

# An extension of the Bloch-Floquet theory to the Heisenberg group and its applications to asymptotic problems for heat kernels and prime closed geodesics

Atsushi Katsuda\*

## Abstract

The Bloch-Floquet theory are popular tools for the investigation of materials with periodic structures. For example, we can show that the spectrum of periodic Schrödinger operators have band structures. Here we shall extend the Bloch-Floquet theory, which is applicable to abelian groups, to the Heisenberg group. Our method is based on a combination of the representations of the discrete Heisenberg groups and of the Heisenberg Lie group. We apply this method to asymptotic problems for heat kernels and for counting prime closed geodesics. In this application, we need additional ingredients, the semi-classical analysis and the Chen's iterated integrals. As a by-product, we give another mathematical explanation of the semi-classical asymptotic expansion formula for the Hofstadter butterfly of Wilkinson, which is originally due to Helffer-Sjöstrand.

## 1 Introduction

As in the abstract, although our main concern in this paper is an extension of the Bloch Floquet theory to the Heisenberg group, we start from the principal application here.

Let  $M$  be a compact Riemannian manifold of negative curvature and let  $\pi(x)$  denote the number of prime closed geodesics on  $M$  whose length is at most  $x$ . Celebrated results of Selberg [38], Huber [16], Margulis [27],[28], Parry and Pollicott [29] asserts that

---

2020 *Mathematics Subject Classification*. Primary 58J50 Secondary 58J37, 58J35, 30F99

*Key words and phrases*. Bloch-Floquet theory, Heisenberg group, Asymptotic expansion

\*This work was supported by JSPS KAKENHI Grant Number JP18K03282 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

THEOREM 1.1.

$$\pi(x) \sim \frac{e^{hx}}{hx},$$

where  $h > 0$  is the topological entropy of the geodesics flow on  $M$ .

This is called the prime geodesic theorem which is geometric analogue of the prime number theorem. As a variants, there are several results in geometry, which are analogue of the Dirichlet density theorem for arithmetic progressions or the Chebotarev density theorem for algebraic extensions of number fields.

Let us formulate problems precisely. Taking into account that there are one to one correspondence among the following triplet, closed geodesics of  $M$ , free homotopy classes of closed curves in  $M$  and conjugacy classes of an element in the fundamental group  $\pi_1(M)$  of  $M$ , we consider a surjective homomorphism

$$\Phi : \pi_1(M) \rightarrow \Gamma$$

for a finitely generated discrete group  $\Gamma$ ,

For a conjugacy class  $\alpha$  in  $\Gamma$ , let  $\pi(x, \alpha)$  denote the number of prime closed geodesics  $\gamma$  on  $M$  whose length is at most  $x$  and satisfies  $\Phi([\gamma]) \subset \alpha$ , where  $[\gamma]$  denotes the corresponding conjugacy class of  $\gamma$  in  $\pi_1(M)$ . Then our problems is as follows:

PROBLEM 1.2. What is the asymptotic behavior of  $\pi(x, \alpha)$  as  $x \rightarrow \infty$ ?

When  $\Gamma$  is the trivial group,  $\pi(x, \alpha) = \pi(x)$  and the answer of the above problem is given by the prime geodesic theorem. First answer of the above problem is given when  $\Gamma$  is a finite group by Parry and Pollicott [30] and Adachi and Sunada [1] independently as follows:

THEOREM 1.3.

$$\pi(x, \alpha) \sim \frac{\#\alpha}{\#\Gamma} \frac{e^{hx}}{hx},$$

where  $\#\alpha$  and  $\#\Gamma$  denote the cardinals of  $\alpha$  and  $\Gamma$  respectively.

This theorem can be considered as a geomtric analogue of usual Chebotarev density theorem for finite extensions of number fields.

Next answer is given when  $\Gamma$  is an infinite abelian group. In the case when  $M$  is a compact Riemann surface of genus  $g$  with constant negative curvature  $-1$ , the following asymptotic result is given by Phillips and Sarnak [31],

THEOREM 1.4.

$$\pi(x, \alpha) \sim (g-1)^g \frac{e^x}{x^{g+1}} \left( 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots \right)$$

The leading term in the above theorem also obtained by Katsuda and Sunada[19] independently. The essential invariant to the numerator  $(g-1)^g$  of the leading term is the volume of the Jacobi torus which is equal to 1 in this case. The genus  $g$  in the exponent of the denominator is the half of the first Betti number i.e. the rank of the first integral homology group  $H_1(M, \mathbb{Z})$  of  $M$ .

This result is generalized to the case of prime closed geodesics in compact Riemannian manifolds with variable negative curvature or more generally, prime closed orbits of weakly mixing Anosov flows on compact Riemannian manifolds by Lalley [26], Pollicott Pollicott1, Katsuda and Sunada [20] independently for the leading terms and Anantharaman [3], Pollicott and Sharp [33] and Kotani [21] for asymptotic expansions. Furthermore, there are generalizations to the central limit theorems and the large deviations [26], [6], [4].

The case we wish to consider here is that  $\Gamma$  is a discrete nilpotent group. This generalization seems to be natural by the following reason: when  $\Gamma$  is an abelian group, it is a quotient of the maximal abelian group which is isomorphic to  $H_1(M, \mathbb{Z})$ . By the Hurewicz theorem, we have  $H_1(M, \mathbb{Z}) \simeq \pi_1(M)/[\pi_1(M), \pi_1(M)]$  where  $[\pi_1(M), \pi_1(M)]$  denotes the commutator subgroup of  $\pi_1(M)$  and  $\pi_1(M)/[\pi_1(M), \pi_1(M)]$  is the abelianization of  $\pi_1(M)$ . An example of nilpotent group naturally arises as  $\Gamma \simeq \pi_1(M)/[\pi_1(M), [\pi_1(M), \pi_1(M)]]$  where  $[\pi_1(M), [\pi_1(M), \pi_1(M)]]$  denotes the double commutator subgroup of  $\pi_1(M)$ .

For general nilpotent groups, we have the following conjecture:

**CONJECTURE 1.5.** Let  $\Gamma$  be a finitely generated nilpotent group and  $\alpha$  be a conjugacy class of a central element of  $\Gamma$ ,

$$\pi(x, \alpha) \sim C \frac{e^{hx}}{x^{1+d/2}} \left( 1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots \right)$$

, where  $C$  is a constant depending on the geometry of  $M$  and the special values  $\zeta_H(d/2)$  of the spectral zeta function of a hypo-elliptic operator  $H$  related to some irreducible representations of the simply connected nilpotent Lie group  $G$  which contains  $\Gamma$  as a lattice subgroup, i.e.  $G$  is the Malcev completion of  $\Gamma$ , and  $d$  is the polynomial growth order of  $\Gamma$ . Moreover constants  $c_1, c_2, \dots$  are also written in the quantities of  $M$  related to the Chen's iterated integrals.

Note that for a finitely generated discrete nilpotent group  $\Gamma$ , the polynomial growth order  $d$  of  $\Gamma$  can be defined as follows: Take a finite generating set  $S$  of  $\Gamma$  and let  $\omega(k)$  be the number of elements of  $\Gamma$  whose word length with respect to  $S$  are less than or equal to  $k$ . Then it is known that there are constants  $C_1, C_2 > 0$  and positive integer  $d$  such that

$$C_1 k^d \leq \omega(k) \leq C_2 k^d$$

and  $d$  is independent to the choice of  $S$ .

As an application of our extension of Bloch-Floquet theory, we can show that Conjecture 1.5 holds in the case when  $\Gamma$  is the three dimensional discrete Heisenberg group  $\text{Heis}_3(\mathbb{Z})$  and  $M$  is a compact Riemann surface with constant negative curvature  $-1$  as follows:

**THEOREM 1.6.** *Let  $M$  be a compact Riemann surface with the constant negative curvature  $-1$  and  $\Gamma$  be  $\text{Heis}_3(\mathbb{Z})$ . For a conjugacy class  $\alpha$  of a central element of  $\Gamma$ , we have the following asymptotic expansion:*

$$\pi(x, \alpha) \sim \frac{Ce^x}{x^3} \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots\right),$$

where constants  $C, c_1, c_2 \dots$  are described in Conjecture 1.5.

We notice that there is a surjective homomorphism from  $\pi_1(M)$  of Riemannian surface  $M$  of genus  $g \geq 2$  to  $\Gamma = \text{Heis}_3(\mathbb{Z})$  through the surjective homomorphism from  $\pi_1(M)$  to the free group  $F_g$  of rank  $g$ . Since the polynomial growth order  $d$  is 4 in this case and thus, the exponent of denominator of the leading term is  $3 = 1 + 4/2$ . Moreover, the hypoelliptic operator  $H$  in Conjecture 1.5 is essentially the harmonic oscillator  $\mathcal{H} := -\frac{d^2}{du^2} + u^2$  and the special value  $\zeta_H(2)$  can be expressed by the value  $\zeta(2) = \pi^2/6$  of the Riemann zeta function  $\zeta(s)$ .

If a conjugacy class  $\alpha$  is not coming from central elements, then the asymptotic behavior of  $\pi(x, \alpha)$  can be reduced to the analysis of the case for abelian groups, which is the following.

**PROPOSITION 1.7.** *Let  $M$  be a compact Riemann surface with the constant negative curvature  $-1$  and  $\Gamma$  be  $\text{Heis}_3(\mathbb{Z})$ . For a conjugacy class  $\alpha$  of a non-central element of  $\Gamma$ , we have the following asymptotic expansion:*

$$\pi(x, \alpha) \sim \frac{Ce^x}{x^2} \left(1 + \frac{c_1}{x} + \frac{c_2}{x^2} + \dots\right). \tag{1.1}$$

In this case, the exponent of denominator of the leading term is express as  $2 = 1 + 2/2$  whose numerator of the second term 2 is the rank of an abelian group  $\Gamma/[\Gamma, \Gamma]$ .

Next we consider the long time asymptotics of the heat kernels on  $\Gamma$ -covering manifolds. We can also apply our extension of the Bloch-Floquet analysis. In the case when  $\Gamma$  is an abelian group, the similarity of the analysis of the closed geodesics and the heat kernels was pointed by Sunada. In the case when  $\Gamma$  is nilpotent, Boulanger [9] reduces the asymptotics of the prime closed geodesics for manifolds with constant negative curvature to that of the heat kernels.

Let  $M$  be a compact Riemannian manifolds or a finite unoriented graph. For a discrete group  $\Gamma$ , take a normal covering  $\Pi : X \rightarrow M$  with the covering transformation group

$\Gamma$ . We denote by  $k_X(t, p, q)$  the heat kernel (resp. the transition probability of simple random walks) of  $X$  when  $M$  is a compact Riemannian manifold (resp. a finite graph). For the brevity of the description of results, we assume that  $M$  is not a bipartite graph.

Our problem here is as follows:

PROBLEM 1.8. What is the asymptotic behavior of  $k_X(t, p, q)$  as  $t \rightarrow \infty$ ?

In the case when  $\Gamma$  is an abelian group, the following results are known.

THEOREM 1.9.

$$k_X(t, p, q) \sim \frac{C}{t^{d/2}} \left( 1 + \frac{c_1}{t} + \frac{c_2}{t^2} + \dots \right), \quad (1.2)$$

where  $d$  is the rank of abelian group  $\Gamma$  and  $C, c_1, c_2, \dots$  are constants depending on the geometry of  $M$ .

This result is due to Kotani and Sunada [23]. The leading term of the above theorem is obtained by several authors [14], [18], [25] and [39] independently. In particular, Kotani, Shirai and Sunada [22] focus on the geometric nature of  $C$  which is written in terms of the volume of ‘‘Jacobi torus’’. This was inspired by the corresponding case for closed geodesics in [31], [19] as in Theorem 1.4.

In the case of nilpotent groups, we have the following conjecture for heat kernels, which corresponds to Conjecture 1.5,

CONJECTURE 1.10. Let  $\Gamma$  be a finitely generated nilpotent group.

$$k_X(t, p, q) \sim \frac{C}{t^{d/2}} \left( 1 + \frac{c_1}{t} + \frac{c_2}{t^2} + \dots \right),$$

where constants  $C, d, c_1, c_2, \dots$  are similar to Conjecture 1.5.

In the case when  $\Gamma = \text{Heis}_3(\mathbb{Z})$ , our result for heat kernels is the following, which corresponds Theorem 1.6.

THEOREM 1.11. Let  $\Gamma$  be  $\text{Heis}_3(\mathbb{Z})$ .

$$k_X(t, p, q) \sim \frac{C}{t^2} \left( 1 + \frac{c_1}{t} + \frac{c_2}{t^2} + \dots \right),$$

where constants  $C, c_1, c_2, \dots$  are similar to Theorem 1.6.

It should be noted that in the case when  $M$  is a finite graph, the asymptotics for the leading term is already obtained by a combination of Alexopoulos [2] and Ishiwata [17] for general nilpotent groups  $\Gamma$ . However, their method seems not to give geometric nature of the leading coefficient  $C$ . In fact, their results is a comparison between  $k_X(t, p, q)$  and the heat kernel of a stratified nilpotent Lie groups but gives no information for the latter.

Finally, we point out a relation to the our methods for the Heisenberg group and the analysis of the discrete magnetic Laplacian or the Harper operator on the square lattice  $\mathbb{Z}^2$ . Spectrum of the latter operators are expressed by the celebrated Hofstadter butterfly as the following figure:

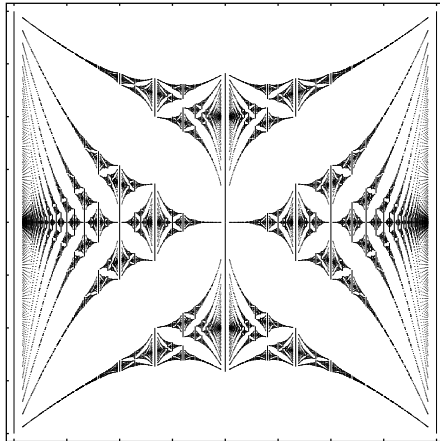


Figure 1: the Hofstadter's butterfly (created by Hisashi Naito). The horizontal axis express the value of  $\theta$  corresponding to the strength of a magnetic field or the magnetic flux density in the interval  $[0, 2\pi]$  and the vertical axis express the spectrum of  $H_\theta$  which is a subset of the interval  $[-4, 4]$ .

The Harper operator  $H_\theta : \ell^2(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z}^2)$  is defined as follows:

$$(H_\theta u)(m, n) = u(m+1, n) + u(m-1, n) + e^{\sqrt{-1}n\theta} u(m, n+1) + e^{-\sqrt{-1}n\theta} u(m, n-1)$$

This operator is a discrete analogue of the Laplacian on the plane under constant magnetic field.

The structure of the spectrum of  $H_\theta$  are already extensively studied (cf.[40]). For example, if the parameter  $\theta$  is a rational number, then the spectrum of  $H_\theta$  is a union of finite intervals, namely it has band structure and  $\theta$  is a irrational number, it was conjectured that the spectrum is a Cantor set, which was known as M. Kac's "Ten Martini Problem" and was finally settled by Avila and Jitomirskaya [5]. Our interest here is the following semi-classical asymptotic expansion formula of the spectrum of  $H_\theta$  as  $\theta \rightarrow 0$  where we regard  $\theta$  as semi classical parameters (3.9 [35], similar formulas (6.9 in [43] and 6.3.1 in [15]),

$$E_n = -4 + (2n+1)\theta + O(\theta^2) \quad n = 0, 1, 2, \dots \quad (1.3)$$

This formula first obtained by Wilkinson [43] by WKB arguments. Rammal and Bellissard [35] also derived this formula by a similar method. Formal derivation of the above

expansion is not difficult as follows: it is known that  $H_\theta$  is unitally equivalent to the following operator acting on  $L^2(\mathbb{R})$ ,

$$h_\theta = -2 \cos\left(\sqrt{\theta} \frac{d}{\sqrt{-1} du}\right) - 2 \cos(\sqrt{\theta} u).$$

By the (formal) Taylor expansion formula, this operator can be expressed as

$$h_\theta = -4 + \left(-\frac{d^2}{du^2} + u^2\right) \theta + O(\theta^2). \quad (1.4)$$

The coefficient of the linear part in  $\theta$  is the harmonic oscillator  $\mathcal{H} := -\frac{d^2}{du^2} + u^2$  whose eigenvalues are  $n + \frac{1}{2}, n = 0, 1, 2, \dots$ , which implies the above expansion (1.3). However, (1.4) is a form that a bounded operator  $h_\theta$  is approximated by a unbounded operator  $\mathcal{H}$ . We need to justify the above arguments mathematically. Note that there are other reasons of necessity for mathematical justifications.

This is first given by Helffer and Sjöstrand [15] using semi-classical analysis. Their method is described very roughly as follows: If the domain  $L^2(\mathbb{R})$  of both operators  $h_\theta$  and  $\mathcal{H}$  are exhausted by a sequence of common invariant finite dimensional spaces  $V_k, k = 1, 2, \dots$ , then the expansion 1.4 can be considered as a limit of Taylor expansion of the restriction  $h_\theta|_{V_k}$  of  $h_\theta$ . However there is no such sequence. In stead of  $V_k$ , Helffer and Sjöstrand used the space  $\tilde{V}_k$  consisting of eigenfunctions of  $\mathcal{H}$  associated with eigenvalues less than or equal to  $k$ . This space is not invariant by  $h_\theta$  but the error is  $O(h^\infty)$  and thus does not effect the asymptotic expansion 1.3.

We shall give another proof based on a comparison between unitary representations of  $\text{Heis}_3(\mathbb{Z})$  and the Heisenberg Lie group  $\text{Heis}_3(\mathbb{R})$ . We believe our method has some advantage to the original proof, which we shall explain in later.

## 2 Outline of the proof for the asymptotics of the heat kernels and other results

Since we take the similar strategy in a broad sense, as in the case when  $\Gamma$  is an abelian group, let us recall the proof of Theorem 1.9 for asymptotics of the leading term.

Our concern is a long time asymptotics of the heat kernel  $k_X(t, p, q)$  on  $X$  for a  $\Gamma$ -covering  $\pi : X \rightarrow M$ . We start from easier case that  $\Gamma$  is a finite group. In this case,  $X$  is compact and thus,  $k_X(t, p, q)$  can be expressed as follows:

$$k_X(t, p, q) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(p) \varphi_i(q), \quad (2.1)$$

where

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

are eigenvalues of the Laplacian  $\Delta_X$  on  $X$  and  $\{\varphi_i\}_{i=0}^\infty$  is a complete orthonormal system of eigenfunctions  $\varphi_i$  associated to eigenvalues  $\lambda_i$ . Note that the 0-th (normalized) eigenfunction  $\varphi_0$  is a constant function  $1/\sqrt{\text{vol}(X)}$  with the volume  $\text{vol}(X)$  of  $X$ . If  $i \geq 1$ , then  $\lambda_i > 0$  and thus  $e^{-\lambda_i t}$  decay to 0 if  $t \rightarrow \infty$ . Then, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} k_X(t, p, q) &= \lim_{t \rightarrow \infty} e^{-\lambda_0 t} \varphi_0(p) \varphi_0(q) \\ &= \varphi_0(p) \varphi_0(q) = \frac{1}{\text{vol} X}. \end{aligned}$$

Next we consider the case that  $\Gamma$  is an infinite abelian group. For simplicity, we assume  $\Gamma = \mathbb{Z}^d = H_1(M, \mathbb{Z})$ . Essential points in the arguments are already appeared in this case and it is not so difficult to extend to general finitely generated abelian groups. Here  $X$  is noncompact. Although there exists a similar formula to (2.1) in this case using the spectral decomposition of  $\Delta_X$ , it seems useless for our problem. In stead of this formula, we decompose  $L^2(X)$  into in some sense “ $L^2(M) \otimes L^2(\Gamma)$ ”. In the case when  $\Gamma$  is abelian, the latter space can be analyzed by the usual Bloch-Floquet theory in conjunction with some perturbation arguments, which we explain in the following two steps:

The first step is a decomposition of  $L^2(X)$  as above. For this purpose, we formulate the Bloch-Floquet theory geometrically as follows: First we identify  $L^2(X)$  with the space  $L^2(E_R)$  of sections of the flat vector bundle  $E_R$  over  $M$  associated with the right regular representation  $R$  of  $\Gamma$ . Here  $E_R$  is described as follows;

$$E_R \simeq X \times L^2(\Gamma) / \sim, \quad (p, \varphi) \sim (\gamma p, R(\gamma^{-1})\varphi)$$

for  $\gamma \in \Gamma$ . Since  $\Gamma \simeq \mathbb{Z}^d$ ,  $R$  can be written as a direct integral of one dimensional irreducible unitary representations, i.e. characters over the unitary dual  $\hat{\Gamma}$  of  $\Gamma$  which is the space of equivalence class of characters  $\chi : \Gamma \rightarrow U(1)$ . Note that  $\hat{\Gamma}$  is isomorphic to a  $d$ -dimensional torus  $U(1)^d$ , which can be identified with the Brillouin zone in usual terminology of the Bloch-Floquet theory in condensed matter physics. Associated with this formula, we have the following decomposition

$$L^2(E_R) \simeq \int_{\hat{\Gamma}}^{\oplus} L^2(E_\chi) d\chi,$$

where  $L^2(E_\chi)$  is the space of sections of the flat line bundle associated with  $\chi$ , which is similarly defined as  $L^2(E_R)$ . The space  $L^2(E_\chi)$  can be identified with the space

$$H_\chi = \{f : X \rightarrow \mathbb{C} \mid f(\gamma p) = \chi(\gamma)f(p) \text{ for all } \gamma \in \Gamma\}.$$



Using this identification, the heat kernel  $k_X(t, p, q)$  can be decomposed as

$$k_X(t, p, q) = \int_{\hat{\Gamma}} k_\chi(t, p, q) d\chi \quad (2.2)$$

where

$$\begin{aligned} k_\chi(t, p, q) &= \sum_{\gamma \in \Gamma} \chi(\gamma) k_X(t, p, \gamma q) \\ &= \sum_{i=0}^{\infty} e^{-\lambda_i(\chi)t} \varphi_{i,\chi}(p) \varphi_{i,\chi}(q) \end{aligned} \quad (2.3)$$

where  $\lambda_i(\chi)$  and  $\varphi_{i,\chi}$  are  $i$ -th eigenvalue and eigenfunction of the twisted Laplacian  $\Delta_\chi := \Delta_X|_{H_\chi}$ . Note that  $\lambda_i(\chi)$  are discrete as eigenvalues  $\lambda_i$  of the usual Laplacian  $\Delta_M$  on  $M$  and  $\lambda_i = \lambda_i(\mathbf{1})$  for the trivial character  $\mathbf{1}$ .

The second step is an analysis of eigenvalues  $\lambda_i(\chi)$  of the twisted Laplacian  $\Delta_\chi$  with respect to  $\chi$ . It is not difficult to show that there is a positive constant  $c$  such that  $\lambda_i(\chi) \geq c$  for  $i \geq 1$ . We also have  $\lambda_0(\chi) \geq 0$  and the equality holds if and only if  $\chi = \mathbf{1}$ ,  $\lambda_0(\chi)$  depend smoothly on  $\chi$  and the first derivative with respect to  $\chi$  at  $\mathbf{1}$  is zero. The essential point is to show the hessian of  $\lambda_0(\chi)$  at  $\chi = \mathbf{1}$  is positive definite.

To prove this, we use perturbation arguments as follows: Although the usual setting of the perturbation theory is that operators defined on the fixed domain are varied, our situation here are converse that defining domains  $H_\chi$  are varied but the operator  $\Delta_\chi$  are fixed since it is the restriction of the Laplacian  $\Delta_X$  on  $L^2(X)$  to  $H_\chi$ .

To reduce usual setting of the perturbation theory, we construct a canonical section  $s_\chi$  of the line bundle  $E_\chi$  and identify  $L^2(M)$  with  $L^2(E_\chi)$  by the correspondence

$$L^2(M) \ni f \longleftrightarrow f s_\chi \in L^2(E_\chi) \simeq H_\chi.$$

Associated with this correspondence, we have a unitary equivalence between the twisted operator  $L_\chi := s_\chi^{-1} \circ \Delta_\chi \circ s_\chi$  acting on  $L^2(M)$  and the twisted Laplacian  $\Delta_\chi$  acting on  $L^2(E_\chi) \simeq H_\chi$ .

To construct  $s_\chi$ , by de Rham-Hodge Theorem, we can take a harmonic 1-form  $\omega$  satisfying

$$\chi([\gamma]) := \chi_\omega([\gamma]) := \exp(2\pi\sqrt{-1} \int_\gamma \omega)$$

for a closed curve  $\gamma$  and its homology class  $[\gamma] \in H_1(M, \mathbb{Z})$ .

Since  $\chi$  is a homomorphism from  $\Gamma \simeq H_1(M, \mathbb{Z}) \simeq \pi_1/[\pi_1, \pi_1]$  to an abelian group  $U(1)$  where  $\pi_1/[\pi_1, \pi_1]$  is the abelianization of the fundamental group  $\pi_1 = \pi_1(M)$  of  $M$ , we can lift  $\chi$  to a character  $\tilde{\chi} : \pi_1(M) \rightarrow U(1)$  canonically. Then, taking a lift  $\tilde{\omega}$  of  $\omega$  to the universal covering  $\tilde{M}$  of  $M$ , we define a function  $\tilde{s}_{\tilde{\omega}}$  on  $\tilde{M}$  by

$$\tilde{s}_{\tilde{\omega}}(p) = \exp(2\pi\sqrt{-1} \int_{p_0}^p \tilde{\omega}) \quad (2.4)$$

for some reference point  $p_0 \in \tilde{M}$ . Note that this is well defined since the line integral  $\int_{p_0}^p \tilde{\omega}$  does not depend on the choice of a curve from  $p_0$  to  $p$  by the homotopy invariance property due to the fact that  $\tilde{\omega}$  is a closed form. Since  $\tilde{s}_{\tilde{\omega}}(\gamma p) = \chi_{\omega}(\gamma) \tilde{s}_{\tilde{\omega}}(p)$ , this function can be identified with a section  $s_{\omega} = s_{\chi_{\omega}}$  of  $L^2(E_{\chi})$ .

Then  $L_{\chi} = s_{\chi_{\omega}}^{-1} \circ \Delta_{\chi_{\omega}} \circ s_{\chi_{\omega}}$  can be written as

$$L_{\chi} f = \Delta_M f - 4\pi \langle \omega, df \rangle + 4\pi^2 |\omega|^2 f$$

where  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  are the inner product and the norm of cotangent space induced from the Riemannian metric of  $M$  respectively.

From this, we can compute the hessian  $\text{Hess}_0 \lambda_0$  of  $\lambda_0(\chi)$  at  $\chi = \chi_{\omega} = \mathbf{1}$  (i.e.  $\omega = 0$ ) which is written as

$$\text{Hess}_0 \lambda_0(\omega, \omega) = \frac{8\pi^2}{\text{vol}(M)} \int_M |\omega|^2 dv_g$$

where  $\text{vol}(M)$  is the Riemannian volume and  $dv_g$  is the Riemannian measure of  $M$ .

By the Morse lemma, we can take a local coordinates  $(U, (x^1, x^2, \dots, x^d))$  near the trivial character  $\mathbf{1}$  on  $\hat{\Gamma} \simeq (U(1))^d$  such that

$$\lambda_0(\chi) = \lambda_0(x^1, x^2, \dots, x^d) = \sum_{k=1}^d (x^k)^2.$$

Then we get the conclusion by the following computation

$$\begin{aligned} k_X(t, p, q) &= \int_{\hat{\Gamma}} k_{\chi}(t, p, q) d\chi \\ &= \int_{\hat{\Gamma}} \sum_{n=0}^{\infty} e^{-\lambda_n(\chi)t} \varphi_{n,\chi}(p) \varphi_{n,\chi}(q) d\chi \\ &= \int_U e^{-\sum_{i=1}^d (x^i)^2 t} dx^1 dx^2 \dots dx^d + \text{error term} \\ &\sim \prod_{i=1}^d \left\{ \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} e^{-y^2} dy^i \right\} \\ &= \frac{C}{t^{d/2}}. \end{aligned}$$

where  $U$  is a neighborhood of  $\mathbf{1}$  in  $\hat{\Gamma}$ .

For generalization of the above arguments to nilpotent groups, we point out arising difficulties and give a strategy to overcome them in the case when  $\Gamma$  is the three dimensional discrete Heisenberg group  $\text{Heis}_3(\mathbb{Z})$ .

As for the first step in previous arguments, we have used the irreducible decomposition formula of the right regular representation  $R$  of abelian groups. However, if  $\Gamma$  is non abelian nilpotent, its representation theory is not so called of type I. In this case, although

there exists abstract decomposition formula of the regular representation, it is not unique and the unitary dual is a wild space. As a conclusion, it seems that there is no computable formula in practice for full unitary dual  $\hat{\Gamma}$ .

In the case that  $\Gamma = \text{Heis}_3(\mathbb{Z})$ , if we restrict to finite dimensional unitary representations, we can use the Plancherel formula due to Pytlik [34]. In this formula, the Plancherel measure  $\mu$  is finitely additive and supported on finite dimensional irreducible representations. This formula is indeed useful. However dimensions of representations appearing in the formula are varied, which makes some difficulties to apply perturbation arguments similar to the second step in the above. To overcome this point, we relate these finite dimensional representations to infinite dimensional irreducible unitary representations which are called the Schrödinger representations of the Heisenberg Lie group  $\text{Heis}_3(\mathbb{R})$ . These representations  $\rho_h$  are parametrized by  $h \in \mathbb{R} \setminus \{0\}$  and their representation spaces are common  $L^2(\mathbb{R})$  for all  $h \in \mathbb{R} \setminus \{0\}$ .

Let us explain briefly the above relation. Finite dimensional irreducible unitary representations  $\rho_{\text{fin},x}$  of  $\Gamma$  are parametrized by  $x = (x_1, x_2, x_3) \in \hat{X} := [0, 1] \times [0, 1] \times (\mathbb{Q} \cap [0, 1])$  and the above Plancherel measure  $\mu$  is the product measure of the Lebesgue measure  $m$  on the interval  $[0, 1]$  for first and second factors and the finitely additive measure  $\tilde{m}$  on the third factor  $\mathbb{Q} \cap [0, 1]$  determined by

$$\tilde{m}(\mathbb{Q} \cap [a, b]) = b - a.$$

The representations  $\rho_{\text{fin},x}$  are essentially determined from  $x_3$  which is, in some sense, a discrete version of the Schrödinger representation. If we express  $x_3$  as irreducible fraction  $p/q$ , then the dimension of  $\rho_{\text{fin},x}$  is  $q$  and the role of  $x_1, x_2$  for  $\rho_{\text{fin},x}, x = (x_1, x_2, x_3)$  are abelian perturbations of  $\rho_{\text{fin},(0,0,x_3)}$ . The point of our arguments is the fluctuation by  $x_1, x_2$  is like  $O(1/q)$  which is independent to the numerator  $p$  of  $x_3 = p/q$ . Moreover, the subset of  $X$  consisting  $x = (x_1, x_2, x_3)$  whose third component  $x_3 = p/q$  has the denominator smaller than a fixed constant is a null set with respect to the above Plancherel measure  $\mu$ . To relate the Schrödinger representation  $\rho_h$ , we note that if we restrict  $\rho_h$  to the discrete subgroup  $\Gamma$  when the parameter  $h$  is a rational number  $p/q$ , the restriction  $\rho_h|_{\Gamma}$  to  $\Gamma$  of  $\rho_h$  can be decomposed as a direct integral

$$\rho_h|_{\Gamma} = \int_0^1 \int_0^1 \rho_{\text{fin},(x_1, \{qx_3x_2\}, x_3)} dm(x_1) dm(x_2),$$

where  $\{a\}$  denotes the fractional part  $a - [a]$ . We view this decomposition formula as the approximation formula of the left hand side by  $\rho_{\text{fin},x}$ . It means that the left hand side is the “mean” of the integrand in the right hand side and the fluctuation of the integrand are  $O(1/q)$ . Moreover, for any  $h$  near 0, it can be approximated by rational numbers with arbitrarily large denominators, which implies an infinite dimensional representation  $\rho_h$  can be approximated by finite dimensional representations  $\rho_{\text{fin},x}$  by arbitrarily

orders. This means that for a mathematical justification of the arguments (e.g. problem of convergence), we use the above Plancherel formula which include only finite dimensional representations but for some formal computation of the eigenvalues, we can use the Schrödinger representation since it varies smoothly in  $h$ . Namely we can exchange freely finite dimensional representations and infinite dimensional representations. This is what we call the extension of the Bloch-Floquet theory to the Heisenberg group.

As in the second step of previous arguments for abelian groups, we need to investigate the behavior of the eigenvalues of Laplacian  $\Delta_{\rho_{\text{fin},x}}$  twisted by  $\rho_{\text{fin},x}$  near the trivial representation  $\mathbf{1}$ . By the first step, It can reduced to formal computation replacing  $\Delta_{\rho_{\text{fin},x}}$  to the twisted Laplacian  $\Delta_{\rho_h}$  associated to  $\rho_h$ , which are, in spirits, same as abelian case. However, in technically, it is necessary to modify the arguments. One of them is to replace the line integral  $\int_{p_0}^p \tilde{\omega}$  in (2.4) by the Lie integral of the Lie algebra valued 1-form (cf. [7]), which is equivalent to Chen's iterated integrals [10]. Other part of the actual computation is somewhat complicated since we need to carry out in infinite rank vector bundles. For example, in this procedure, the hessian  $\text{Hess}_0 \lambda_0$  should be replaced by the quadratic forms associated with the harmonic oscillator acting on the fiber  $L^2(\mathbb{R})$  of  $E_{\rho_h}(L^2(\mathbb{R}))$ . However, the arguments are essentially a hybrid of the so called Schrödinger method in physics literatures (cf. [35],[36]) and a computation carried in [23] in abelian case.

For the asymptotic formula for counting prime geodesics in Theorem 1.6, we use the Selberg trace formula as in the proof of Theorem 1.4 for abelian groups. The spectral side of the trace formula can be treated similarly as the above heat kernel asymptotics. In the geometric side, the assumption that  $\alpha$  is a conjugacy class of a central element of  $\Gamma$  in Theorem 1.4 are used in the fact that the following formula holds for such  $\alpha$

$$\text{tr}(\rho_{\text{fin},x}(f)\rho_{\text{fin},x}(\alpha)) = \text{tr}(\rho_{\text{fin},x}(f))\text{tr}(\rho_{\text{fin},x}(\alpha))$$

where  $\text{tr}(A) = \frac{1}{\dim A} \text{Tr}(A)$  is the normalized trace of the matrix  $A$  which is normalization of the usual matrix trace  $\text{Tr}(A)$  satisfying  $\text{tr}(I) = 1$  for the identity matrix  $I$  and  $\rho_{\text{fin},x}(f)$  is the Fourier transform

$$\rho_{\text{fin},x}(f) = \int_{\Gamma} f(\sigma)\rho_{\text{fin},x}(\sigma)d\sigma = \sum_{\sigma \in \Gamma} f(\sigma)\rho_{\text{fin},x}(\sigma).$$

Then, similar arguments to abelian case in [31], which is essentially due to Dirichlet, implies Theorem 1.6.

In the case when  $\alpha$  is coming from non-central elements  $\sigma$ , then we can identify the conjugacy class  $\alpha = [\sigma]$  with  $\Pi(\alpha)$  for the canonical projection  $\Pi : \Gamma \rightarrow \Gamma/[\Gamma, \Gamma] \simeq \mathbb{Z}^2$ . Therefore we can apply same arguments as in abelian case.

As for our mathematical justification of the semiclassical expansion (1.4) and (1.3), we first point out that analysis of the Harper operator corresponds the case when  $M$  is

the graph  $\mathcal{G} = (V, E)$  such that  $V$  has only one element  $p$  and  $E$  has two loops  $u, v$  at  $p$ , namely if we realize  $G$  as a one dimensional complex,  $G$  is the bouquet of two  $S^1$ 's, i.e. one point sum of two circles. If  $\Gamma = \text{Heis}_3(\mathbb{Z})$ , then the  $\Gamma$  covering space  $X$  is the Cayley graph of  $\Gamma$  with generators  $\{u, v\}$ ,

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

which is described by the following figure:

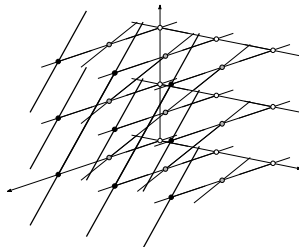


Figure 2: created by Satoshi Ishiwata

Moreover, the square lattice  $\mathbb{Z}^2$  appears as an intermediate covering  $\varpi_1 : X \rightarrow \mathbb{Z}^2$  and  $\varpi_2 : \mathbb{Z}^2 \rightarrow M$  of  $\pi = \varpi_2 \circ \varpi_1 : X \rightarrow M$ . Moreover for  $\theta \in \mathbb{Q}$ , the Harper operator  $H_\theta$  is a lift (with a shift by  $4 \times \text{identity}$ ) of the twist discrete Laplacian  $\Delta_{\rho_{\text{fin},x}}$  on  $M$  with  $x = (0, 0, \theta/2\pi) \in \hat{X}$ . Then, the semiclassical expansion (1.4) and 1.3) is essentially same as the above investigation of the behavior of the eigenvalue of the twisted Laplacian  $\Delta_{\rho_{\text{fin},x}}$  and  $\Delta_{\rho_h}$ . Namely our mathematical justification comes from a simple application of the above freeness of the interchange between  $\rho_h$  and  $\rho_{\text{fin},x}$ .

We think our method has some advantage to the original proof in [15] since we can justify approximations of the differential operator  $\text{fracd}\sqrt{-1}du$  and the multiplication operator  $u$  separately while the original case is an approximation conjointly, i.e. by the harmonic oscillator  $\mathcal{H} = -\frac{d^2}{du^2} + u^2$ .

## References

- [1] T. ADACHI AND T. SUNADA, Twisted Perron-Frobenius theorem and L -functions. J. Funct. Anal. 71 (1987), no. 1, 1–46.
- [2] G. ALEXOPOULOS, Convolution powers on discrete groups of polynomial volume growth. Harmonic analysis and number theory (Montreal, PQ, 1996), 31–57, CMS Conf. Proc., 21, Amer. Math. Soc., 1997

- [3] N. ANANTHARAMAN, Precise counting results for closed orbits of Anosov flows. *Ann. Sci. École Norm. Sup. (4)* 33 (2000), no. 1, 33–56.
- [4] N. ANANTHARAMAN, Counting geodesics which are optimal in homology. *Ergodic Theory Dynam. Systems* 23 (2003), no. 2, 353–388.
- [5] A. AVILA AND S. JITOMIRSKAYA, The Ten Martini Problem. *Ann. of Math. (2)* 170 (2009), no. 1, 303–342
- [6] M. BABILOT AND F. LEDRAPPIER, Lalley’s theorem on periodic orbits of hyperbolic flows. *Ergodic Theory Dynam. Systems* 18 (1998), no. 1, 17–39.
- [7] D. BENARDETE AND J. MITCHELL, Asymptotic homotopy cycles for flows and  $\Pi_1$  de Rham theory. *Trans. Amer. Math. Soc.* 338 (1993), no. 2, 495–535.
- [8] A. BOULANGER, A stochastic approach to counting problems, arXiv:1811.02899.
- [9] A. BOULANGER, Heat kernel, Selberg’s pre-trace formula and counting problems, arXiv:1902.06580
- [10] K. T. CHEN, Iterated path integrals, *Bull. Amer. Math. Soc.* 83 (1977), 831–878
- [11] L. J. CORWIN AND F. P. GREENLEAF, Representations of nilpotent Lie groups and their applications. Part I. Basic theory and examples. *Cambridge Studies in Advanced Mathematics*, 18. Cambridge University Press, Cambridge, 1990.
- [12] K. R. DAVIDSON,  $C^*$ -algebras by example. *Fields Institute Monographs*, 6. Amer. Math. Soc., 1996.
- [13] A. GRIGOR’YAN, Heat kernel and Analysis on manifolds, *Studies in Advanced Mathematics*, 47, Amer. Math. Soc., 2009.
- [14] Y. GUIVARC’H, Application d’un théorème limite local à la transience et à la récurrence de marches de Markov. *Théorie du potentiel (Orsay, 1983)*, 301–332, *Lecture Notes in Math.*, 1096, Springer, 1984.
- [15] B. HELFFER AND J. SJÖSTRAND, Analyse semi-classique pour l’équation de Harper. II: comportement semi-classique près d’un rationnel *Mém. Soc. Math. France (N.S.)* No. 40 (1990).
- [16] H. HUBER, Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen, *Math. Ann.* 138 (1959), 1–26.

- [17] S. ISHIWATA, A central limit theorem on a covering graph with a transformation group of polynomial growth. *J. Math. Soc. Japan* 55 (2003), no. 3, 837–853.
- [18] J. JACOD, Théorème de renouvellement et classification pour les chaînes semi-markoviennes. *Ann. Inst. H. Poincaré Sect. B (N.S.)* 7 (1971), 83–129.
- [19] A. KATSUDA AND T. SUNADA, Homology and closed geodesics in a compact Riemann surface. *Amer. J. Math.* 110 (1988), no. 1, 145–155.
- [20] A. KATSUDA AND T. SUNADA, Closed orbits in homology classes. *Publ. IHES Math. No.* 71 (1990), 5–32.
- [21] M. KOTANI, A note on asymptotic expansions for closed geodesics in homology classes. *Math. Ann.* 320 (2001), no. 3, 507–529.
- [22] M. KOTANI, T. SHIRAI AND T. SUNADA Asymptotic behavior of the transition probability of a random walk on an infinite graph. *J. Funct. Anal.* 159 (1998), no. 2, 664–689.
- [23] M. KOTANI AND T. SUNADA, Albanese maps and off diagonal long time asymptotics for the heat kernel. *Comm. Math. Phys.* 209 (2000), no. 3, 633–670.
- [24] M. KOTANI AND T. SUNADA, Large deviation and the tangent cone at infinity of a crystal lattice. *Math. Z.* 254 (2006), no. 4, 837–870.
- [25] A. KRÁMLI AND D. SZÁSZ, Random walks with internal degrees of freedom. I. Local limit theorems. *Z. Wahrsch. Verw. Gebiete* 63 (1983), no. 1, 85–95.
- [26] S. LALLEY, Closed geodesics in homology classes on surfaces of variable negative curvature, *Duke Math. J.* 58 (1989), 795–821.
- [27] G. A. MARGULIS, Certain applications of ergodic theory to the investigation of manifolds of negative curvature *Funkcional. Anal. i Priložen.* 3 1969 no. 4, 89–90.
- [28] G. A. MARGULIS, On some aspects of the theory of Anosov systems. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows. Translated from the Russian by Valentina Vladimirovna Szulikowska. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.
- [29] W. PARRY AND M. POLLICOTT, An analogue of the prime number theorem for closed orbits of Axiom A flows. *Ann. of Math. (2)* 118 (1983), no. 3, 573–591.
- [30] W. PARRY AND M. POLLICOTT, The Chebotarov theorem for Galois coverings of Axiom A flows. *Ergodic Theory Dynam. Systems* 6 (1986), no. 1, 133–148.

- [31] R. PHILLIPS AND P. SARNAK, Geodesics in homology classes. *Duke Math. J.* 55 (1987), no. 2, 287–297.
- [32] M. POLLICOTT, Homology and closed geodesics in a compact negatively curved surface. *Amer. J. Math.* 113 (1991), no. 3, 379–385.
- [33] M. POLLICOTT AND R. SHARP, Asymptotic expansions for closed orbits in homology classes. *Geom. Dedicata* 87 (2001), no. 1–3, 123–160.
- [34] T. PYTLIK, A Plancherel measure for the discrete Heisenberg group. *Colloq. Math.* 42 (1979), 355–359.
- [35] R. RAMMAL AND J. BELLISSARD, An Algebraic Semi-Classical Approach to Bloch Electrons in a Magnetic Field, *J. de Phys. France*, 51, 1803–1830, (1990).
- [36] J. SAKURAI AND J. NAPOLITANO, *Modern quantum mechanics* (2nd ed.), Addison Wesley, 2011
- [37] L. SALOFF-COSTE, The heat kernel and its estimates. *Probabilistic approach to geometry*, 405–436, *Adv. Stud. Pure Math.*, 57, Math. Soc. Japan, Tokyo, 2010.
- [38] A. SELBERG, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)* 20 (1956), 47–87.
- [39] YA. G. SINAI, Random Walks and Some Problems Concerning Lorentz Gas, *Sūrikaiseikikenkyūsho Kōkyūroku No.* 439 (1981), 6–17
- [40] I. SATIJA, *Butterfly in the Quantum World*, Morgan and Claypool Publishers, 2016
- [41] M. SPIVAK, *A comprehensive introduction to differential geometry. Vol. II. Second edition.* Publish or Perish, Inc., Wilmington, Del., 1979.
- [42] T. SUNADA, Unitary representations of fundamental groups and the spectrum of twisted Laplacians. *Topology* 28 (1989), no. 2, 125–132.
- [43] M. WILKINSON, An example of phase holonomy in WKB theory, *J. Phys. A. Math. Gen.* 17 (1984), 3459–3476.



DEPARTMENT OF MATHEMATICS

KYUSHU UNIVERSITY

FUKUOKA 980-8578

JAPAN

*E-mail address:* [katsuda@math.kyushu-u.ac.jp](mailto:katsuda@math.kyushu-u.ac.jp)