Complex symmetric operators and isotropic vectors in Banach spaces via linear functionals

by

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Abstract

We generalize the concept of complex symmetric operators to Banach spaces via their dual spaces. With this extension we show the existence of isotropic vectors on Banach spaces whose dimension is at least two and the relation between the simplicity of an eigenvalue and the non-existence of its isotropic eigenvectors. All this work is based on [1].

1 Conjugations and isotropic vectors

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ with its inner product $\langle \cdot, \cdot \rangle$ which is linear on the first and antilinear on the second. A conjugation C on $\mathcal{H}$ is an antilinear isometric involution on $\mathcal{H}$. In other words, for any vectors x and y in $\mathcal{H}$, the equality

$$\langle Cx, Cy \rangle = \langle y, x \rangle \tag{1}$$

holds. In particular, $C^2 = I$ where I is the identity operator on $\mathcal{H}$.

Recently, Chō and Tanahashi [2] extended the concept of conjugations to a complex Banach space $\mathcal{X}$ (with its norm $\| \cdot \|$) as antilinear involutions whose operator norms are at most 1. More precisely, any operator $C : \mathcal{X} \to \mathcal{X}$ is called a conjugation on $\mathcal{X}$, if C satisfies

$$C^2 = I, \quad \|C\| \leq 1, \quad C(x+y) = Cx + Cy, \quad C(\lambda x) = \overline{\lambda} Cx, \tag{2}$$

where x and y are in $\mathcal{X}$ and λ is a complex number. Note that (2) implies that $\|Cx\| = \|x\|$ for all $x \in \mathcal{X}$. They showed that the definition above is the same as the usual one for conjugations in a complex Hilbert space $\mathcal{H}$.

A vector $x \in \mathcal{H}$ is called *isotropic* (with respect to a conjugation C) if $\langle Cx, x \rangle = 0$. For example, when $\mathcal{H} = \mathbb{C}^2$ and C is the canonical conjugation on $\mathbb{C}^2$, i.e., $C(x, y) = (\overline{x}, \overline{y})$, any vector of the form $(a, \pm ia)$ with any complex number a becomes isotropic with respect to the canonical conjugation C.

Due to the lack of an inner product, we cannot directly use the definition of isotropic vectors above in $\mathcal{H}$. In order to extend them to a complex Banach space $\mathcal{X}$ let us first observe that $\langle Cx, x \rangle = 0 \Longleftrightarrow Cx \perp x$.

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As a generalization of orthogonality to $\mathcal{X}$, we adapt the orthogonality in the Birkhoff-James sense on [2, 5, 7]. For given two elements x and y in $\mathcal{X}$, x is called *orthogonal to y in the Birkhoff-James sense*, in short $x \perp_B y$, if

$$\|x + \lambda y\| \geq \|x\| \quad \text{for all } \lambda \in \mathbb{C}. \tag{3}$$

Note that, by the Hahn-Banach theorem, $x \perp_B y$ if and only if there is a norm-one linear functional f in $\mathcal{X}^*$ such that $f(x) = \|x\|$ and $f(y) = 0$.

Based on our discussion, isotropic vectors on $\mathcal{X}$ are defined as follows:

Definition 1.1. A vector x in $\mathcal{X}$ is called isotropic, if $Cx \perp_B x$, or equivalently, for all $\lambda \in \mathbb{C}$,

$$\|Cx + \lambda x\| \geq \|Cx\|.$$

Since C is norm-preserving, $Cx \perp_B x$ implies that $x \perp_B Cx$ (which is notable since the orthogonality in the Birkhoff-James sense is not symmetric). Again, by the Hahn-Banach theorem, $x \perp_B Cx$ if and only if there is f in $\mathcal{X}^*$ such that $\|f\| = 1, f(x) = \|x\|$ and $f(Cx) = 0$.

Let us now focus on the existence of isotropic vectors. On these isotropic vectors on $\mathcal{H}$, Garcia et al. [3, 4] showed two well-known results: the existence of isotropic vectors in any subspace of $\mathcal{H}$ whose dimension is at least two (which is Lemma 4.11 in [3] or Lemma 1.2 in this article) and the relation between the simplicity of an eigenvalue λ of T and the non-existence of its isotropic eigenvectors for λ when T is complex symmetric (which is Theorem 4.12 in [3]). Here an eigenvalue λ is called *simple* if its algebraic multiplicity is 1, or equivalently $\dim \ker(\lambda I - T) = 1$.

Lemma 1.2. [3, Lemma 4.11] If $C : \mathcal{H} \to \mathcal{H}$ is a conjugation, then every subspace whose dimension is at least two contains isotropic vectors for the bilinear form $\langle \cdot, C \cdot \rangle$.

In order to show our affirmative answer on Banach-space version of the lemma above, introduce the Gram-Schmidt process on $\mathcal{X}$ via linear functionals in [6]. Note that this result is an easy application of Hahn-Banach theorem.

Lemma 1.3. [6, Proposition 2.1] Let x_1, x_2 be vectors of $\mathcal{X}$. Then the following are equivalent:
(i) $\{x_1, x_2\}$ is linearly independent.
(ii) There exist functionals f_1, f_2 in $\mathcal{X}^*$ such that $f_1(x_1) \neq 0$, $f_2(x_1) = 0$, and $f_2(x_2) \neq 0$.

Based on [1], we now see the existence of isotropic vectors on $\mathcal{X}$ via a similar idea of Lemma 1.2.

Theorem 1.4. [1, Theorem 2.4] If $C : \mathcal{X} \to \mathcal{X}$ is a conjugation, then every subspace whose dimension is at least two contains isotropic vectors for the conjugation C.

Therefore, we have the following corollary.

Corollary 1.5. Let $T \in \mathcal{L}(\mathcal{X})$ be complex symmetric and let λ be an eigenvalue of T. If T has no isotropic eigenvectors for λ, then $\dim\ker(T-\lambda) = 1$.

Proof. Suppose not, that is, let $\dim\ker(T-\lambda) \geq 2$. Then Theorem 1.4 indicates that the subspace $\ker(T-\lambda)$ contains an isotropic eigenvector which is impossible. $\square$

We now consider a natural dual element related to a given conjugation C. Let C be a conjugation on $\mathcal{X}$. Define a dual conjugation C^* of C defined by

$$\big(C^*(f)\big)(x) := \overline{f(Cx)}. \tag{4}$$

Then C^* is a conjugation on the dual space $\mathcal{X}^*$ of $\mathcal{X}$. See [2] for more details. Then we have the following proposition:

Proposition 1.6. Let x be a unit vector of a Banach space $\mathcal{X}$ and let C be a conjugation on $\mathcal{X}$. Let f be a linear functional on $\mathcal{X}$ such that $\|f\| = f(x) = \|x\| = 1$ and $f(Cx) = 0$. If $x \perp_B Cx$, then $f \perp_B C^*(f)$. In particular, x is isotropic.

Note that this natural dual conjugation C^* of C in (4) will be discussed and used in Section 2.

2 C-symmetric operators, shortly CSOs

In this section, we would like to extend the concept of complex symmetric operators or more generally C-symmetric operators from $\mathcal{H}$ to $\mathcal{X}$, based on [1]. Let us first recall that, in $\mathcal{H}$, the C-symmetry of a bounded linear operator T on $\mathcal{H}$ is expressed by $CT^*C = T$. In particular, with the aid of (1), for any vectors x and y in $\mathcal{H}$,

$$\langle CTy, x\rangle = \langle CTCCy, x\rangle = \langle T^*Cy, x\rangle = \langle Cy, Tx\rangle = \langle CTx, y\rangle. \tag{5}$$

However, due to the lack of an inner product again, we cannot use the condition to define complex symmetric operators on $\mathcal{X}$. Instead, we interpret (5) via linear functionals in the dual space $\mathcal{H}^*$ of $\mathcal{H}$ in order to generalize C-symmetric operators to $\mathcal{X}$.

Proposition 2.1. For a bounded linear operator T on $\mathcal{H}$, $CT^*C = T$ if and only if for every pair of unit vectors, say x and y, there are two functionals f and g in $\mathcal{H}^*$ such that $||f|| = ||g|| = 1$, $f(x) = g(y) = 1$, $f(Cy) = g(Cx)$ and $f(CTy) = g(CTx)$.

Note that, in Proposition 2.1, we used the fact that, for given a unit vector $x \in \mathcal{H}$, there is a *unique* norm-one linear functional f with $f(x) = 1$, which is just $f(\cdot) = \langle \cdot, x\rangle$. Based on this observation, we would like to extend the C-symmetry of linear operators to Banach spaces and in particular to those satisfying the property above, that is, for each unit vector x, there is a unique norm-one functional f with $f(x) = 1$. It is then well-known that Phelps [8] and Taylor [9] characterize the condition for such a property on $\mathcal{X}$ as follows:

Theorem 2.2 ([8, 9])**.** If $\mathcal{X}^*$ is strictly convex, then every subspace, say $\mathcal{M}$, has a unique norm-preserving extension of continuous linear functionals on $\mathcal{M}$. The converse holds when $\mathcal{X}$ is reflexive.

Recall that a normed space $\mathcal{X}$ is called *strictly convex*, if $x, y \in S$ and $x \neq y$ imply $\|\lambda x + (1-\lambda)y\| < 1$ for $0 < \lambda < 1$, where S is the unit sphere $\{x \in \mathcal{X} \mid \|x\| = 1\}$ in $\mathcal{X}$, i.e., S contains no line segments. It is worth to mention that, due to the theorem above, when $\mathcal{X}^*$ is strictly convex, then for given a unit vector x there exists the unique norm-one functional f in $\mathcal{X}^*$ satisfying $f(x) = 1$.

A typical example of such a complex Banach space in our mind is $L^p(\mathbb{T}, d\mu)$ with $p > 1$ but $p \neq 2$, where $\mathbb{T}$ is the unit circle and $d\mu$ is a finite measure on $\mathbb{T}$. More precisely, by Riesz representation theorem for $L^p(\mathbb{T})$ says that, for a given norm-one linear functional ϕ, there exists unique $g \in L^q(\mathbb{T}, d\theta)$ (essentially up to measure $d\mu$) with $1/p + 1/q = 1$, such that $||g||_q = ||\phi|| = 1$ and

$$\phi(f) = \int_{\mathbb{T}} fg\, d\mu, \quad f \in L^p(\mathbb{T}, d\mu).$$

Based on Proposition 2.1 and the observation above, we extend the C-symmetry of linear operators to complex Banach spaces $\mathcal{X}$ via their dual spaces as follows:

Definition 2.3. Let $T \in \mathcal{L}(\mathcal{X})$ and let C be a conjugation on $\mathcal{X}$. Then T is called C-symmetric in the sense of $\mathcal{X}^*$, if, for every pair of unit vectors x and y in $\mathcal{X}$, there exist two norm-one functionals f and g in $\mathcal{X}^*$ such that $f(x) = g(y) = 1$, $f(Cy) = g(Cx)$ and $f(CTy) = g(CTx)$.

Here the functionals are sometimes denoted by $f_{x,T}$ and $g_{y,T}$, especially when we emphasize the dependence of these functionals on x, y and T. However, in many cases when they are unambiguous, let us ignore these subscripts for convenience.

Remarks. 1. Even though the definition above seems a very natural extension of the C-symmetry of linear operators from $\mathcal{H}$ to $\mathcal{X}$ via its dual space, it would not be clear if the identity operator I of $\mathcal{X}$ would be C-symmetric in the sense of $\mathcal{X}^*$.
2. It is worthy mentioning that, if there exists $T \in \mathcal{L}(\mathcal{X})$ which is C-symmetric in the sense of $\mathcal{X}^*$, then so is the identity operator I. The functionals f and g can be chosen for both T and I at the same time (i.e., $f_{x,T} = f_{x,I}$ and $g_{y,T} = g_{y,I}$).
3. Even though T and S are C-symmetric in the sense of $\mathcal{X}^*$, we do not know if $f_{x,T} = f_{x,S}$ nor if $f(CTy) = g(CTx)$ implies $f(TCy) = g(TCx)$ in general.
4. On general complex Banach spaces, Definition 2.3 seems weaker. One of the reasons is that $\langle y, x\rangle = \langle Cx, Cy\rangle$ in $\mathcal{H}$, which can be re-written to

$$f_x(y) := \langle y, x\rangle = \langle Cx, Cy\rangle =: g_{Cy}(Cx)$$

via linear functionals in $\mathcal{X}^*$. This would, however, be hard to achieve on general Banach spaces.

To overcome the weakness of Definition 2.3 (which was addressed on 3 and 4 in Remarks above), in most cases we will assume that $\mathcal{X}^*$ is strictly convex. Then, due to Theorem 2.2, these functionals $f_{x,T}$ and $g_{y,T}$ are uniquely chosen independently of C-symmetric operators T in the sense of $\mathcal{X}^*$. Besides this, the following proposition expresses several useful properties when $\mathcal{X}^*$ is strictly convex:

Proposition 2.4. [1, Proposition 3.4] Let $\mathcal{X}^*$ be strictly convex and let the identity operator I on $\mathcal{X}$ be C-symmetric in the sense of $\mathcal{X}^*$. Let $T \in \mathcal{L}(\mathcal{X})$. For two given unit vectors x and y denote by f and g two functionals satisfying $f(x) = g(y) = 1$ and $f(Cy) = g(Cx)$. Then the following statements hold:
(i) The (unique) norm-one functional whose function value at Cx is 1 is C^*f. (Note that $\|Cx\| = 1$ again.)
(ii) $f(y) = \overline{g(x)}$ and $f(y) = (C^*g)(Cx)$.
(iii) If T is C-symmetric in the sense of $\mathcal{X}^*$, then $f(TCy) = g(TCx)$.
(iv) T is C-symmetric in the sense of $\mathcal{X}^*$ if and only if CTC is also C-symmetric in the sense of $\mathcal{X}^*$.

In any Hilbert space these are very easy to show. For example, (ii) is true due to $f(y) = \langle y, x\rangle = \overline{\langle x, y\rangle} = \overline{g(x)}$ and $f(y) = \langle y, x\rangle = \langle Cx, Cy\rangle = (C^*g)(Cx)$.

From now on let us see some properties of C-symmetric operators in the sense of $\mathcal{X}^*$.

Proposition 2.5. *If $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric in the sense of $\mathcal{X}^*$, then the following properties hold;*
(i) *λT and $T - \lambda I$ are C-symmetric in the sense of $\mathcal{X}^*$ for any complex number λ (and vice versa).*
(ii) *$T|_{\mathcal{M}}$ is C-symmetric in the sense of $\mathcal{X}^*$ for any nonzero subspace $\mathcal{M}$ of $\mathcal{X}$.*

Note that in the proposition above we do not assume that $\mathcal{X}^*$ is strictly convex.

Theorem 2.6. [1, Theorem 3.6] Let $\mathcal{X}^*$ be strictly convex and let $T, S \in \mathcal{L}(\mathcal{X})$ be C-symmetric in the sense of $\mathcal{X}^*$. Then the following statements hold.
(i) $T + S$ is C-symmetric in the sense of $\mathcal{X}^*$.
(ii) If $TS = ST$, then TS is C-symmetric in the sense of $\mathcal{X}^*$.

As an easy consequence of the theorem above, we have the following:

Corollary 2.7. Let $\mathcal{X}^*$ be strictly convex. If $T \in \mathcal{L}(\mathcal{X})$ is C-symmetric in the sense of $\mathcal{X}^*$, then so is $p(T)$ for any polynomial $p(z)$.

Proposition 2.8. Let $\mathcal{X}^*$ be strictly convex and let $T \in \mathcal{L}(\mathcal{X})$ be C-symmetric in the sense of $\mathcal{X}^*$. If T^{-1} exists, then it is also C-symmetric in the sense of $\mathcal{X}^*$.

It is worth to mention that this proof is similar to that of (ii) in Theorem 2.6.

Proposition 2.9. Let $\mathcal{X}^*$ be strictly convex. If the sequence $\{T_n\}$ of C-symmetric operators in the sense of $\mathcal{X}^*$ converges to $T \in \mathcal{L}(\mathcal{X})$ in the strong operator norm topology, then T is also C-symmetric in the sense of $\mathcal{X}^*$.

Proof. With a similar argument in the proof of Theorem 2.6 it suffices to show that $f(CTy) = g(CTx)$ for given two unit vectors x and y. Due to the continuity of f and g and the assumption on the convergence above, it follows that

$$f(CTy) = \lim_{n\to\infty} f(CT_ny) = \lim_{n\to\infty} g(CT_nx) = g(CTx).$$

□

Let us extend Theorem 4.12 in [3] on $\mathcal{H}$ which shows the relation between the simplicity of an eigenvalue and the non-existence of its isotropic eigenvectors. For this, we generalize so-called C-projections to $\mathcal{X}$.

Definition 2.10. A projection P is called a C-projection in the sense of $\mathcal{X}^*$ if it is C-symmetric in the sense of $\mathcal{X}^*$.

Let λ be an isolated eigenvalue of a bounded linear operator T on $\mathcal{X}$. Then the *Riesz idempotent* E of T with respect to λ is defined by

$$E := \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (z-T)^{-1}dz$$

where $\mathbb{D}$ is an open disk centered at λ with $\overline{\mathbb{D}} \cup \sigma(T) = \{\lambda\}$. It is then well-known that $E^2 = E, ET = TE, \sigma(T|_{\mathrm{ran}(E)}) = \{\lambda\}$ and $\ker(T-\lambda I) \subset \mathrm{ran}\,(E)$. When T is C-symmetric in the sense of $\mathcal{X}^*$, Theorem 2.6 and Proposition 2.9 imply that E is a C-projection in the sense of $\mathcal{X}^*$, i.e., for given two unit vectors x and y, there exist two norm-one linear functionals f and g satisfying $f(x) = g(y) = 1$, $f(Cy) = g(Cx)$ and $f(CEy) = g(CEx)$.

The following theorem is a generalization of Theorem 4.12 in [3] with the C-symmetry in the sense of $\mathcal{X}^*$:

Theorem 2.11. [1, Theorem 3.11] Let $T \in \mathcal{L}(\mathcal{X})$ be C-symmetric in the sense of $\mathcal{X}^*$ and λ an isolated eigenvalue of T. If T has no isotropic eigenvectors for λ, then λ is simple. Moreover, the converse is true if $\mathcal{X}^*$ is strictly convex.

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