_p(\Gamma)\} \subset H^2(BT),$$

where $E_p(\Gamma)$ is the set of out-going edges from the vertex p. Note that $|E_p(\Gamma)| = m$ because we assume Γ is an m-valent graph. An *axial function* on Γ is the function $\alpha : E(\Gamma) \to H^2(BT^n)$ for $n \leq m$ which satisfies the following three conditions:

(1): $\alpha(e) = \pm\alpha(\overline{e})$, where $\overline{e}$ is the edge e with the reversed orientation;
(2): for each vertex $p \in V(\Gamma)$, the set $\alpha_{(p)}$ is *pairwise linearly independent*, i.e., each pair of elements in $\alpha_{(p)}$ is linearly independent in $H^2(BT)$;
(3): for all $e \in E(\Gamma)$, there exists a bijective map $\nabla_e : E_{i(e)}(\Gamma) \to E_{t(e)}(\Gamma)$ from the out-going edges on the initial vertex $i(e)$ of e to the out-going edges on the terminal vertex $t(e)$ of e such that
(1) $\nabla_{\overline{e}} = \nabla_e^{-1}$,
(2) $\nabla_e(e) = \overline{e}$, and
(3) for each $e' \in E_{i(e)}(\Gamma)$, there exists an integer $c_e(e')$ such that

$$\alpha(\nabla_e(e')) - \alpha(e') = c_e(e')\alpha(e) \in H^2(BT). \tag{1.1}$$

The collection $\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ is called a *connection* on the labelled graph (Γ, α); we denote the labelled graph with connection as (Γ, α, ∇), and the equation (1.1) is called a *congruence relation.* We call the integer $c_e(e')$ in the congruence relation an *Euler number of e' over e.* The conditions as above are called an *axiom of axial function.*

Definition 1.1 (GKM graph [**GZ01**]). If an m-valent graph Γ is labeled by an axial function $\alpha : E(\Gamma) \to H^2(BT^n)$ for some $n \leq m$, then such labeled graph is said to be an (abstract) *GKM graph*, and denoted as (Γ, α, ∇) (or (Γ, α) if the connection ∇ is obviously determined).

In addtion, we often assume the following condition:

(4): for each $p \in V(\Gamma)$, the set $\alpha_{(p)}$ spans $H^2(BT)$.

The axial function which satisfies (4) is called an *effective* axial function.

Definition 1.2 ((m, n)-type GKM graph). Let (Γ, α, ∇) be an abstract GKM graph. If the axial function α is effective, (Γ, α, ∇) is said to be an *(m, n)-type GKM graph.*

1.2. Equivariant vector bundle over a GKM manifold. We next recall the equivariant (complex) vector bundle (see e.g. [**Ka88, Ka91**]). In particular, we introduce the equivariant vector bundle over a GKM manifold. Let M be a smooth manifold with T-action. Note that the T-action induces the diffeomorphism

$$t : M \to M$$

for each element $t \in T$. We often denote

$$t \cdot p := t(p)$$

for the map from $p \in M$ to $t(p) \in M$ by the diffeomorphism $t \in T$. Let ξ be a complex vector bundle over M. We use the following notations:

- $E(\xi)$ denotes the total space of ξ;
- $\pi : E(\xi) \to M$ denotes the projection of the vector bundle;
- $F_p(\xi) := \pi^{-1}(p)$ denotes the fibre over $p \in M$.

We call ξ an *equivariant (complex) vector bundle* over M if it satisfies the following three conditions:

(1) $E(\xi)$ also has a T-action;
(2) The projection $\pi : E(\xi) \to M$ is T-equivariant; therefore, for $p \in M$ and $t \in T$, the diffeomorphism $p \mapsto t \cdot p$ induces the map on fibres $t^* : F_p(\xi) \to F_{t\cdot p}(\xi)$;
(3) The induced map $t^* : F_p(\xi) \to F_{t\cdot p}(\xi)$ is a complex linear isomorphism for every $p \in M$ and $t \in T$.

If M is a GKM manifold and $E(\xi)$ be its equivariant complex rank r vector bundle, then there is the following irreducible decomposition for the fibre on a fixed point $p \in M^T$:

$$F_p(\xi) \simeq V(\eta_{p,1}) \oplus \cdots \oplus V(\eta_{p,r}), \tag{1.2}$$

for $j = 1, \dots, r$, where $\eta_{p,j} : T \to S^1$ is a one-dimensional (possibly trivial) representation. Note that the orbit space of each factor may be regarded as $V(\eta_{p,j})/T^n \simeq \mathbb{R}_+$ (half line), i.e., leg with the initial vertex p. Therefore, we may define the r-legs with labels on each fixed point p. This property is the motivation to define the equivariant vector bundle over a GKM graph.

2. Equivariant vector bundle over a GKM graph and its projectivization

In this section, we define the equivariant vector bundle over a GKM graph and its projectiviation.

2.1. Equivariant vector bundle over a GKM graph. Let $\mathcal{G} := (\Gamma, \alpha, \nabla)$ be an m-valent GKM graph with the axial function $\alpha : E(\Gamma) \to H^2(BT^n)$, where we denote $E = E(\Gamma)$ and $V = V(\Gamma)$. In this paper, we assume that there is no legs in E (see [**KU**]), i.e., Γ is a compact graph.

By Section 1.2, we may define a(n) (equivariant complex) *rank r vector bundle* $\widetilde{\mathcal{G}} := (\widetilde{\Gamma}, \widetilde{\alpha}, \widetilde{\nabla})$ over $\mathcal{G}$ as follows:

(1) the abstract (non-compact) graph $\widetilde{\Gamma}$ consists of $V(\widetilde{\Gamma}) = V$ and $E(\widetilde{\Gamma}) = E \cup L$, where L is the set of legs such that $L_p = L \cap E_p(\widetilde{\Gamma}) = \{l_{p,1}, \dots, l_{p,r}\}$ for all $p \in V$;
(2) the label $\widetilde{\alpha} : E \cup L \to H^2(BT^n)$ such that $\widetilde{\alpha}|_E = \alpha$ and $\alpha(l_{p,j}) = \eta_{p,j} \in H^2(BT^n)$;
(3) the connection $\widetilde{\nabla} = \{\widetilde{\nabla}_e \mid e \in E\}$ is defined by the collection of bijective maps $\widetilde{\nabla}_e : E_{i(e)} \cup L_{i(e)} \to E_{t(e)} \cup L_{t(e)}$ for the initial vertex $i(e)$ and the terminal vertex $t(e)$ of the edge e such that $\widetilde{\nabla}_e|_{E_{i(e)}} = \nabla_e$.

Remark 2.1. Note that the labels on L_p might not be pairwise linearly independent (see Figure 1). Therefore, the vector bundle over a GKM graph might not be defined by the one-skelton of the (non-compact) manifold with torus actions (also see [**GZ01**] for the geometric meaning of pairwise linearly independence).

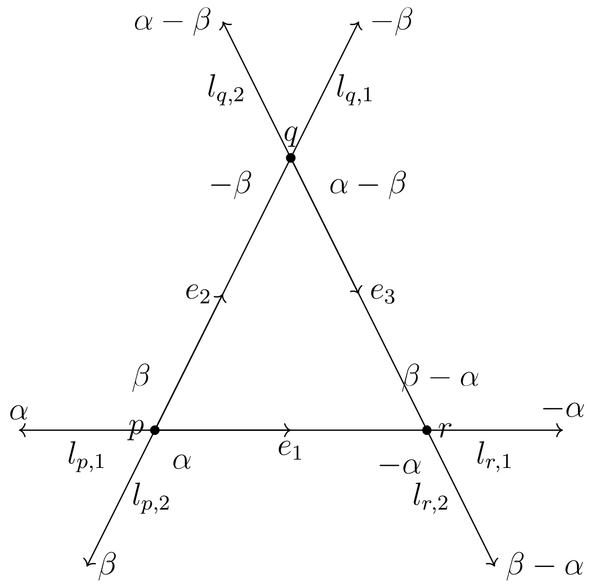

FIGURE 1. The labeled graph of the equivariant vector bundle which is induced from the tangent bundle over $\mathbb{C}P^2$ with the standard T^2-action. The middle triangle represents the GKM graph of $\mathbb{C}P^2$ with the standard T^2-action. Note that this labeled graph does not satisfy the pairwise linearly independent; for example around p, $\widetilde{\alpha}(e_1) = \alpha = \widetilde{\alpha}(l_{p,1})$, $\widetilde{\alpha}(e_2) = \beta = \widetilde{\alpha}(l_{p,2})$.

2.2. Projectivization of an equivariant vector bundle over a GKM graph. Let $\widetilde{\mathcal{G}} = (\widetilde{\Gamma}, \widetilde{\alpha}, \widetilde{\nabla})$ be a rank $r+1$ vector bundle over a GKM graph $\mathcal{G} = (\Gamma, \alpha, \nabla)$ for some $r \geq 0$. We define the projectivization $P(\widetilde{\mathcal{G}}) := (\Gamma', \alpha', \nabla')$.

The graph Γ' of $P(\widetilde{\mathcal{G}})$ consists of the following vertices and edges:

(1) $V(\Gamma') = L$, i.e., each leg becomes a vertex of $P(\widetilde{\mathcal{G}})$;
(2) two legs $l_{p,i}$, $l_{q,j}$ are connecting by the edge if one of the following holds:
  - $p = q$, i.e., $l_{p,i}, l_{p,j} \in L_p$;
  - there exists an edge $e \in E$ such that $i(e) = p, t(e) = q$ and $\widetilde{\nabla}_e(l_{p,i}) = l_{q,j}$.

It is easy to check that Γ' is an $m+r$ valent graph. We attach the label $\alpha' : E' \to H^2(BT^n)$ on every edge as follows:

(1) if $e \in E'$ satisfies $i(e) = l_{p,i}$, $t(e) = l_{p,j}$, then
$$\alpha'(e) = \widetilde{\alpha}(l_{p,i}) - \widetilde{\alpha}(l_{p,j});$$
(2) if $e \in E'$ satisfies $i(e) = l_{p,i}$, $t(e) = \widetilde{\nabla}_f(l_{p,i})$ for some edge $f \in E$, then
$$\alpha'(e) = \widetilde{\alpha}(f) = \alpha(f).$$

We can define the connection $\nabla'_e : E'_{i(e)} \to E'_{t(e)}$ which satisfies the congruence relations as follows:

(1) if $e \in E'$ is the edge with $i(e) = l_{p,i}$, $t(e) = l_{p,j}$, then for $f \in E'_{i(e)}$ the edge $\nabla'_e(f)$ is the edge which satisfies that
   (a) if $t(f) = l_{p,k}$, then $i(\nabla'_e(f)) = t(e) = l_{p,j}$ and $t(\nabla'_e(f)) = t(f) = l_{p,k}$;
   (b) if $t(f) = \widetilde{\nabla}_g(l_{p,i})$ for some $g \in E$ in Γ such that $i(g) = p$, $t(g) = q$, then $i(\nabla'_e(f)) = t(e) = l_{p,j}$ and $t(\nabla'_e(f)) = \widetilde{\nabla}_g(l_{p,j})$.

(2) if $e \in E'$ is the edge with $i(e) = l_{p,i}$, $t(e) = \widetilde{\nabla}_f(l_{p,i})$ for some edge $f \in E$, then for $g \in E'_{i(e)}$ the edge $\nabla'_e(g)$ is the edge which satisfies that
   (a) if $t(g) = l_{p,j}$, then $i(\nabla'_e(g)) = t(e) = \widetilde{\nabla}_f(l_{p,i})$ and $t(\nabla'_e(g)) = \widetilde{\nabla}_f(l_{p,j})$;
   (b) if $t(g) = \widetilde{\nabla}_h(l_{p,i})$ for some $h \in E$ in Γ such that $i(h) = p$, $t(h) = q$, then $i(\nabla'_e(g)) = t(e) = \widetilde{\nabla}_f(l_{p,i})$ and $t(\nabla'_e(g)) = \widetilde{\nabla}_{\nabla_h(f)} \circ \widetilde{\nabla}_f(l_{p,i})$.

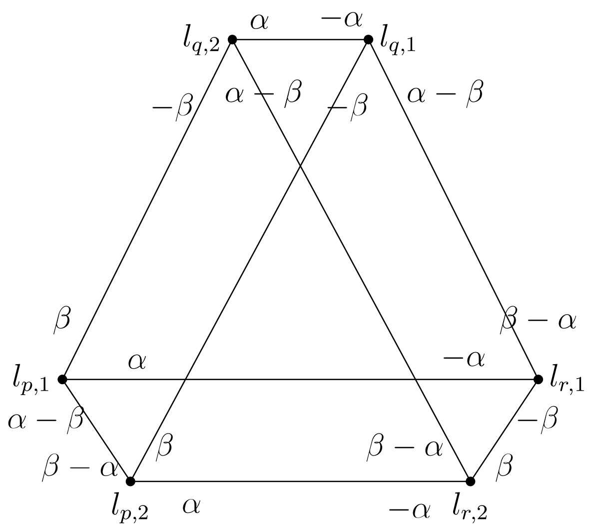

FIGURE 2. The projectivization of the vector bundle in Figure 1. Geometrically this is nothing but the projectivization of the tangent bundle over $\mathbb{C}P^2$, i.e., $P(T\mathbb{C}P^2)$. We can also check that there is an equivariant diffeomorphism $P(T\mathbb{C}P^2) \simeq \mathcal{F}l(\mathbb{C}^3)$, i.e., the 6-dimensional flag manifold with T^2-action.

3. Combinatorial Borel-Hirzebruch formula and Leray-Hirsh theorem

In this section, we will state the main theorem. Namely, we translate the Borel-Hirzebruch formula for the projectivization of complex vector bundle and the Leray-Hirsh theorem for the complex projective bundle to the combinatoral theorem for GKM graphs. In this section, we put

- $\mathcal{G} = (\Gamma, \alpha, \nabla)$ is an m-valent GKM graph with $\alpha : E \to H^2(BT^n)$, where $\Gamma = (V, E)$;

- $\widetilde{\mathcal{G}} = (\widetilde{\Gamma}, \widetilde{\alpha}, \widetilde{\nabla})$ is a rank $r+1$ vector bundle over $\mathcal{G}$, where $L_p := \{l_{p,1}, \ldots, l_{p,r+1}\}$ is the legs over $p \in V$;
- $P(\widetilde{\mathcal{G}}) = (\Gamma', \alpha', \nabla')$ is the projectivization of $\widetilde{\mathcal{G}}$.

Recall that the cohomology ring $H^*(\mathcal{G})$ of a GKM graph $\mathcal{G}$ is defined as follows:

$$H^*(\mathcal{G}) := \{f : V \to H^*(BT^n) \mid f(i(e)) - f(t(e)) \equiv 0 \mod \alpha(e)\},$$

where $H^*(BT^n) = \mathbb{Z}[a_1, \ldots, a_n]$. If $P(\widetilde{\mathcal{G}})$ is a GKM graph, then there is the natural embedding from $H^*(\mathcal{G})$ to $H^*(P(\widetilde{\mathcal{G}}))$ by taking $f(p) = f(l_{p,i})$ for all $i = 1, \ldots, r+1$, i.e.,

$$H^*(\mathcal{G}) \ni \bigoplus_{p\in V} f(p) \mapsto \bigoplus_{p\in V} \left(\bigoplus_{i=1}^{r+1} f(p) \right) \in H^*(P(\widetilde{\mathcal{G}})). \tag{3.1}$$

3.1. Preliminary. To state the main theorem, we need to prepare some notations. We first define the ith Chern class of $\widetilde{\mathcal{G}}$, say $c_i^T(\widetilde{\mathcal{G}}) \in H^{2i}(\mathcal{G})$, for $i = 0, \ldots, r+1$. Put

$$\widetilde{\alpha}(l_{p,j}) = \eta_{p,j} \in H^2(BT^n)$$

for all $j = 1, \ldots, r+1$ on $p \in V$. We define the following ith symmetric polynomial in $H^{2i}(BT^n)$:

$$\begin{aligned}\sigma_{p,i}(\widetilde{\mathcal{G}}) &:= \sigma_i(\eta_{p,1}, \ldots, \eta_{p,r+1}) \\ &= \sum_{k_1+\cdots+k_{r+1}=i} \eta_{p,1}^{k_1} \cdots \eta_{p,r+1}^{k_{r+1}} \in H^{2i}(BT^n) \subset H^*(BT^n).\end{aligned}$$

Set

$$c_i^T(\widetilde{\mathcal{G}}) := \bigoplus_{p\in V} \sigma_{p,i}(\widetilde{\mathcal{G}}) \in \bigoplus_{p\in V} H^*(BT^n).$$

By using the GKM conditions, we have the following lemma:

Lemma 3.1. $c_i^T(\widetilde{\mathcal{G}}) \in H^*(\mathcal{G})$.

We next define the 1st Chern class of the tautological line bundle of $P(\widetilde{\mathcal{G}})$, say $c_1^T(\gamma_{\widetilde{\mathcal{G}}}) \in H^2(P(\widetilde{\mathcal{G}}))$. The element $c_1^T(\gamma_{\widetilde{\mathcal{G}}}) : V' \to H^2(BT^n)$ is defined as follows:

$$c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{p,j}) := \alpha(l_{p,j}) = \eta_{p,j} \in H^2(BT^n).$$

By the definition of the projectivization, we have the following lemma:

Lemma 3.2. *If $P(\widetilde{\mathcal{G}})$ is a GKM graph, then $c_1^T(\gamma_{\widetilde{\mathcal{G}}}) \in H^*(P(\widetilde{\mathcal{G}}))$.*

Remark 3.3. Note that in order to state the main theorem, we only need the 1st Chern class of $\gamma_{\widetilde{\mathcal{G}}}$. So in this article, we do not define $\gamma_{\widetilde{\mathcal{G}}}$. The tutological line bundle $\gamma_{\widetilde{\mathcal{G}}}$ will be defined in [**Ku21**].

3.2. Main theorem. Note that by the embedding (3.1), we may regard the ith Chern class $c_i^T(\widetilde{\mathcal{G}}) \in H^*(\mathcal{G})$ as an element of $c_i^T(\widetilde{\mathcal{G}}) \in H^*(P(\mathcal{G}))$. Moreover, we may regard

$$H^*(\mathcal{G}) \subset H^*(P(\mathcal{G})).$$

Now we may state the main result.

THEOREM 3.4 (Combinatorial Leray-Hirsh theorem). *Let $\widetilde{\mathcal{G}} = (\widetilde{\Gamma}, \widetilde{\alpha}, \widetilde{\nabla})$ be a rank $r+1$ equivariant vector bundle over a GKM graph $\mathcal{G} = (\Gamma, \alpha, \nabla)$. Assume that the projectivization $P(\widetilde{\mathcal{G}}) := (\Gamma', \alpha', \nabla')$ satisfies the GKM conditions. Then its equivariant cohomology $H^*(P(\widetilde{\mathcal{G}}))$ is isomorphic to the following algebra over $H^*(\mathcal{G})$:*

$$H^*(P(\widetilde{\mathcal{G}})) \simeq H^*(\mathcal{G})[c_1^T(\gamma_{\widetilde{\mathcal{G}}})]/\langle \sum_{i=0}^{r+1} (-1)^i c_i^T(\widetilde{\mathcal{G}}) c_1^T(\gamma_{\widetilde{\mathcal{G}}})^{r+1-i} \rangle,$$

where $c_i^T(\widetilde{\mathcal{G}}) \in H^{2i}(P(\widetilde{\mathcal{G}}))$ is the ith Chern class of $\widetilde{\mathcal{G}}$ and $c_1^T(\gamma_{\widetilde{\mathcal{G}}}) \in H^2(P(\widetilde{\mathcal{G}}))$ is the 1st Chern class of the tautological line bundle of $P(\widetilde{\mathcal{G}})$.

Namely, the equivariant cohomology of $P(\widetilde{\mathcal{G}})$ is generated by $c_1^T(\gamma_{\widetilde{\mathcal{G}}})$ and there is the following unique relation:

$$\sum_{i=0}^{r+1} (-1)^i c_i^T(\widetilde{\mathcal{G}}) c_1^T(\gamma_{\widetilde{\mathcal{G}}})^{r+1-i} = 0. \tag{3.2}$$

The relation (3.2) is also called a *Borel-Hirzebruch formula* for the ordinary projectivization of the complex vector bundle. So (3.2) may be regarded as a *combinatorial Borel-Hirzebruch formula* from GKM theoretical point of view..

3.3. Example. Let $P(\widetilde{\mathcal{G}})$ be the projectivization in Figure 2. In this final section, we check Theorem 3.4 by example in Figure 2.

The 1st Chern class $c_1^T(\gamma_{\widetilde{\mathcal{G}}})$ of the tautological line bundle of $P(\widetilde{\mathcal{G}})$ is given by the following equation by Figure 1 (see Figure 3):

$$\begin{aligned}
c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{q,2}) &= \widetilde{\alpha}(l_{q,2}) = \alpha - \beta;\\
c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{q,1}) &= \widetilde{\alpha}(l_{q,1}) = -\beta;\\
c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{r,1}) &= \widetilde{\alpha}(l_{r,1}) = -\alpha;\\
c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{r,2}) &= \widetilde{\alpha}(l_{r,2}) = \beta - \alpha;\\
c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{p,2}) &= \widetilde{\alpha}(l_{p,2}) = \beta;\\
c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{p,1}) &= \widetilde{\alpha}(l_{p,1}) = \alpha.
\end{aligned}$$

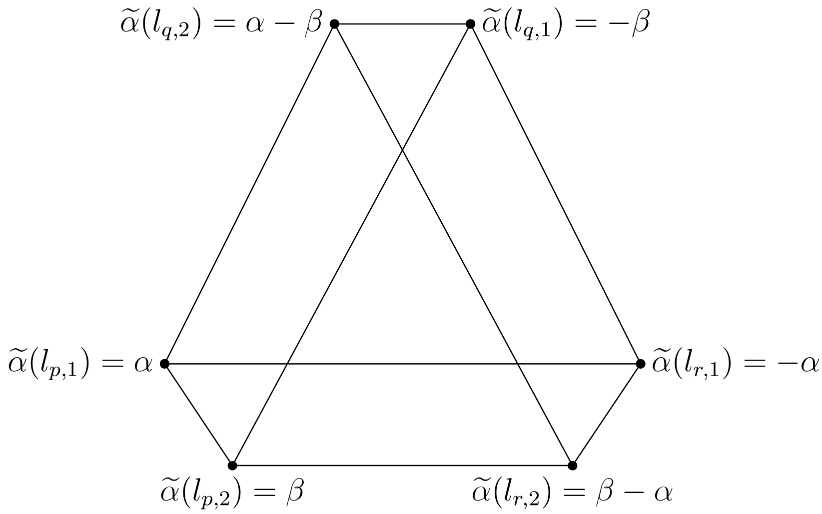

FIGURE 3. The 1st Chern class of the tautological line bundle $c_1^T(\gamma_{\widetilde{\mathcal{G}}})$ for Figure 2.

The 1st Chern class $c_1^T(\widetilde{\mathcal{G}}) \in H^2(P(\widetilde{\mathcal{G}}))$ of the vector bundle $\widetilde{\mathcal{G}}$ is given by the following equation by Figure 1 (see Figure 4):

$$c_1^T(\widetilde{\mathcal{G}})(l_{q,1}) = c_1^T(\widetilde{\mathcal{G}})(l_{q,2}) = \widetilde{\alpha}(l_{q,1}) + \widetilde{\alpha}(l_{q,2}) = -\beta + (\alpha - \beta) = \alpha - 2\beta;$$
$$c_1^T(\widetilde{\mathcal{G}})(l_{r,1}) = c_1^T(\widetilde{\mathcal{G}})(l_{r,2}) = \widetilde{\alpha}(l_{r,1}) + \widetilde{\alpha}(l_{r,2}) = -\alpha + (\beta - \alpha) = \beta - 2\alpha;$$
$$c_1^T(\widetilde{\mathcal{G}})(l_{p,1}) = c_1^T(\widetilde{\mathcal{G}})(l_{p,2}) = \widetilde{\alpha}(l_{p,1}) + \widetilde{\alpha}(l_{p,2}) = \alpha + \beta.$$

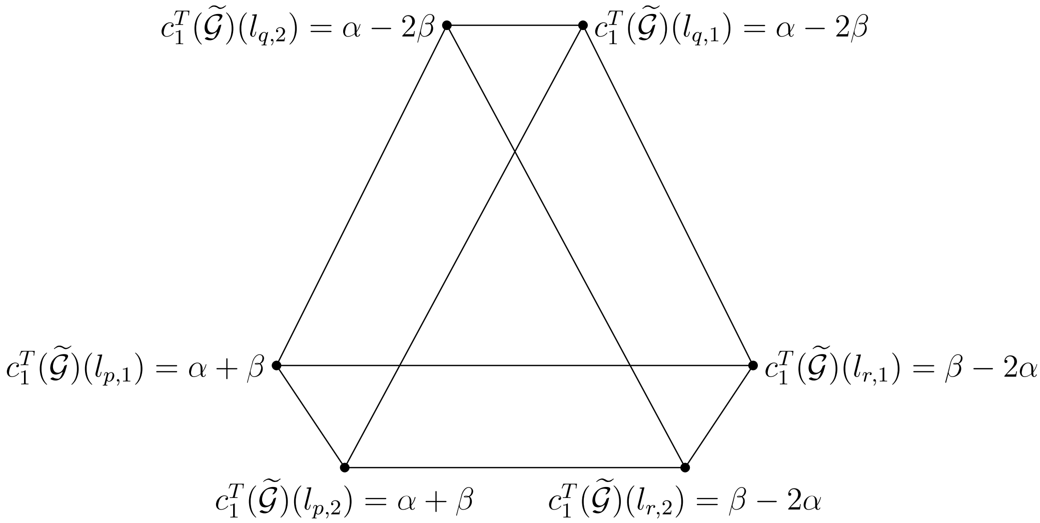

FIGURE 4. The 1st Chern class $c_1^T(\widetilde{\mathcal{G}})$ of Figure 1.

The 2nd Chern class $c_2^T(\widetilde{\mathcal{G}}) \in H^2(P(\widetilde{\mathcal{G}}))$ of the vector bundle $\widetilde{\mathcal{G}}$ is given by the following equation by Figure 1 (see Figure 5):

$$\begin{aligned}
c_2^T(\widetilde{\mathcal{G}})(l_{q,1}) = c_2^T(\widetilde{\mathcal{G}})(l_{q,2}) = \widetilde{\alpha}(l_{q,1}) \cdot \widetilde{\alpha}(l_{q,2}) = -\beta(\alpha - \beta);\\
c_2^T(\widetilde{\mathcal{G}})(l_{r,1}) = c_2^T(\widetilde{\mathcal{G}})(l_{r,2}) = \widetilde{\alpha}(l_{r,1}) \cdot \widetilde{\alpha}(l_{r,2}) = -\alpha(\beta - \alpha);\\
c_2^T(\widetilde{\mathcal{G}})(l_{p,1}) = c_2^T(\widetilde{\mathcal{G}})(l_{p,2}) = \widetilde{\alpha}(l_{p,1}) \cdot \widetilde{\alpha}(l_{p,2}) = \alpha\beta.
\end{aligned}$$

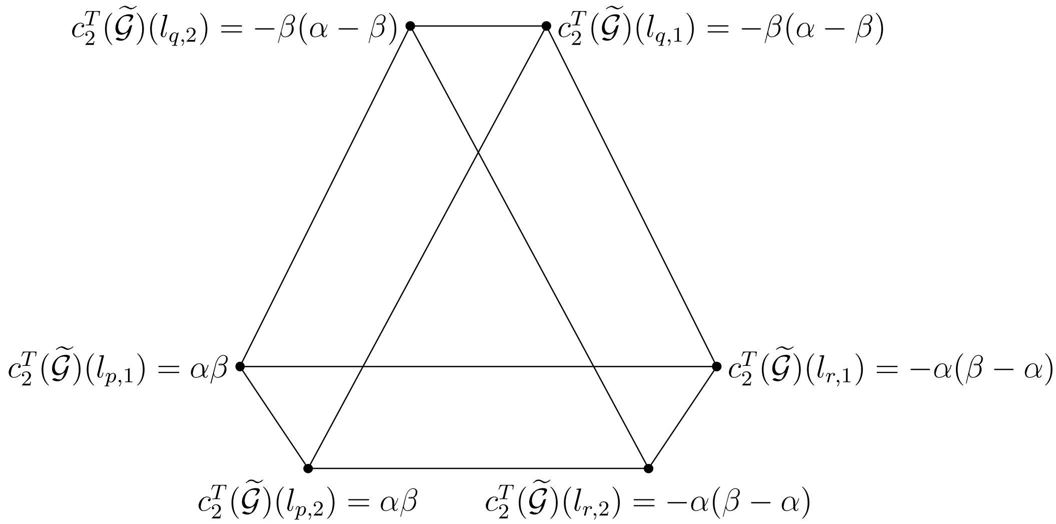

FIGURE 5. The 2nd Chern class $c_2^T(\widetilde{\mathcal{G}})$ of Figure 1.

Then we can check the following equation on the vertex $l_{q.2} \in V'$:

$$\begin{aligned}
&\left(\sum_{i=0}^{2}(-1)^i c_i^T(\widetilde{\mathcal{G}}) c_1^T(\gamma_{\widetilde{\mathcal{G}}})^{2-i}\right)(l_{q,2})\\
=&(c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{q,2}))^2 - c_1^T(\widetilde{\mathcal{G}})(l_{q,2}) \cdot c_1^T(\gamma_{\widetilde{\mathcal{G}}})(l_{q,2}) + c_2^T(\widetilde{\mathcal{G}})(l_{q,2})\\
=&(\alpha - \beta)^2 - (\alpha - \beta)(\alpha - 2\beta) + (-\beta(\alpha - \beta))\\
=&0.
\end{aligned}$$

It is also easy to check the similar equations for all vertices V'. This shows that the combinatorial Borel-Hirzebruch formula (3.2) is true for $H^*(P(\widetilde{\mathcal{G}}))$ of $P(\widetilde{\mathcal{G}})$ in Figure 2.

By using [**GKM98**] and the well-known results of $H^*_{T^2}(\mathbb{C}P^2)$, we have the following application to the geometry:

Corollary 3.5. The equivariant cohomology of the T^2-action on $T\mathbb{C}P^2 \simeq \mathcal{F}l(\mathbb{C}^3)$ is isomorphic to the following ring:

$$\begin{aligned} H^*_{T^2}(\mathcal{F}l(\mathbb{C}^3)) \simeq & H^*(P(\widetilde{\mathcal{G}})) \\ \simeq & H^*(\mathcal{G})[c_1^T(\gamma_{\widetilde{\mathcal{G}}})]/\langle c_1^T(\gamma_{\widetilde{\mathcal{G}}})^2 - c_1^T(\widetilde{\mathcal{G}}) \cdot c_1^T(\gamma_{\widetilde{\mathcal{G}}}) + c_2^T(\widetilde{\mathcal{G}})\rangle \\ \simeq & \mathbb{Z}[\tau_1,\ \tau_2,\ \tau_3,\ c_1^T(\gamma_{\widetilde{\mathcal{G}}})]/\langle \tau_1\tau_2\tau_3,\ c_1^T(\gamma_{\widetilde{\mathcal{G}}})^2 - c_1^T(\widetilde{\mathcal{G}}) \cdot c_1^T(\gamma_{\widetilde{\mathcal{G}}}) + c_2^T(\widetilde{\mathcal{G}})\rangle, \end{aligned}$$

where τ_i's are Thom class of the GKM subgraph of the GKM graph $\mathcal{G}$ of $\mathbb{C}P^2$.

This is the computation of the equivariant cohomology of flag manifolds by using the Borel-Hirzebruch formula (also see [**KLSS**]).

The proof of Theorem 3.4 will be given in [**Ku21**]

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