The heat kernel of the quantum Rabi model and related models

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1 Introduction

The quantum Rabi model (QRM) is one of the simplest models used to describe quantum interaction. It represents the interaction of a two level system (typically and two-level atom or qubit) with a harmonic oscillator (e.g. a light field). This fundamental model, along with its generalizations, has seen a resurgence of interest since the previous decade due to advances in theoretical and experimental techniques and in also in part due to the prospects of applications to quantum computing (see e.g. [6, 5, 22] and references therein). While a number of mathematical studies of the QRM and its generalizations have appeared during the years, the mathematical study of diverse aspects of these interactions models is still developing. A reason is that despite the simplicity of its definition, explicit computations of the QRM tend to be quite complicated.

Let us introduce the QRM by considering one of its simplest generalizations. The Hamiltonian H_ϵ of the asymmetric quantum Rabi model (AQRM) is given by

$$H_\epsilon := \omega a^\dagger a + \Delta\sigma_z + g(a + a^\dagger)\sigma_x + \epsilon\sigma_x,$$

where $a^\dagger$ and a are the creation and annihilation operators of quantum harmonic oscillator (i.e. $[a, a^\dagger] = 1$) of frequency ω (we set $\omega = 1$ hereafter) and σ_x, σ_z are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The system parameters are $g, \Delta > 0$ and $\epsilon \in \mathbb{R}$. The AQRM was introduced in [3] (and the online [4]), where the fundamental result of the exact solvability of the QRM was first presented.

The Hamiltonian of the QRM is given by the case $\epsilon = 0$. The additional parameter ϵ makes the spectrum of the Hamiltonian H_ϵ be multiplicity free for the cases $2\epsilon \notin \mathbb{Z}$. For the case of the QRM ($\epsilon = 0$) or, in general, for half-integer ϵ (i.e. $2\epsilon \in \mathbb{Z}$), for fixed Δ there are parameters $g > 0$ such that the spectrum has multiplicity 2 eigenvalues, in other words, there are crossings of the spectral curves with respect to g. This phenomenon is generally considered to be the consequence of the presence of a symmetry, that is, an operator commuting with the Hamiltonian. In the case of the QRM, the mechanism is well understood and leads to a separation of the ambient Hilbert space into invariant "parity" subspaces, but in the case of the AQRM (the so-called hidden symmetry), the subject is still open despite recent advances (see e.g. [2, 11, 14, 21]).

In a recent paper [15], the author and Masato Wakayama obtained an expression for the heat kernel of the QRM given explicitly as a power series in terms of elementary functions. For completeness sake, let us recall the definition of the heat kernel for the case of the AQRM.

The heat kernel $K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta)$ is the integral kernel of the fundamental solution of the heat equation associated to the AQRM:

$$\left(\frac{\partial}{\partial t} + H_\epsilon\right)\phi = 0.$$

that is, it satisfies

$$e^{-tH_\epsilon}\phi(x) = \int_{-\infty}^{\infty} K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta)\phi(y)dy$$

for a compactly supported smooth function $\phi : \mathbb{R} \to \mathbb{C}^2$ (see e.g. [7] for more details on the general theory of the heat kernel). Equivalently, the function $K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta)$ satisfies the conditions

- $\lim_{t\to 0} K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta) = \delta_x(y)\boldsymbol{I}_2$ for $x,y \in \mathbb{R}$,
- $\frac{\partial}{\partial t}K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta) = -H_\epsilon K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta)$ for all $t > 0$.

The computation in [15] is done by directly from the Trotter (or Trotter-Kato) product formula by overcoming several obstacles appearing in the evaluation of the limit. While the resulting expression appears to be quite complicated, it may be used to provide upper bounds with respect to the variables or the system parameters for several applications. For instance, in [16] the analytic continuation of the partition function is used to give an alternative proof of the meromorphic continuation to the complex plane of the spectral zeta function of the QRM (shown originally in [19] by a different method).

The formula was later generalized to the AQRM in [18]. The introduction of the addition term containing the parameter ϵ in the Hamiltonian exposes certain features that do not appear in the derivation for the symmetric QRM case. In this report we discuss some of the general ideas behind the computation of the heat kernel of the QRM and AQRM. In particular, we discuss some of the issues encountered when the method was generalized to the AQRM. This document may be considered to be a summary or complement of the aforementioned papers. We direct the reader to the papers for the details and proofs.

We would also like to mention some related results. The heat kernel of the QRM was obtained by the Feynman-Kac formula by Hirokawa and Hiroshima in [8]. However, it appears that to obtain an explicit expression from the Feynman-Kac formula it requires a similar amount of effort than the present computation. For the Hamiltonian associated to the Kondo problem (sd-model), a special matrix coefficients of the heat kernel were obtained by Anderson, Yuval and Hamann in [1] as a power series in one of the parameters with coefficients given by iterated integrals. The result for the sd-model suggests that the method used for the QRM may be extended to a wide range of models, but this is still an open problem.

2 Main results

In this section we state some results from the papers [15, 18]. The main result is the explicit formula for the heat kernel of the AQRM Hamiltonian H_ϵ. The heat kernel $K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta)$ of H_ϵ is given by a power series on the parameter Δ, namely

$$K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta) = K_0(x,y,t;g)\sum_{\lambda=0}^{\infty}(t\Delta)^{\lambda}\Phi_{\lambda}^{(\epsilon)}(x,y,t;g).$$

We now describe each of the involved functions. The function $K_0(x,y,t;g)$ is the Mehler's kernel, the heat kernel of the quantum harmonic oscillator, given explicitly by

$$K_0(x,y,t;g) = \frac{e^{g^2 t}}{\sqrt{\pi(1-e^{-2t})}} \exp\left(-\frac{1+e^{-2t}}{2(1-e^{-2t})}(x^2+y^2) + \frac{2e^{-t}xy}{1-e^{-2t}} \right).$$

The presence of the Mehler's kernel is natural due to the prominent place of the quantum harmonic oscillator in the AQRM Hamiltonian. It also appears naturally during the computations from an early stage due to the choice of pair of Hamiltonians for the Trotter-Kato product formula.

The function $\Phi_\lambda^{(\epsilon)}(x,y,t;g)$ is the main component of the heat kernel formula, it is a 2×2 matrix-valued function given by an iterated integral over a λ-simplex of volume $(\lambda!)^{-1}$. Concretely, we have

$$\begin{aligned}\Phi_\lambda^{(\epsilon)}(x,y,t;g) :=& e^{-2g^2(\coth(\frac{t}{2}))^{(-1)^\lambda}} \int\cdots\int_{0\le\mu_1\le\cdots\le\mu_\lambda\le1} e^{4g^2\frac{\cosh(t(1-\mu_\lambda))}{\sinh(t)}(\frac{1+(-1)^\lambda}{2})+\xi_\lambda(\boldsymbol{\mu_\lambda},t)} \\ &\times \begin{bmatrix} (-1)^\lambda\cosh & (-1)^{\lambda+1}\sinh \\ -\sinh & \cosh \end{bmatrix} (\theta_\lambda(x,y,\boldsymbol{\mu_\lambda},t)+\eta_\lambda(\boldsymbol{\mu_\lambda},t))\, d\boldsymbol{\mu_\lambda}.\end{aligned}$$

We note that $\operatorname{tr}\left(\Phi_\lambda^{(\epsilon)}(x,y,t;g)\right)=0$ for $\lambda\equiv1 \pmod 2$, thus only the even summands appear in the (matrix) trace of $K_{\mathrm{R}}^{(\epsilon)}(x,y,t;g,\Delta)$ (and, consequently, only the even terms contribute in the partition function).

The functions $\theta_\lambda(x,y,\boldsymbol{\mu_\lambda},t)$ and $\xi_\lambda(\boldsymbol{\mu_\lambda},t)$ are given essentially in terms of hyperbolic functions. Concretely, for $\boldsymbol{\mu_\lambda}=(\mu_1,\mu_2,\cdots,\mu_\lambda)\in\mathbb{R}^\lambda$, we have

$$\theta_\lambda(x,y,\boldsymbol{\mu_\lambda},t;g) := \sqrt{2}g(x+y)\tanh(t) - \frac{\sqrt{2}gt}{\sinh(t)} s_\lambda(x,y,\boldsymbol{\mu}_\lambda,t)$$

with

$$s_\lambda(x,y,\boldsymbol{\mu_\lambda},t) = \sum_{\gamma=1}^{\frac{\lambda}{2}} \left(x\int_{\mu_{2\gamma-1}}^{\mu_{2\gamma}} (e^{t(1-\alpha)}-e^{t(\alpha-1)})d\alpha + y\int_{\mu_{2\gamma-1}}^{\mu_{2\gamma}} (e^{t\alpha}-e^{-t\alpha})d\alpha \right)$$

if $\lambda\equiv0 \pmod 2$ and for $\lambda\equiv1 \pmod 2$, we have

$$s_\lambda(x,y,\boldsymbol{\mu}_\lambda,t) = \sum_{\gamma=0}^{\frac{\lambda-1}{2}} \left(x\int_{\mu_{2\gamma}}^{\mu_{2\gamma+1}} (e^{t(1-\alpha)}-e^{t(\alpha-1)})d\alpha + y\int_{\mu_{2\gamma}}^{\mu_{2\gamma+1}} (e^{t\alpha}-e^{-t\alpha})d\alpha \right).$$

As we see also in the function $\theta_\lambda(x,y,\boldsymbol{\mu_\lambda},t)$, there is a clear difference between odd and even cases of $\lambda\ge0$.

The function $\xi_\lambda(\boldsymbol{\mu_\lambda},t;g)$ is given by

$$\begin{aligned}\xi_\lambda(\boldsymbol{\mu_\lambda},t;g) :=& -\frac{2g^2e^{-t}}{1-e^{-2t}}\left(e^{\frac12 t(1-\mu_\lambda)}-e^{\frac12 t(\mu_\lambda-1)}\right)^2(-1)^\lambda\sum_{\gamma=0}^{\lambda}(-1)^\gamma(e^{-t\mu_\gamma}+e^{t\mu_\gamma}) \\ &-\frac{2g^2e^{-t}}{1-e^{-2t}}\sum_{\substack{0\le\alpha<\beta\le\lambda-1\\ \beta-\alpha\equiv1 \pmod 2}}\left((e^{t(1-\mu_{\beta+1})}+e^{t(\mu_{\beta+1}-1)})-(e^{t(1-\mu_\beta)}+e^{t(\mu_\beta-1)})\right) \\ &\times((e^{t\mu_\alpha}+e^{-t\mu_\alpha})-(e^{t\mu_{\alpha+1}}+e^{-t\mu_{\alpha+1}})).\end{aligned}$$

We note that in the formulas above, we use the convention $\mu_0 = 0$ whenever it appears.

The dependence of the parameter ϵ is given by the function $\eta_\lambda(\boldsymbol{\mu}_\lambda, t)$, defined by

$$\eta_\lambda(\boldsymbol{\mu}_\lambda, t) := -2\,\epsilon\, t(-1)^\lambda \sum_{\gamma=1}^{\lambda} (-1)^\gamma \mu_\gamma + \epsilon t.$$

Thus, at the level of the heat kernel, the only difference between the QRM and the AQRM case is the presence of the function $\eta_\lambda(\boldsymbol{\mu}_\lambda, t)$.

While the functions above are quite complicated, we would like to remark it is not difficult to obtain useful upper bounds for their absolute value. For instance, let $\lambda \in \mathbb{Z}_{\geq 1}$, for fixed $x, y > 0$ and $t > 0$, there are real functions $C_1(x,y,t;g), C_2(t;g) \geq 0$ bounded in closed intervals of the half plane $\Re(t) > 0$, such that

$$|\theta_\lambda(\boldsymbol{\mu_\lambda}, x, y, t; g)| \leq \left| \frac{\sqrt{2}g}{1-e^{-2t}} \right| C_1(x,y,t;g), \qquad |\xi_\lambda(\boldsymbol{\mu_\lambda}, t; g)| \qquad \leq \left| \frac{2g^2}{1-e^{-2t}} \right| C_3(t;g)\lambda$$

uniformly for $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_\lambda \leq 1$. Using this, we can easily show analytical properties of the heat kernel, like the analytic continuation. We refer to [16] for the detailed discussion.

In addition, the explicit for the partition function $Z_{\text{Rabi}}^{(\epsilon)}(t)$ of the AQRM are explicitly given in [18]. It is obtained in a straightforward way from the heat kernel formula.

3 General method of computation

In this section we give a step-by-step overview of the method of computation of the heat kernel AQRM. By separating the computation into steps we intend to explain the places where the computation differs from the one of the QRM. It is also helpful to clarify the process in light of further generations to a wider range of Hamiltonians. The general method is illustrated in Figure 3.

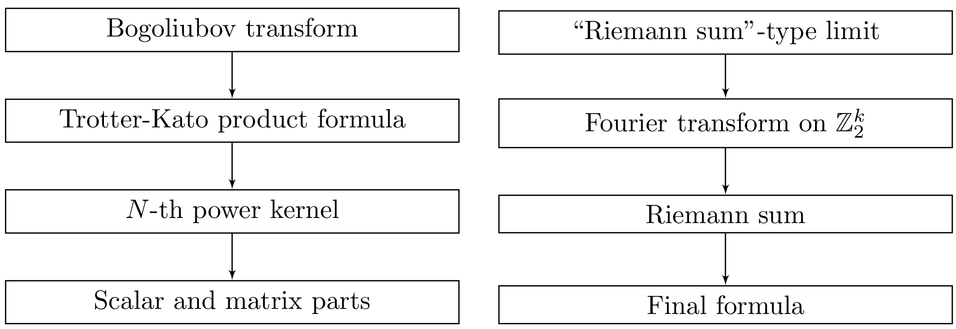

Figure 1: General procedure for the computation of the heat kernel

Bogoliubov tranform

In this part, the Hamiltonian $H = H_\epsilon$ is separated into two parts

$$H = H_1 + H_2,$$

where both Hamiltonians are assumed to be bounded below. The desired property of the choice is that the integral kernel $D(x,y,t)$ of $e^{-tH_1}e^{-tH_2}$ may be easily computed.

For the case of the QRM we take

$$H_1 = b^\dagger b - g^2 = (a^\dagger + g\sigma_x)(a + g\sigma_x) - g^2, \qquad H_2 = \Delta\sigma_z.$$

Then, H_1 is just a matrix version of the quantum harmonic oscillator since the operators b and $b^\dagger$ satisfies the commuting relation $[b, b^\dagger] = \boldsymbol{I}_2$. Thus, the spectrum of H_1 is $\{n - g^2\}_{n\geq 0}$ with each eigenvalue with multiplicity two, and heat kernel $D(x, y, t; g)$ can easily be computed in the standard way in terms of the Mehler kernel.

For the case of the AQRM, we take

$$H_2 = \Delta\sigma_z + \epsilon\sigma_x,$$

and note that for the computations it is convenient to define the matrix

$$C = \begin{bmatrix} \Delta - \mu & \epsilon \\ \Delta + \mu & \epsilon. \end{bmatrix}.$$

with $\mu = \sqrt{\epsilon^2 + \Delta^2}$, so that

$$CH_2C^{-1} = \begin{bmatrix} -\mu & 0 \\ 0 & \mu \end{bmatrix}.$$

The parameter μ takes a similar role as the parameter Δ in the computations of the QRM case.

We also remark that is possible to take different choices in the case of the AQRM. For instance, we can take

$$H_1 = b^\dagger b - g^2 + \epsilon\sigma_x, \qquad H_2 = \Delta\sigma_z,$$

however this choice does not lead to simplification in the computations.

Trotter-Kato product formula

From the discussion above, the operators H_1 and H_2 satisfy the conditions of the Trotter-Kato product formula and we have

$$e^{-tH} = e^{-t(H_1+H_2)} = \lim_{N\to\infty} (e^{-\frac{t}{N}H_1} e^{-\frac{t}{N}H_2})^N,$$

in the strong operator topology. Similarly, we have the limit expression for the heat kernel

$$K_{\mathrm{R}}^{(\epsilon)}(x, y, t; g, \Delta) = \lim_{N\to\infty} D^{(N)}(x, y, t),$$

where for fixed $N \in \mathbb{Z}_{\geq 1}$, $D^{(N)}(x, y, t)$ is the integral kernel of $(e^{-tH_1}e^{-tH_1})^N$, computed as

$$D^{(N)}(x, y, t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} D(x, v_1, t) D(v_1, v_2, t) \cdots D(v_{N-1}, y, t) d\boldsymbol{v}.$$

In order to compute the integral above, we separate the matrix and scalar part of the above product.

N-th power kernel

Setting $u = e^{-t}$, the matrix part of the integrand of $D^{(N)}(x,y,t)$ is given by

$$\overrightarrow{\prod}_{i=1}^{N}\left\{\left[\cosh\left(\sqrt{2}g\frac{1-u}{1+u}(v_{i-1}+v_i)\right)\mathbf{I}-\sinh\left(\sqrt{2}g\frac{1-u}{1+u}g(v_{i-1}+v_i)\right)\sigma_x\right]u^{\Delta\sigma_z+\epsilon\sigma_x}\right\}.$$

Then, using the identity

$$\cosh(\alpha)\,\mathbf{I}-\sinh(\alpha)\,\sigma_x=\frac{1}{2}\left(\mathbf{I}+\sigma_x\right)e^{-\alpha}+\frac{1}{2}\left(\mathbf{I}-\sigma_x\right)e^{\alpha}, \tag{1}$$

and expanding the product, we see that the matrix product is

$$\sum_{\boldsymbol{s}\in\mathbb{Z}_2^N} G_N^{(\epsilon)}(u,\Delta,\boldsymbol{s})\prod_{i=1}^{N}\exp\left((-1)^{\boldsymbol{s}(i)}\sqrt{2}g\frac{1-u}{1+u}(v_i+v_{i-1})\right),$$

where the choice of $\boldsymbol{s}\in\mathbb{Z}_2^N$ depends on the factors sinh and cosh appearing in the expansion.

Here, the function $G_N^{(\epsilon)}(u,\Delta,\boldsymbol{s})$ is given by

$$G_N^{(\epsilon)}(u,\boldsymbol{s}):=\frac{1}{2^N}\overrightarrow{\prod}_{i=1}^{N}[\boldsymbol{I}+(-1)^{1-\boldsymbol{s}(j)}\sigma_x]u^{\Delta\sigma_z+\epsilon\sigma_x}.$$

This expansion exhibits a $\mathbb{Z}_2^N$ structure, which we later exploit for the computation. It is important to note that the $\mathbb{Z}_2^N$ structure here is purely combinatorial, and essentially a consequence of the identity (1). It is a remarkable fact that this combinatorial structure is later seen to be compatible with a group structure allowing the use of harmonic analysis in $\mathbb{Z}_2^N$. The reason why this procedure work is still unclear and its understanding is important to further generalize to the Hamiltonians of other systems.

The kernel $D^{(N)}(x,y,t)$ is then given by

$$D^{(N)}(x,y,t)=\sum_{\boldsymbol{s}\in\mathbb{Z}_2^N}G_N^{(\epsilon)}(t,\Delta,\boldsymbol{s})I_N(x,y,t,\,\boldsymbol{s}),$$

where $I_N(x,y,u,\boldsymbol{s})$ is the scalar part, which includes the iterated integrals in $D^{(N)}(x,y,t)$. We note there that the scalar part does not contain the parameter ϵ and thus it is the same for both QRM and AQRM. This fact simplifies the computations by allowing to use some of the computations in [15] directly.

Scalar and matrix part separation

The objective is to compute the limit

$$K_{\mathrm{R}}^{(\epsilon)}(x,y,t)=\lim_{N\to\infty}\sum_{\boldsymbol{s}\in\mathbb{Z}_2^N}G_N^{(\epsilon)}(\tfrac{t}{N},\Delta,\boldsymbol{s})I_N(x,y,\tfrac{t}{N},\,\boldsymbol{s}). \tag{2}$$

In the current form, there is no apparent way to evaluate the limit. Let us introduce a partition into $\mathbb{Z}_2^N$ in order to rewrite the sum inside the limit.

Definition 3.1. Let $N\in\mathbb{Z}_{\geq 1}$ and $i,j\in\mathbb{Z}_2$.

1. The subset $\mathcal{C}_{ij}^{(N)} \subset \mathbb{Z}_2^N$ is given by

$$\mathcal{C}_{ij}^{(N)} = \{\boldsymbol{s} \in \mathbb{Z}_2^N \mid \boldsymbol{s}(1) = i, \backslash(N) = j\}.$$

2. For $3 \leq k \leq N$ the subset $\mathcal{A}_{ij}^{(k,N)} \subset \mathbb{Z}_2^N$ is given by

$$\mathcal{A}_{ij}^{(k,N)} = \{\boldsymbol{s} \in \mathbb{Z}_2^N \mid \boldsymbol{s}(1) = i, \boldsymbol{s}(k-1) = 1-j, \boldsymbol{s}(n) = j \text{ for } k \leq n \leq N\},$$

3. We have

$$\mathcal{A}_{00}^{(1,N)} = \{(0,0,0,0,\cdots,0)\}, \qquad \mathcal{A}_{01}^{(2,N)} = \{(0,1,1,1,\cdots,1)\},$$
$$\mathcal{A}_{11}^{(1,N)} = \{(1,1,1,,1,\cdots,1)\}, \qquad \mathcal{A}_{10}^{(2,N)} = \{(1,0,0,0,\cdots,0)\},$$

and $\mathcal{A}_{ij}^{(k,N)} = \emptyset$ for $k = 1,2$ if it is not one of the four sets above.

For $N \geq 2$, the sets $\mathcal{A}_{ij}^{(k,N)} \subset \mathbb{Z}_2^N$ form a partition of $\mathbb{Z}_2^N$, that is,

$$\mathbb{Z}_2^N = \bigsqcup_{1 \leq k \leq N} \bigsqcup_{i,j \in \mathbb{Z}_2} \mathcal{A}_{ij}^{(k,N)}.$$

Next, in order to change the sum inside the limit of (2) according to the partition we need to rewrite the functions $G_N(t,\Delta,\boldsymbol{s})$ and $I_N(x,y,t,\,\boldsymbol{s})$ in a compatible way.

Let us first consider the matrix part $G_N^{(\epsilon)}(t,\Delta,\boldsymbol{s})$. For $v,w \in \{0,1\}$, the function $h_{v,w}(\tau)$ is given by

$$h_{v,w}(\tau) := \left(\frac{1+(-1)^{v+w}}{2}\right)(1+\tau) + \left(\frac{(-1)^v+(-1)^w}{2}\right)\frac{\epsilon}{\mu}(1-\tau) + \left(\frac{1-(-1)^{v+w}}{2}\right)\frac{\Delta}{\mu}(1-\tau).$$

Proposition 3.1. *For $\boldsymbol{s} \in \mathbb{Z}_2^k$, we have*

$$G_k^{(\epsilon)}(u,\boldsymbol{s}) = \frac{\prod_{i=1}^{k-1} h_{s(i),s(i+1)}(u^{2\mu})}{u^{(k-1)\mu}2^k}\mathbf{M}_k(\boldsymbol{s})u^{\Delta\sigma_z+\epsilon\sigma_x}, \tag{3}$$

where

$$\mathbf{M}_k(\boldsymbol{s}) = \mathbf{M}_{s(1),s(k)},$$

with

$$\mathbf{M}_{00} := \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \mathbf{M}_{01} := \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \mathbf{M}_{10} := \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{M}_{11} := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The proposition above shows, in particular, that the function $G_k^{(\epsilon)}(u,\boldsymbol{s})$ is multiplicative with respect to the concatenation of elements of $\mathbb{Z}_2^k$. This is the property that allows the rewriting of the sum in (2).

Next, let us consider the scalar factor. It is explicitly evaluated by means of multivariate Gaussian integrals.

Proposition 3.2. *We have*

$$\begin{aligned}
I_N(x,y,u,\,\boldsymbol{s}) &= K_0(x,y,g,u^N) \\
&\times \exp\left(\sqrt{2}g\frac{(1-u)}{(1-u^{2N})}\sum_{j=1}^{N}(-1)^{\boldsymbol{s}(j)}\left(x\Lambda^{(j)}(u) + y\Lambda^{(N-j+1)}(u)\right)\right) \\
&\times \exp\left(\frac{g^2(1-u)^2}{2(1+u)^2(1-u^{2N})}\left(\sum_{i=1}^{N-1}\eta_i(\boldsymbol{s})^2\Omega^{(i,i)}(u) + 2\sum_{i<j}\eta_i(\boldsymbol{s})\eta_j(\boldsymbol{s})\Omega^{(i,j)}(u)\right) - \frac{2Ng^2(1-u)}{1+u}\right),
\end{aligned}$$

with

$$\eta_i(\boldsymbol{s}) := (-1)^{\boldsymbol{s}(i)} + (-1)^{\boldsymbol{s}(i+1)}, \quad \Lambda^{(j)}(u) := u^{j-1}\left(1 - u^{2(N-j)+1}\right),$$
$$\Omega^{(i,j)}(u) = u^{j-i}\left(1-u^{2i}\right)\left(1-u^{2(N-j)}\right).$$

We make two observations on Proposition 3.2. First, in $I_N(x, y, u, \boldsymbol{s})$, and in particular in the function $\eta_i(\boldsymbol{s})$, the factors of the form $(-1)^{s(i)}$ may be interpreted as characters of the group $\mathbb{Z}_2^N$. This is a first indication that it may be fruitful to introduce the harmonic analysis on the groups $\mathbb{Z}_2^N$.

Second, we can factor the function $I_N(x, y, u, \boldsymbol{s})$ as follows. For $\bar{\boldsymbol{s}} \in \mathbb{Z}_2^{k-1}$ with $k \geq 1$, write

$$I_N(x, y, u, \bar{\boldsymbol{s}} \oplus \boldsymbol{0}_{N-k+1}) = K_0(x, y, g, u^N) J_0^{(k,N)}(x, y, u) R_0^{(k,N)}(x, y, u, \bar{\boldsymbol{s}}),$$
$$I_N(x, y, u, \bar{\boldsymbol{s}} \oplus \boldsymbol{1}_{N-k+1}) = K_0(x, y, g, u^N) J_1^{(k,N)}(x, y, u) R_1^{(k,N)}(x, y, u, \bar{\boldsymbol{s}}),$$

where $\oplus$ is the concatenation of elements of $\mathbb{Z}_2^k$, and for appropriate functions $J_\mu^{(k,N)}(x, y, u)$ and $R_\mu^{(k,N)}(x, y, u, \bar{\boldsymbol{s}})$ for $\mu \in \{0, 1\}$ (defined in [15]). We note here that $J_\mu^{(k,N)}(x, y, u)$ does not depend on the vector $\bar{\boldsymbol{s}}$ and only of the tail of zeros (or ones). This factorization corresponds to the multiplicativity property of function $G_k^{(\epsilon)}(u, \boldsymbol{s})$. The next step is to rewrite the sum in the limit of the heat kernel.

Limit expression as "Riemann-sum"-type limit

Putting everything together, we arrive to the limit expression for $K_{\mathrm{R}}^{(\epsilon)}(x, y, t)$, namely, the heat kernel is equal to

$$K_0(x,y,g,u) \lim_{N\to\infty} \left(\frac{1}{2} \sum_{i=0}^{1} \left(\frac{h_{i,i}(u^{\frac{2\mu}{N}})}{2u^{\frac{\mu}{N}}} \right)^{N-1} J_i^{(1,N)}(x,y,u^{\frac{1}{N}},g) \boldsymbol{M}_{i,i} \right.$$
$$+ \left(\frac{h_{0,1}(u^{\frac{2\mu}{N}})}{2u^{\frac{\mu}{N}}} \right) \sum_{i=0}^{1} \sum_{k\geq 2}^{N} \left[\left(\frac{h_{i,i}(u^{\frac{2\mu}{N}})}{2u^{\frac{\mu}{N}}} \right)^{N-k} J_i^{(k,N)}(x,y,u^{\frac{1}{N}},g) \right.$$
$$\left.\left. \times \sum_{v=0}^{1} \boldsymbol{M}_{v0} \sum_{\boldsymbol{s}\in\mathcal{C}_{v,i-1}^{(k-1)}} g_{k-1}^{(\epsilon)}(u^{\frac{1}{N}}, \boldsymbol{s}) R_i^{(k,N)}(u^{\frac{1}{N}}, \boldsymbol{s}) \right] \right),$$

where $g_{k-1}^{(\epsilon)}(u^{\frac{1}{N}}, \boldsymbol{s})$ is the scalar part of $G_{k-1}^{(\epsilon)}(u^{\frac{1}{N}}, \boldsymbol{s})$.

In the sum above, we note that

$$\lim_{N\to\infty} \left(\frac{h_{0,1}(u^{\frac{2\mu}{N}})}{2u^{\frac{\mu}{N}}} \right) = 0$$

and therefore, the limit has the form of a Riemann sum. However it is not possible to control the change of signs appearing in the functions involved (in particular $R_\eta^{(\alpha,\eta)}(u^{\frac{1}{N}}, \boldsymbol{s})$). In order to deal with this problem we use the Fourier transform on the groups $\mathbb{Z}_2^N$.

Fourier transform in finite groups

The main tool is the application of the Parseval identity on the abelian groups $\mathbb{Z}_2^{k-3}$ to the innermost sum in the limit of the heat kernel expression, that is,

$$\sum_{\boldsymbol{s}\in\mathcal{C}_{vw}^{(k-1)}} g_{k-1}(u^{\frac{1}{N}},\boldsymbol{s})R_w^{(k,N)}(u^{\frac{1}{N}},\boldsymbol{s}) = \frac{1}{2^{k-3}}\sum_{\rho\in\mathcal{C}_{vw}^{(k-1)}} \hat{g}_{k-1}(u^{\frac{1}{N}},\rho)\hat{R}_w^{(k,N)}(u^{\frac{1}{N}},\rho), \tag{4}$$

by taking advantage of the fact that the set $\mathcal{C}_{vw}^{(k-1)}$ can be equipped with a $\mathbb{Z}_2^{k-3}$ abelian group structure. The first step is to compute the Fourier transform of the functions $g_{k-1}(u^{\frac{1}{N}},\boldsymbol{s})$ and $R_w^{(k,N)}(u^{\frac{1}{N}},\boldsymbol{s})$.

Let $v,w\in\{0,1\}$. Then, for $\boldsymbol{s}\in\mathbb{Z}_2^k$ with $k\geq 1$, define the function $g_k^{(v,w)}(u,\boldsymbol{s})$ by

$$g_k^{(v,w)}(u,\boldsymbol{s}) := h_{v,s(1)}(u^{2\mu})h_{s(k),w}(u^{2\mu})\prod_{i=1}^{k-1} h_{s(i),s(i+1)}(u^{2\mu}).$$

The main difference between the computation of the case of the QRM and the AQRM is that the Fourier transform of $g_k^{(v,w)}(u,\boldsymbol{s})$ cannot be computed directly. In contrast, we only have an expression as the product of certain matrices given by the factors of the function $g_k^{(v,w)}(u,\boldsymbol{s})$. While this change introduces some complications, the remainder of the computations are not changed in a significant way.

Proposition 3.3. *For $\rho\in\mathbb{Z}_2^k$, we have*

$$\left[\widehat{g_k^{(v,0)}}(\rho),\ \widehat{g_k^{(v,1)}}(\rho)\right] = \left[h_{0,v}(u^{2\mu}),\ h_{1,v}(u^{2\mu})\right]\overrightarrow{\prod}_{i=1}^{k}\boldsymbol{B}(\rho_i),$$

where the matrix-valued function $\boldsymbol{B}(s)$, for $s\in\mathbb{Z}_2$, is given by by

$$\boldsymbol{B}(s) = \begin{bmatrix} h_{0,0}(u^{2\mu}) & h_{0,1}(u^{2\mu}) \\ (-1)^s h_{1,0}(u^{2\mu}) & (-1)^s h_{1,1}(u^{2\mu})\end{bmatrix}.$$

To compute the Fourier transform of the function $R_\eta^{(\alpha,\eta)}(u^{\frac{1}{N}},\boldsymbol{s})$ is necessary to make certain combinatorial considerations. These combinatorial considerations depend essentially on the characters of $\mathbb{Z}_2^N$ appearing in $R_\eta^{(\alpha,\eta)}(u^{\frac{1}{N}},\boldsymbol{s})$ and therefore there is no change from the case of the QRM. We refer the reader to [15] for the full details.

Summing up, we have

$$\begin{aligned}
&\sum_{\boldsymbol{s}\in\mathcal{C}_{vw}^{(k-1)}} g_{k-1}(\boldsymbol{s})R_\eta^{(k,N)}(\boldsymbol{s}) \\
=&\frac{1}{2^{(k-1)}u^{(k-2)\mu}}\left(h_{1-w,1-w}(u^{2\mu})\right)^{k-3}\sum_{\rho\in\mathbb{Z}_2^{k-3}} h_{v,|\rho|+w}(u^{2\mu})\left(\frac{h_{0,1}(u^{\frac{2\mu}{N}})}{h_{1-w,1-w}(u^{2\mu})}\right)^{|\rho|} \\
&\left(\frac{h_{w,w}(u^{2\mu})}{h_{1-w,1-w}(u^{2\mu})}\right)^{\alpha(\rho)}\exp\left(a_0^{(\eta)}+\sum_{m=0}^{k-4}\sum_{j=1}^{k-3-m}(-1)^{(|\rho|+w)\delta_0(m)+\sum_{i=m}^{m+j-1}\rho_i}a_{m,m+j}^{(\eta)}\right),
\end{aligned}$$

where $\alpha(s) = k-\left\lfloor\frac{|s|}{2}\right\rfloor-\varphi(s)$ and $a_{m,m+j}^{(\eta)}$ are the Fourier coefficients of $R_\eta^{(\alpha,\eta)}(u^{\frac{1}{N}},\boldsymbol{s})$ (see [15]).

It is at this point that a defining feature of the method of computation becomes apparent. In the limit above, with the exception of the function $\alpha(\rho)$, the vector ρ appears in the right-hand side only as a function of its norm, that is, it is similar to a radial function.

Next, we see that the function $\alpha(\rho)$, or rather the function $\varphi(s)$ appearing in is definition, has a similar property. Let us first give the definition.

Definition 3.2. The function $\varphi_k : \mathbb{Z}_2^k \to \mathbb{Z}$ is defined by

$$\varphi_k(\rho) = \frac{k}{2} - \frac{1}{2}\left(\sum_{i=1}^{k}(-1)^{\sum_{j=i}^{k}\rho_j}\right).$$

The function φ_k (and thus $\alpha(s)$) controls the commutation relations between the matrices of the Hamiltonian. Let us also define a generalization of the function φ_k by adding an auxiliary parameter.

Definition 3.3. For $k \geq 1$ and $t \in \mathbb{C}$, the function $\varphi_k(\rho; t) : \mathbb{Z}_2^k \to \mathbb{C}$ is defined by

$$\varphi_k(\rho; t) := \frac{1}{2}\sum_{i=1}^{k}\left(1 - (-1)^{\sum_{j=i}^{k}\rho_j}\right) t^{i-1}.$$

The key property is the the formula,

$$\varphi_k(\rho; t) = \sum_{i=1}^{|\rho|}(-1)^{i-1}[j_{|\rho|+1-i}]_t,$$

where j_i is the position of the i-th one in ρ and $[a]_t = \frac{1-t^a}{1-t}$.

Proposition 3.4. *Let $n \leq N \in \mathbb{Z}_{\geq 0}$. There is a bijection*

$$\mathcal{S}^{(n)} := \left\{\rho \in \mathbb{Z}_2^N \ : \ |\rho| = n\right\} \longleftrightarrow \{j_1, j_2, \cdots, j_n \in \mathbb{Z}_{\geq 1}\, ;\, j_1 < j_2 < \cdots < j_n \leq N\} =: \mathcal{J}_n, \tag{5}$$

Let $\rho \in \mathcal{S}^{(n)}$, corresponding to $\boldsymbol{j} \in \mathcal{J}_n$, then we have

$$\varphi(\rho; t) = \phi^{(n)}(\boldsymbol{j}, t),$$

where $\phi^{(n)}(\boldsymbol{j}, t)$ is a q-polynomial in the variable t.

The proposition says that while $\varphi(\rho; t)$ is not a radial function, it is defined, as a q-polynomial by the norm of ρ. Therefore, by fixing $|\rho| = \lambda \in \mathbb{Z}_{\geq 0}$, the expression

$$\sum_{\rho \in \mathbb{Z}_2^{k-3}} \hat{g}_{k-1}^{(\alpha,\eta)}(u^{\frac{1}{N}}, \rho)\hat{R}_{\eta}^{(\alpha,\eta)}(u^{\frac{1}{N}}, \rho),$$

does not depends k, it only depends on $|\rho|$ and can be expressed in terms of the set $\mathcal{J}_n$. In other words, by fixing $|\rho| = \lambda$ we can control the variation of the signs. This is the property that allows us to compute the heat kernel.

Final computations

Returning to the limit expression in the heat kernel, by partitioning the sum according to the norm and changing the order of summation, we obtain a sum of the type

$$\sum_{\lambda=0}^{\infty} \lim_{N\to\infty} \left(\frac{h_{0,1}(u^{\frac{2\mu}{N}})}{2u^{\frac{\mu}{N}}} \right) \sum_{k\geq 2}^{N} J_0^{(k,N)}(x,y,u^{\frac{1}{N}},g) \left(\frac{h_{0,1}(u^{\frac{2\mu}{N}})}{h_{0,0}(u^{\frac{2\mu}{N}})} \right)^{\lambda}$$
$$\times \sum_{\substack{\rho\in\mathbb{Z}_2^{k-3}\\ |\rho|=\lambda}} \left(\frac{h_{1,1}(u^{\frac{2\mu}{N}})}{h_{0,0}(u^{\frac{2\mu}{N}})} \right)^{\alpha(\rho)} e^{\left(a_0^{(0)}+\sum_{m=0}^{k-4}\sum_{j=1}^{k-3-m}(-1)^{(|\rho|+w)\delta_0(m)+\sum_{i=m}^{m+j-1}\rho_i} a_{m,m+j}^{(0)}\right)},$$

for each component.

As long as we can control the innermost sum, we are able to evaluate the heat kernel as a sum of Riemann integrals. Actually, the sum

$$\sum_{\substack{\rho\in\mathbb{Z}_2^{k-3}\\ |\rho|=\lambda}} \left(\frac{h_{1,1}(u^{\frac{2\mu}{N}})}{h_{0,0}(u^{\frac{2\mu}{N}})} \right)^{\alpha(\rho)} e^{\left(a_0^{(0)}+\sum_{m=0}^{k-4}\sum_{j=1}^{k-3-m}(-1)^{(|\rho|+w)\delta_0(m)+\sum_{i=m}^{m+j-1}\rho_i} a_{m,m+j}^{(0)}\right)},$$

by the bijection in (5) and the properties of $\varphi_k(s;t)$, may be expressed expressed by a multivariate integral in a standard way by using Riemann-Stieltjes integration. After that, it is a straightforward, but laborious process, to complete the computation of the heat kernel by evaluating the limit as a usual Riemann integral to obtain the final formula.

4 Remarks on other models

The computation of heat kernel of QRM offers little understanding on the actual reason why it is successful. Considering the AQRM allows us to abstract certain computations related to the Fourier transform of the functions to a certain extend.

However, in order to further advance the generalization, we need to reinterpret other parts of the computation. For instance, the family of group $\mathbb{Z}_2^N$ in the computation can be identified with the inductive limit $\mathbb{Z}_2^{\infty} = \varinjlim_n \mathbb{Z}_2^n$.

There is a natural action of the infinite symmetric group $\mathfrak{S}_\infty := \varinjlim_n \mathfrak{S}_n$ (see e.g. [12]) where the orbits are given by

$$\mathcal{O}_\lambda := \{\sigma \in \mathbb{Z}_2^{\infty} \,:\, |\sigma| = \lambda\},$$

for $\lambda \geq 0$.

In general, we expect to have the action of $\mathfrak{S}_\infty$ on an induced family of finite groups and an appropriate orbit invariant. Using this idea, we may rearrange the limit for the main body of heat kernel of the QRM into a sum of the form

$$\sum_{\lambda=0}^{\infty} \lim_{N\to\infty} \left(\frac{1-u^{\frac{2\Delta}{N}}}{2u^{\frac{\Delta}{N}}} \right) \sum_{k\geq\lambda}^{N} \left(\frac{1+u^{\frac{2\Delta}{N}}}{2u^{\frac{\Delta}{N}}} \right)^{N-k} h_\eta^{(k,N)}(x,y,u^{\frac{1}{N}},g) \int_{\mathcal{O}_\lambda^k} f_{k-1}^{(\alpha,\eta)}(u^{\frac{1}{N}},\mu) d\mu_\lambda,$$

where the integral (an "orbit integral") is given by

$$\int_{\mathcal{O}_\lambda^k} f_{k-1}^{(\alpha,\eta)}(u^{\frac{1}{N}},\mu) d\mu_\lambda = \sum_{1\leq j_1<j_2<\cdots<j_\lambda\leq k} f_{k-1}^{(\alpha,\eta)}(u^{\frac{1}{N}},\boldsymbol{j}).$$

Therefore, if for a given Hamiltonian we are able to find the appropriate group family from the structure of the matrices of the Hamiltonian, define the corresponding symmetric group action and orbit invariant (in case of QRM and AQRM it is given by $|\cdot|$), we can use the framework of the computation of the QRM with small modifications.

For simpler modifications AQRM, two-photon Rabi, Dicke model, we can use this method with minimal changes, however some of the computations involved are non-trivial. The generalization is the subject of a forthcoming paper of the author with Masato Wakayama [17].

5 Spectral zeta function

We conclude by mentioning that by following the method of [15], we can give the meromorphic continuation of the spectral zeta function of the AQRM. This result was announced without proof in [10].

The function $\Omega^{(\epsilon)}(t)$ is the numerator of the partition function $Z_{\text{Rabi}}^{(\epsilon)}(t)$ of the AQRM. Concretely, we define $\Omega^{(\epsilon)}(t)$ implicitly by

$$Z_{\text{Rabi}}^{(\epsilon)}(t) = \frac{\Omega^{(\epsilon)}(t)}{(1-e^{-t})}.$$

The function $\Omega^{(\epsilon)}(t)$ is explicitly given by

$$\Omega^{(\epsilon)}(t) = 2e^{g^2 t}\Bigg[\cosh(\epsilon t) + e^{-2g^2\coth(\frac{t}{2})}\sum_{\lambda=1}^{\infty}(t\Delta)^{2\lambda} \\ \times \int\cdots\int_{0\le\mu_1\le\cdots\le\mu_{2\lambda}\le 1} \cosh(\epsilon\,(t+\eta_{2\lambda}(\boldsymbol{\mu_{2\lambda}},t)))e^{4g^2\frac{\cosh(t(1-\mu_{2\lambda}))}{\sinh(t)}+\xi_{2\lambda}(\boldsymbol{\mu_{2\lambda}},t)+\psi^{-}_{2\lambda}(\boldsymbol{\mu_{2\lambda}},t)}d\boldsymbol{\mu_{2\lambda}}\Bigg],$$

where the function $\psi^{-}_{\lambda}(\boldsymbol{\mu_\lambda},t)$ is given by

$$\psi^{-}_{\lambda}(\boldsymbol{\mu_\lambda},t) := \frac{2g^2e^{-t}}{1-e^{-2t}}\left[\sum_{\gamma=0}^{\lambda}(-1)^{\gamma}\left(e^{t\left(\frac{1}{2}-\mu_\gamma\right)}-e^{t\left(\mu_\gamma-\frac{1}{2}\right)}\right)\right]^2.$$

for $\lambda\ge 1$ and $\boldsymbol{\mu_\lambda}=(\mu_1,\mu_2,\cdots,\mu_\lambda)$ and where $\mu_0=0$.

The main result is the expression of the spectral zeta function of the AQRM as a contour integral, as in one of the classical proofs of the meromorphic continuation of the Riemann zeta function (see e.g. [20] or [9]).

Theorem 5.1. *We have*

$$\zeta_{QRM}^{(\epsilon)}(s;\tau) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\infty}^{(0+)}\frac{(-w)^{s-1}\Omega^{(\epsilon)}(w)e^{-\tau w}}{1-e^{-w}}dw. \tag{6}$$

Here the contour integral is given by the path which starts at ∞ on the real axis, encircles the origin (with a radius smaller than 2π) in the positive direction and returns to the starting point and it is assumed $|\arg(-w)|\le\pi$. This gives a meromorphic continuation of $\zeta_{QRM}(s;\tau)$ to the whole plane where the only singularity is a simple pole with residue 2 at $s=1$. □

It is clear from the formulas that

$$\zeta_{\text{QRM}}^{(\epsilon)}(s;\tau) = \zeta_{\text{QRM}}^{(-\epsilon)}(s;\tau),$$

as expected by the well-known result that the spectrum is invariant under the transformation $\epsilon \mapsto -\epsilon$ (see for instance [10]).

The meromorphic continuation of the spectral zeta function is not the only application of the explicit formulas of the heat kernel and partition function of the QRM (and AQRM). For other applications we refer the reader to [16].

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