On unstable twisted rational cohomology groups of the automorphism groups of free groups

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Abstract

In this article, we consider unstable twisted rational cohomology groups of the automorphism groups of free groups. First, we exposit that there are non-trivial unstable twisted 2-cocycles which are constructed by Kawazumi's cocycles. Second, we exposit a calculation of the second and the third cohomology groups of the automorphism group of free group of rank three with coefficients in the exterior products of the abelianization of the free group.

Let F_n be a free group of rank $n \geq 2$ with basis $x_1, \ldots, x_n$, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. The study of the (co)homology groups of the automorphism groups of free groups has a long history. To the best of our knowledge, the first contribution goes back to a work of Nielsen [36] in 1924, who obtained the first finite presentation for $\mathrm{Aut}\, F_n$ and showed $H_1(\mathrm{Aut}\, F_n, \mathbf{Q}) = 0$ for $n \geq 2$. In 1984, by constructing the free group analogue of the Steinberg group, Gersten [22] showed $H_2(\mathrm{Aut}\, F_n, \mathbf{Q}) = 0$ for $n \geq 5$. In 1996, by introducing "non-abelian K-theory", Kiralis [24] showed $H_2(\mathrm{Aut}\, F_4, \mathbf{Q}) = 0$. In 1986, Culler-Vogtmann [8] introduced Outer spaces, and made a breakthrough in computation of (co)homology groups of the outer automorphism groups of free groups. To put it briefly, the Outer space $\mathcal{K}_n$ is a finite dimensional contractible CW-complex on which the outer automorphism group $\mathrm{Out}\, F_n$ naturally acts properly discontinuously and cocompactly with finite cell stabilizers. The space $\mathcal{K}_n$ is an analogue of the Teichmüller space on which the mapping class group of a surface naturally acts. It follows immediately from the structure of the Outer space that $H_i(\mathrm{Aut}\, F_n, \mathbf{Q}) = 0$ for $i > 2n-2$. Together with the development of computer technology, the Outer space enables one to compute unstable (co)homology groups. For example, Hatcher-Vogtmann [15] computed $H_4(\mathrm{Aut}\, F_4, \mathbf{Q}) = \mathbf{Q}$, and Vogtmann [47] showed $H_4(\mathrm{Out}\, F_4, \mathbf{Q}) = \mathbf{Q}$. In their doctoral thesis, Gerlits [21] computed $H_7(\mathrm{Aut}\, F_5, \mathbf{Q}) = \mathbf{Q}$ in 2002, and Ohashi [37] computed $H_8(\mathrm{Out}\, F_6, \mathbf{Q}) = \mathbf{Q}$ in 2007 respectively. By using sophisticated homotopy theory, Galatius [20] showed that the

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stable integral homology groups of $\mathrm{Aut}\, F_n$ are isomorphic to those of the symmetric group $\mathfrak{S}_n$ of degree n, in particular, $H_i(\mathrm{Aut}\, F_n, \mathbf{Q}) = 0$ for $n \geq 2i+1$.

In an unstable range, the (co) homology groups of $\mathrm{Aut}\, F_n$ and $\mathrm{Out}\, F_n$ behaves in much complicated and mysterious way. The first systematic construction of unstable (co)homology classes was given by Morita [33] in 1999. He constructed a series of unstable homology classes $\mu_k \in H_{4k}(\mathrm{Out}\, F_{2k+2}, \mathbf{Q})$ for $k \geq 1$ by using Kontsevich's results [25] and [26]. (See also [34].) Today, these homology classes are called the Morita classes of the outer automorphism groups of free groups. It is known that the first and the second one are generators of $H_4(\mathrm{Out}\, F_4, \mathbf{Q})$ and $H_8(\mathrm{Out}\, F_6, \mathbf{Q})$ respectively. (See [34] and [7] respectively.) Furthermore, Morita-Sakasai-Suzuki [35] showed that show the integral Euler characteristic of $\mathrm{Out}\, F_{11}$ is -1202. Recently, Borinsky and Vogtmann [2] showed that the rational Euler characteristic of $\mathrm{Out}\, F_n$ is always negative. From these fact it seems that the unstable rational (co)homology groups of $\mathrm{Out}\, F_n$ are quite large and complicated. We should remark that in [6], Conant-Hatcher-Kassabov-Vogtmann gave a construction of many nontrivial unstable homology classes of $\mathrm{Aut}\, F_n$ and $\mathrm{Out}\, F_n$, and studies the Morita classes.

To our best knowledge, the origin of the study of twisted (co)homology groups of $\mathrm{Aut}\, F_n$ goes back to a work of Kawazumi [23]. Let H be the abelianization of F_n, and set $H^* := \mathrm{Hom}_{\mathbf{Z}}(H, \mathbf{Z})$. Inspired by Morita's previous works [30] and [32] for the mapping class groups of surfaces, he constructed a crossed homomorphism on $\mathrm{Aut}\, F_n$ being the unique extension of the first Johnson homomorphism, and studied the structure of it cup products. (For details, see below.) Especially, he obtained a series of non-trivial rational cohomology classes $\zeta_p(\tau_1^{\otimes p}) \in H^p(\mathrm{Aut}\, F_n, H^* \otimes_{\mathbf{Z}} H^{\otimes p+1})$. In our previous research, also inspired by Morita's works [30] and [32], we computed $H^1(\mathrm{Aut}\, F_n, H) = \mathbf{Z}$ for $n \geq 2$ in [40], and $H^1(\mathrm{Aut}\, F_n, H^* \otimes_{\mathbf{Z}} \wedge^2 H) = \mathbf{Z}^{\oplus 2}$ for $n \geq 6$ [42] by using a presentation of $\mathrm{Aut}\, F_n$. Moreover we showed that the generators of these cohomology groups are given by Morita's cocycle and Kawazumi's cocycle.

In a series of their works, Djament-Vespa [11], Vespa [46], and Djament [10] established a homological algebraic method to compute of stable twisted cohomology groups of $\mathrm{Aut}\, F_n$ with functor homology theory. On the other hand, around the same time, Randal-Williams [38] also established a method to compute stable twisted cohomology groups of $\mathrm{Aut}\, F_n$ by using topology and representation theory. For instance, from their independent works, we see

$$H^k(\mathrm{Aut}\, F_n, \wedge^k H_{\mathbf{Q}}) = \mathbf{Q}^{\oplus p(k)}$$

for $n \geq 2k+3$ where $p(k)$ is the number of partitions of k, and the subscript $\mathbf{Q}$ means tensoring with $\mathbf{Q}$ over $\mathbf{Z}$. Moreover, it is known that a generating system of $H^k(\mathrm{Aut}\, F_n, \wedge^k H_{\mathbf{Q}})$ is constructed from Kawazumi's cohomology classes. (For details, see below.)

The abelianization $F_n \to H$ induces the surjective homomorphism $\mathrm{Aut}\, F_n \to \mathrm{Aut}\, H$, Here we identify $\mathrm{Aut}\, H$ with $\mathrm{GL}(n, \mathbf{Z})$ by fixing the basis of H induced from $x_1, \ldots, x_n$. Let IA_n be the kernel of $\mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbf{Z})$. The group IA_n is called the IA-automorphism group of F_n. By observing the Lyndon-Hochshild-Serre spectral sequence of the group extension

$$1 \to \mathrm{IA}_n \to \mathrm{Aut}\,F_n \to \mathrm{GL}(n,\mathbf{Z}) \to 1,$$

we see that the twisted (co)homology groups of $\mathrm{Aut}\,F_n$ is closely related to the untwisted (co)homology groups of IA_n. More precisely, for an $\mathrm{Aut}\,F_n$-module M on which $\mathrm{Aut}\,F_n$ acts via $\mathrm{GL}(n,\mathbf{Z})$, we have

$$E_2^{p,q} = H^p(\mathrm{GL}(n,\mathbf{Z}), H^q(\mathrm{IA}_n, M)) \Longrightarrow H^{p+q}(\mathrm{Aut}\,F_n, M).$$

In particular, the fact that $H^1(\mathrm{Aut}\,F_n, H^* \otimes_{\mathbf{Z}} \wedge^2 H) = \mathbf{Z}^{\oplus 2}$ comes from

$$H^0(\mathrm{GL}(n,\mathbf{Z}), H^1(\mathrm{IA}_n, M)) = (H^1(\mathrm{IA}_n,\mathbf{Z}) \otimes_{\mathbf{Z}} M)^{\mathrm{GL}(n,\mathbf{Z})}.$$

Hence it is important to investigate the structure of $H^p(\mathrm{IA}_n,\mathbf{Z})$ from a viewpoint of the study of twisted (co)homology groups of $\mathrm{Aut}\,F_n$.

Today, only the first integral homology group of IA_n is completely determined by independent works of Cohen-Pakianathan [4, 5], Farb [12] and Kawazumi [23]. It is isomorphic to the abelianization of IA_n, and is the free abelian group generated by the Magnus generators obtained by Magnus [28] in 1935. Krstić-McCool [27] showed that IA_3 is not finitely presentable. This shows that there is a possibility that the second homology group $H_2(\mathrm{IA}_3,\mathbf{Z})$ is not finitely generated. In fact, this follows by a work of Bestvina-Bux-Margalit [1]. More precisely, by using Outer space, they showed that the quotient group of IA_n by the inner automorphism group $\mathrm{Inn}\,F_n$ has a $2n-4$-dimensional Eilenberg-Maclane space, and that $H_{2n-4}(\mathrm{IA}_n/\mathrm{Inn}\,F_n,\mathbf{Z})$ is not finitely generated. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not. Namely, at the present stage, even $H_2(\mathrm{IA}_n,\mathbf{Z})$ is not determined explicitly. Pettet [39] determined the image of the rational cup product $\cup_{\mathbf{Q}} : \Lambda^2 H^1(\mathrm{IA}_n,\mathbf{Q}) \to H^2(\mathrm{IA}_n,\mathbf{Q})$, and gave its irreducible GL-decomposition for $n \geq 3$. Furthermore, Day-Putman [9] obtained an explicit finite set of generators for $H_2(\mathrm{IA}_n,\mathbf{Z})$ as a $\mathrm{GL}(n,\mathbf{Z})$-module. In our previous paper [44], for $n = 3$, we detected a non-trivial GL-irreducible component in $H^2(\mathrm{IA}_3,\mathbf{Q})/\mathrm{Im}(\cup_{\mathbf{Q}})$, and showed that the image of the triple cup product $\cup^3_{\mathrm{IA}_3} : \Lambda^3 H^1(\mathrm{IA}_3,\mathbf{Q}) \to H^3(\mathrm{IA}_3,\mathbf{Q})$ is trivial

In this article, we consider Kawazumi's cocycles. They are constructed from the extension of the first Johnson homomorphism of $\mathrm{Aut}\,F_n$. Let $F_n = \Gamma_n(1) \supset \Gamma_n(2) \supset \cdots$ be the lower central series of F_n, and $\mathcal{L}_n(k) := \Gamma_n(k)/\Gamma_n(k+1)$ its k-th successive quotient. The graded sum $\bigoplus_{k\geq 1} \mathcal{L}_n(k)$ has the graded Lie algebra structure with the Lie bracket induced from the commutator bracket of F_n, and is isomorphic to the free Lie algebra generated by $\mathcal{L}_n(1) = H$ due to a classical work of Magnus. (See [29] for example.) The first Johnson homomorphism

$$\tau_1 : \mathrm{IA}_n \to H^* \otimes \wedge^2 H \hookrightarrow H^* \otimes H^{\otimes 2}$$

is defined by

$$\tau_1(\sigma)(x \pmod{\Gamma_n(2)}) = x^{-1}x^{\sigma} \pmod{\Gamma_n(3)} \in \mathcal{L}_n(2) = \wedge^2 H$$

where the second equality is induced from the natural injection $\wedge^2 H \hookrightarrow H^{\otimes 2}$. Originally, in a series of his works [16, 17, 18, 19], the Johnson homomorphisms of the mapping class groups were introduced by Johnson who determined the abelianization of the Torelli group by using the first Johnson homomorphism. Today, the study of the Johnson homomorphisms of the mapping class group has achieved a good progress by many authors including Morita [31], Hain [13] and so on. For surveys for the Johnson homomorphisms, see [43] and [14] for example. Here we should remark that Kawazumi [23] showed that tau_1 extends to $\mathrm{Aut}\, F_n$ as a crossed homomorphism by using the magnus expansions of F_n.

For any $k \geq 1$, let $\zeta_k : (H^* \otimes H^{\otimes 2})^{\otimes k} \to H^* \otimes H^{\otimes (k+1)}$ be the map defined by

$$u_1 \otimes \cdots \otimes u_k \mapsto (u_1 \otimes \mathrm{id}^{\otimes (k-1)}) \circ (u_2 \otimes \mathrm{id}^{\otimes (k-2)}) \circ \cdots \circ u_k.$$

Namely, ζ_k is defined by taking the contractions recursively. By considering the cup product of the twisted 1-cocycle τ_1 of $\mathrm{Aut}\, F_n$, Kawazumi [23] constructed twisted cocycles $\zeta_k \circ (\tau_1^{\otimes k}) \in H^k(\mathrm{Aut}\, F_n, H^* \otimes H^{\otimes (k+1)})$. In the present article, at first we show the following theorem.

Theorem 1. *For $n \geq 7$,*

$$H^2(\mathrm{Aut}\, F_n, \mathrm{Im}(\cup_{\mathbf{Q}})^*) \supset \mathbf{Q}^{\oplus d_n}$$

where d_n is the number of the GL-irreducible components of $\mathrm{Im}(\cup_{\mathbf{Q}})$.

In particular, we show the above theorem by constructing linearly independent second cocycles by using Kawazumi's cocycle $\zeta_2 \circ (\tau_1^{\otimes 2})$.

Next, for $k \geq 2$, we slightly improve the fact that $H^k(\mathrm{Aut}\, F_n, \wedge^k H_{\mathbf{Q}}) = \mathbf{Q}^{\oplus p(k)}$ for $n \geq 2k+3$, obtained by the independent works of Djament, Vespa and Randal-Williams as mentioned above. After taking the contraction with respect to the first and second component and the natural projection $H^{\otimes k} \to \wedge^k H$, we denote by $\mapsto h_k \in H^k(\mathrm{Aut}\, F_n, \wedge^k H)$ the image of $\zeta_k \circ (\tau_1^{\otimes k})$ by the induced map between the coefficients. For any partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of $k \geq 1$, set

$$h_\lambda := h_{\lambda_1} \wedge h_{\lambda_2} \wedge \cdots \wedge h_{\lambda_m} \in H^k(\mathrm{Aut}\, F_n, \wedge^k H_{\mathbf{Q}}).$$

It is known that $\{h_\lambda \,|\, \lambda \vdash k\}$ generates $H^k(\mathrm{Aut}\, F_n, \wedge^k H_{\mathbf{Q}})$ for $n \geq 2k+3$. Then we show the following theorem.

Theorem 2. *For $k \geq 2$ and $n \geq 2k$, the set $\{h_\lambda \,|\, \lambda \vdash k\}$ is linearly independent on $H^k(\mathrm{IA}_n, \wedge^k H_{\mathbf{Q}})$.*

Finally, we show an explicit calculation for the case of $n = 3$ and $k = 2$. In 1993, by using the Outer space, Brady [3] computed the integral cohomology groups of $\mathrm{Out}\, F_3$. By applying his method to the computation of twisted rational homology groups of $\mathrm{Out}\, F_3$, we obtain the following result.

Theorem 3.

$$H^q(\operatorname{Out} F_3, \wedge^2 H_{\mathbf{Q}}) = \begin{cases} 0, & q \neq 2, \\ \mathbf{Q}, & q = 2, \end{cases}$$
$$H^q(\operatorname{Out} F_3, \wedge^3 H_{\mathbf{Q}}) = 0, \quad q \geq 0.$$

Furthermore, by using the Lyndon-Hochshild-Serre spectral sequence of the group extension

$$1 \to \operatorname{Inn} F_n \to \operatorname{Aut} F_n \to \operatorname{Out} F_n \to 1,$$

and the fact that

$$H^1(\operatorname{Out} F_3, (H^* \otimes \wedge^2 H)_{\mathbf{Q}}) = \mathbf{Q}, \quad H^2(\operatorname{Out} F_3, (H^* \otimes \wedge^3 H)_{\mathbf{Q}}) = \mathbf{Q},$$

we obtain the following.

Theorem 4.

$$H^2(\operatorname{Aut} F_3, \wedge^2 H_{\mathbf{Q}}) = \mathbf{Q}^2, \quad H^3(\operatorname{Aut} F_3, \wedge^3 H_{\mathbf{Q}}) = \mathbf{Q}.$$

We remark that for $n \geq 3$ the irreducible decomposition of $\operatorname{Im}(\cup_{\mathbf{Q}})^*$ as a GL-module is given by

$$\operatorname{Im}(\cup_{\mathbf{Q}})^* = [1,1] \oplus (D^{-1} \otimes [3,2])$$

where D means the determinant representation, and $[\lambda]$ means the irreducible module correspond to a Young tableau λ. Namely, the multiplicity of $\wedge^2 H_{\mathbf{Q}}$ in $\operatorname{Im}(\cup_{\mathbf{Q}})^*$ is one. It shows that the equality in Theorem 1 does not hold in the unstable range in general. Roughly speaking, the reason why the dimension of $H^2(\operatorname{Aut} F_3, \wedge^2 H_{\mathbf{Q}})$ is different from the multiplicity of $\wedge^2 H_{\mathbf{Q}}$ in $\operatorname{Im}(\cup_{\mathbf{Q}})^*$ comes from the following fact. For a general $n \geq 3$, we have $H^2(\operatorname{Aut} F_n, \wedge^2 H_{\mathbf{Q}})$ is isomorphic to $H^2(\mathrm{IA}_n, \wedge^2 H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})}$. The natural map

$$(\operatorname{Im}(\cup_{\mathbf{Q}}) \otimes \wedge^2 H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})} \to (H^2(\mathrm{IA}_n, \mathbf{Q}) \otimes \wedge^2 H_{\mathbf{Q}})^{\mathrm{GL}(n,\mathbf{Z})}$$

is surjective for $n \geq 6$ from our results, but not surjective for $n = 3$. We show this by using combinatorial group theory and representation theory. We also remark that this seems to imply that for $n = 3$, Kawazumi cocycles $h_{[2]}$ and $h_{[1,1]} \in H^2(\operatorname{Aut} F_3, \wedge^2 H_{\mathbf{Q}})$ are also linearly independent.

This article is an announcement of our recent results. For the details of the proofs, see the forthcoming paper [45].

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