TORIC RICHARDSON VARIETIES OF CATALAN TYPE AND WEDDERBURN–ETHERINGTON NUMBERS

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1. Background and main result

This is a summary of the paper [7] which is joint work with Eunjeong Lee and Seonjeong Park.

For a pair (v,w) of elements in the symmetric group $\mathfrak{S}_{n+1}$ on $n+1$ letters satisfying $v \le w$ in the Bruhat order, Kodama and Williams [4] introduced the *Bruhat interval polytope*

$$\mathsf{Q}_{v,w} := \text{Convex hull of } \{(z(1),\dots,z(n+1)) \in \mathbb{R}^{n+1} \mid v \le z \le w\}.$$

The combinatorial properties of Bruhat interval polytopes were further investigated by Tsukerman and Williams in [10]. On the other hand, for a pair (v,w) with $v \le w$, the *Richardson variety* $X_{w^{-1}}^{v^{-1}}$, which is a subvariety of the flag variety $\mathrm{Flag}(\mathbb{C}^{n+1})$, is the intersection of the Schubert variety $X_{w^{-1}}$ and the opposite Schubert variety $w_0 X_{w_0 v^{-1}}$, where w_0 is the longest element in $\mathfrak{S}_{n+1}$. Note that $X_{w^{-1}}^{v^{-1}} = X_{w^{-1}}$ when v is the identity element.

The two objects $\mathsf{Q}_{v,w}$ and $X_{w^{-1}}^{v^{-1}}$ are related through the standard moment map

$$\mu\colon \mathrm{Flag}(\mathbb{C}^{n+1}) \to \mathbb{R}^{n+1}.$$

Indeed, $\mathsf{Q}_{v,w} = \mu(X_{w^{-1}}^{v^{-1}})$. It is known that

$$\dim_{\mathbb{R}} \mathsf{Q}_{v,w} \le \ell(w) - \ell(v) = \dim_{\mathbb{C}} X_{w^{-1}}^{v^{-1}},$$

where ℓ is the length function on $\mathfrak{S}_{n+1}$. The Richardson variety $X_{w^{-1}}^{v^{-1}}$ is a toric variety with respect to the standard action of $(\mathbb{C}^*)^{n+1}$ on $\mathrm{Flag}(\mathbb{C}^{n+1})$ if and only if $\dim_{\mathbb{R}} \mathsf{Q}_{v,w} = \ell(w) - \ell(v)$. In this case, the fan of $X_{w^{-1}}^{v^{-1}}$ is the normal fan of $\mathsf{Q}_{v,w}$. Every *toric* Schubert variety is smooth, but a *toric* Richardson variety is not necessarily smooth. It is smooth if and only if the corresponding Bruhat interval polytope $\mathsf{Q}_{v,w}$ is combinatorially equivalent to a cube ([6, Proposition 5.6]). This means that a smooth toric Richardson variety is a Bott manifold, that is the total space of an iterated $\mathbb{C}P^1$-bundle over a point where each $\mathbb{C}P^1$-bundle is the projectivization of the Whitney sum of two line bundles. Hirzebruch surfaces are complex 2-dimensional Bott manifolds.

When v is the identity element e, the Bruhat interval polytope $\mathsf{Q}_{e,w}$ is combinatorially equivalent to a cube if and only if w is a product of distinct simple transpositions s_i interchanging i and $i+1$ (see [2, 3, 5]). Such characterization of w is not known for general v but there are many pairs (v,w) such that $\mathsf{Q}_{v,w}$ is combinatorially equivalent to a cube (see [6]). Among them, the following pair of v and w is of the simplest form:

$$w = vs_1s_2\cdots s_n \quad (\text{or } w = vs_n\cdots s_2s_1) \quad \text{and} \quad \ell(w)-\ell(v)=n. \tag{1.1}$$

We associate a complete non-singular fan of dimension n with a triangulation of a convex $(n+2)$-gon P_{n+2} and see that such a fan is the normal fan of $\mathsf{Q}_{v,w}$ for the pair (v,w) in (1.1) and vice versa. As is well-known, the number of triangulations of P_{n+2} is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$, so we say that a toric Richardson variety (or a toric variety) is *of Catalan type* if its fan is associated with a polygon triangulation. A toric (Richardson) variety of Catalan type is not only a Bott manifold but also Fano.

There is a well-known bijection between the set of triangulations of P_{n+2} and the set of (rooted) binary trees with n vertices. We note that a binary tree is *ordered*, which means that an ordering is specified for the children of each vertex. We show that two polygon triangulations produce isomorphic toric (Richardson) varieties of Catalan type if and only if the corresponding binary trees are isomorphic as rooted trees when we forget the orderings. Namely, our main result is the following.

Theorem 1.1. *The set of isomorphism classes of n-dimensional toric* (*Richardson*) *varieties of Catalan type bijectively corresponds to the set of* unordered *binary trees with n vertices, where the cardinality of the latter set is known as the Wedderburn–Etherington number b_{n+1}.*

The Wedderburn–Etherington number b_n $(n \geq 1)$ is the number of ways of parenthesizing a string of n letters, subject to a commutative (but nonassociative) binary operation and appears in counting several different objects (see Sequence A001190 in OEIS [8], [9, A56 in p.133]). The generating function $B(x) = \sum_{n\geq 1} b_n x^n$ of the Wedderburn–Etherington numbers satisfies the functional equation

$$B(x) = x + \frac{1}{2}B(x)^2 + \frac{1}{2}B(x^2),$$

which was the motivation of Wedderburn in his work [11] and was considered by Etherington [1]. This functional equation is equivalent to the recurrence relation

$$b_{2m-1} = \sum_{i=1}^{m-1} b_i b_{2m-i-1} \quad (m \geq 2), \qquad b_{2m} = b_m(b_m+1)/2 + \sum_{i=1}^{m-1} b_i b_{2m-i}$$

with $b_1 = 1$. Using this recurrence relation, one can calculate the Wedderburn–Etherington numbers, see Table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
b_n	1	1	1	2	3	6	11	23	46	98	207	451	983	2179	4850

TABLE 1. Wedderburn–Etherington numbers b_n for small values of n

Flag varieties and Richardson varieties are defined for any Lie type and it would be interesting to find a combinatorial object which describes and classifies toric Richardson varieties in other Lie types for the pairs (v,w) in (1.1).

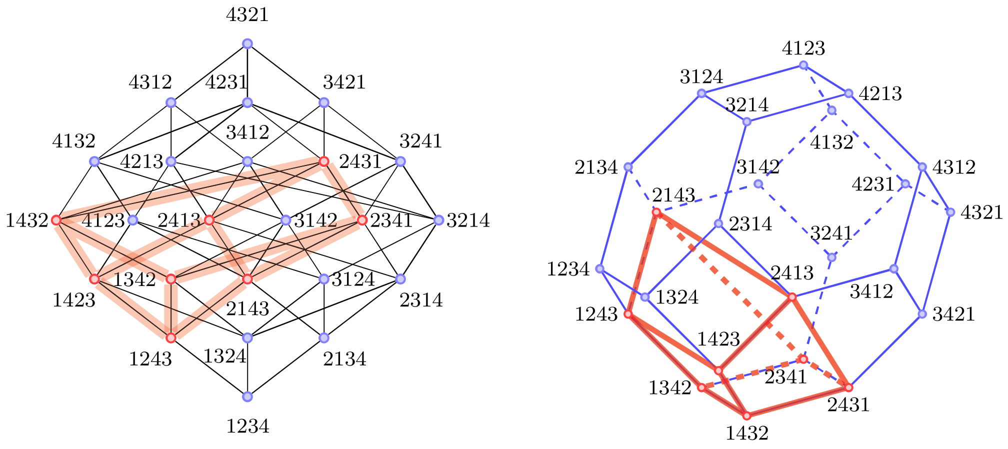
2. An example illustrating the idea

We shall explain with an example how a polygon triangulation appears from a Bruhat interval polytope $\mathsf{Q}_{v,w}$ for a pair (v, w) satisfying (1.1).

Consider two permutations $v = 1243$ and $w = 2431$ in $\mathfrak{S}_4$, where v and w are written in one-line notation. Note that the pair (v, w) satisfies the condition in (1.1). The Bruhat interval $[1243, 2431]$ consists of 8 permutations in the red part of Figure 1(1) and the Bruhat interval polytope $\mathsf{Q}_{1243,2431}$ is a 3-cube drawn in red and thick in Figure 1(2). Here, for permutations v and w in $\mathfrak{S}_{n+1}$, the Bruhat interval $[v, w]$ is defined to be

$$[v, w] \coloneqq \{z \in \mathfrak{S}_{n+1} \mid v \leq z \leq w\}.$$

The entire polytope in Figure 1(2) is the 3-dimensional permutohedron, where the vertices are all the permutations in $\mathfrak{S}_4$ and the label on a vertex, say 2431, shows that the coordinate of the vertex is $(2, 4, 3, 1) \in \mathbb{R}^4$.



(1) Bruhat interval $[1243, 2431]$. (2) Bruhat interval polytope $\mathsf{Q}_{1243,2431}$.

Figure 1. A Bruhat interval polytope which is a 3-cube.

One sees from Figure 1(2) that there are three edges emanating from the vertex $v = 1243$ (resp. $w = 2431$) and their primitive edge vectors are

$$\begin{aligned} &\mathbf{p}_1 = e_1 - e_2, \quad \mathbf{p}_2 = e_2 - e_3, \quad \mathbf{p}_3 = e_2 - e_4 \\ &(\text{resp. } \mathbf{q}_1 = -e_1 + e_4,\ \mathbf{q}_2 = -e_2 + e_3,\ \mathbf{q}_3 = -e_3 + e_4), \end{aligned} \tag{2.1}$$

where e_1, e_2, e_3, e_4 denote the standard basis of $\mathbb{R}^4$. These primitive vectors correspond to the atoms and coatoms of the Bruhat interval $[1243, 2431]$. More precisely, the following pairs $\{i, j\}$

$$\{1,2\},\ \{2,3\},\ \{2,4\} \quad (\text{resp. } \{1,4\},\ \{2,3\},\ \{3,4\}). \tag{2.2}$$

satisfy that $vt_{i,j}$ covers v and $vt_{i,j} \leq w$ (resp. $wt_{i,j}$ is covered by w and $v \leq wt_{i,j}$), where $t_{i,j}$ denotes the transposition interchanging i and j.[1] These correspond

[1]For permutations x and y, we say y *covers* x (or equivalently, x *is covered by* y) if there does not exist z such that $x < z < y$.

to the primitive vectors in (2.1). We subtract 1 from the first three pairs above (corresponding to the atoms) in each element, so that we obtain

$$\{0,1\},\ \{1,2\},\ \{1,3\} \quad (\text{resp. } \{1,4\},\ \{2,3\},\ \{3,4\}) \tag{2.3}$$

and we may regard them as edges or diagonals of the pentagon P_5 with vertices labelled from 0 to 4 in counterclockwise order. The result is shown in Figure 2(1), where edges or diagonals of P_5 obtained from the first three pairs in (2.3) are shown by blue solid lines while those obtained from the latter three pairs are shown by red dashed lines. They form a triangulation of P_5.

(1) A triangulation of P_5 (2) Facet normal vectors

FIGURE 2. Triangulation and facet normal vectors obtained from $\mathsf{Q}_{1243,2431}$

The reader may wonder why we subtract 1 from the first three pairs in (2.2) corresponding to the atoms. Here is the reason. The permutation $v = 1243$ (resp. $w = 2431$) is obtained from 243 by putting the number 1 at the head (resp. at the tail). If we regard the positions of the numbers $2,4,3$ in 243 as the 1st, the 2nd, the 3rd in this order, then the positions of the numbers $1,2,4,3$ in $v = 1243$ are the 0th, the 1st, the 2nd, the 3rd in this order. In this regard, a permutation which covers v, say 2143, is obtained from $v = 1243$ by interchanging the 0th position and the 1st position, so that we obtain the pair $\{0,1\}$ in (2.3). This is the reason why we subtract 1 from the pairs in (2.2) corresponding to the atoms but leave the pairs corresponding to the coatoms unchanged.

The primitive edge vectors $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ (resp. $\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3$) in (2.1) form a basis of the sublattice M of $\mathbb{Z}^4$ with the sum of the coordinates equal to zero. Through the dot product on $\mathbb{Z}^4$, we can think of the dual lattice of M as the quotient lattice $N = \mathbb{Z}^4/\mathbb{Z}(1,1,1,1)$. Let ϖ_k $(k=1,2,3)$ be the quotient image of $\sum_{i=1}^k e_i$ in N and set $\varpi_0 = \varpi_4 = \mathbf{0}$. Then the dual basis of $\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3$ (resp. $\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3$) is given by

$$\begin{gathered} \mathbf{v}_1 = \varpi_1 = \varpi_1 - \varpi_4, \quad \mathbf{v}_2 = \varpi_2 - \varpi_3, \quad \mathbf{v}_3 = \varpi_3 - \varpi_4, \\ (\text{resp. } \mathbf{w}_1 = -\varpi_1 = \varpi_0 - \varpi_1,\ \mathbf{w}_2 = \varpi_1 - \varpi_2,\ \mathbf{w}_3 = \varpi_1 - \varpi_3). \end{gathered} \tag{2.4}$$

These vectors may be regarded as the primitive inward facet normal vectors of the 3-cube $\mathsf{Q}_{1243,2431}$, so they are ray generators of the normal fan of $\mathsf{Q}_{1243,2431}$. Finally, we note that $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3$ in (2.4) can be assigned to edges or diagonals of P_5 as shown in Figure 2(2) by looking at the suffixes of ϖ_i's in their expression (2.4). Then the three relations

$$\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{0}, \quad \mathbf{v}_2 + \mathbf{w}_2 = \mathbf{w}_3, \quad \mathbf{v}_3 + \mathbf{w}_3 = \mathbf{v}_1$$

obtained from (2.4) correspond to the three triangles in our triangulation of P_5 as is seen in Figure 2(2), where we understand that the zero vector $\mathbf{0}$ is assigned to the distinguished edge connecting the vertices 0 and 4.

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