A GENERALIZATION OF THE ALUTHGE TRANSFORMATION IN THE VIEWPOINT OF OPERATOR MEANS

TAKEAKI YAMAZAKI

ABSTRACT. The Aluthge transformation is generalized in the viewpoint of the axiom of operator means by using double operator integrals. It includes the mean transformation which is defined by S. H. Lee, W. Y. Lee and Yoon. Next we shall give some properties of it. Especially, we shall show that the n-th iteration of mean transformation of an invertible matrix converges to a normal matrix.

1. INTRODUCTION

Let $B(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. Let $T = U|T| \in B(\mathcal{H})$ be the polar decomposition of T. The **Aluthge transformation** $\Delta(T)$ of T is defined in [1] as follows.

$$(1.1) \qquad \Delta(T) := |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}.$$

Several properties of the Aluthge transformation has been studied, for example, (i) $\sigma(\Delta(T)) = \sigma(T)$, where $\sigma(T)$ is the spectrum of $T \in B(\mathcal{H})$ [14], (ii) $\Delta(T)$ has a non-trivial invariant subspace if and only if T does so [15], and (iii) if T is semi-hyponormal (i.e., $|T^*| \leq |T|$), then $\Delta(T)$ is hyponormal (i.e., $\Delta(T)\Delta(T)^* \leq \Delta(T)^*\Delta(T)$) [1], where "$\leq$" means the Loewner partial order. By considering the Loewner-Heinz inequality, hyponormality of an operator implies semi-hyponormality, but the converse implication does not hold in general. Hence the Aluthge transformation $\Delta(T)$ may have better properties than T. Recently, a related new operator transformation has been defined in [18] and studied in [7, 19] which is called the **mean transformation** $\hat{T}$ of T. The definition is

$$\hat{T} := \frac{U|T| + |T|U}{2}.$$

The aim of this report is to unify these operator transformations in the viewpoint of operator means, and give some properties.

An operator mean is a binary operation on positive semi-definite operators. It was defined by Kubo-Ando as follows. Let $B(\mathcal{H})^+$ and $B(\mathcal{H})^{++}$ be the sets of positive semi-definite and positive invertible operators, respectively.

Definition 1 (Operator mean, [17])**.** *Let* $\mathfrak{M} : B(\mathcal{H})^+ \times B(\mathcal{H})^+ \to B(\mathcal{H})^+$ *be a binary operation. If* $\mathfrak{M}$ *satisfies the following four conditions, then* $\mathfrak{M}$ *is called an* **operator mean**.

(1) *If* $A \leq C$ *and* $B \leq D$, *then* $\mathfrak{M}(A, B) \leq \mathfrak{M}(C, D)$,
(2) $X^*\mathfrak{M}(A, B)X \leq \mathfrak{M}(X^*AX, X^*BX)$ *for all* $X \in B(\mathcal{H})$,

(3) $A_n \searrow A$ and $B_n \searrow B$ imply $\mathfrak{M}(A_n, B_n) \searrow \mathfrak{M}(A, B)$ in the strong operator topology,

(4) $\mathfrak{M}(I, I) = I$, where I means the identity operator in $B(\mathcal{H})$.

To get a concrete formula of an operator mean, the following relation is very important. Let f be a real-valued function defined on an interval $J \subseteq (0, \infty)$. Then f is said to be **operator monotone** if $A \leq B$ for self-adjoint operators $A, B \in B(\mathcal{H})$ whose spectra are contained in J, then $f(A) \leq f(B)$, where $f(A)$ and $f(B)$ are defined by the functional calculus.

Theorem A ([17]). *Let $\mathfrak{M}$ be an operator mean. Then there exists an operator monotone function f on $(0, \infty)$ such that $f(1) = 1$ and*

$$\mathfrak{M}(A, B) = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$

for all $A \in B(\mathcal{H})^{++}$ and $B \in B(\mathcal{H})^{+}$.

If $A \in B(\mathcal{H})^{+}$, we can obtain $\mathfrak{M}(A, B) = \lim_{\varepsilon \searrow 0} \mathfrak{M}(A + \varepsilon I, B)$ because $A + \varepsilon I \in B(\mathcal{H})^{++}$ for $\varepsilon > 0$ and Definition 1 (3). The function f is called a **representing function** of an operator mean $\mathfrak{M}$. Throughout this report, we note $\mathfrak{M}_f$ for an operator mean with a representing function f. In this case, $f'(1) = \lambda \in [0, 1]$ holds (cf. [11]), and we sometimes call $\mathfrak{M}_f$ a λ-weighted operator mean. Moreover, if $\lambda = f'(1) \in (0, 1)$, then

(1.2) $$[1 - \lambda + \lambda x^{-1}]^{-1} \leq f(x) \leq 1 - \lambda + \lambda x$$

holds for all $x > 0$ (cf. [20]).

Typical examples of operator means are the λ-weighted geometric and λ-weighted power means. These representing functions are $f(x) = x^\lambda$ and $f(x) = [1 - \lambda + \lambda x^t]^{\frac{1}{t}}$, respectively, where $\lambda \in [0, 1]$ and $t \in [-1, 1]$ (in the case $t = 0$, we consider $t \to 0$). The weighted power mean interpolates arithmetic, geometric and harmonic means by putting $t = 1, 0, -1$, respectively.

The aim of this report is to apply operator means to generalize the Aluthge transformation. To do this, firstly, we shall explain how to generalize the Aluthge transformation in the matrices case in Section 2. Then to extend the discussion in Section 2 into Hilbert space operators, we will introduce the double operator integrals in Section 3. We show that all operator means can be applied to double operator integrals. In Section 4, we shall generalize the Aluthge transformation respect to an arbitrary operator mean via double operator integrals. In Section 5, we shall consider n-th iterated generalized Aluthge transformation. We divide this discussion into finite and infinite Hilbert space cases. More precisely, we shall show that n-th iterated mean transformation of every invertible matrix converges to a normal matrix, and show that there is a weighted unilateral shift on ℓ^2 such that n-th iterated generalized Aluthge transformation does not converge in a week operator topology.

2. A GENERALIZATION OF THE ALUTHGE TRANSFORMATION IN THE MATRICES CASE

In this section, we shall generalize the Aluthge transformation in the matrices case which is a motivation of this report. Let $\mathcal{M}_m$ be a set of all m–by–m matrices. It is known that $\mathcal{M}_m$ is a Hilbert space with an inner product $\langle A, B \rangle := \mathrm{trace}\, AB^*$. For $A, B \in \mathcal{M}_m$, let $\mathbb{L}_A$ and $\mathbb{R}_B$ be bounded linear operators on $\mathcal{M}_m$ defined as follows:

$$\mathbb{L}_A(X) = AX \quad \text{and} \quad \mathbb{R}_B(X) = XB$$

for $X \in \mathcal{M}_m$. They are called the **left and right multiplications**, respectively. If $A, B \in \mathcal{M}_m$ are positive semi-definite (resp. positive invertible) matrices, then $\mathbb{L}_A$ and $\mathbb{R}_B$ are positive semi-definite (resp. positive invertible) operators on $\mathcal{M}_m$, too. It is easy to see that $\mathbb{L}_A$ and $\mathbb{R}_B$ are commuting on the product, i.e.,

$$\mathbb{L}_A \mathbb{R}_B(X) = \mathbb{R}_B \mathbb{L}_A(X) = AXB$$

holds for all $X \in \mathcal{M}_m$. Moreover, for each Hermitian $A \in \mathcal{M}_m$, $f(\mathbb{L}_A)(X) = \mathbb{L}_{f(A)}(X)$ (resp. $f(\mathbb{R}_A)(X) = \mathbb{R}_{f(A)}(X)$) holds for all analytic functions f if $f(A)$ is defined. Hence we can consider operator means of $\mathbb{L}_A$ and $\mathbb{R}_B$. For example, the arithmetic mean $\mathfrak{A}$ of $\mathbb{L}_A$ and $\mathbb{R}_B$ is computed by

$$\mathfrak{A}(\mathbb{L}_A, \mathbb{R}_B)(X) = \frac{\mathbb{L}_A + \mathbb{R}_B}{2}(X) = \frac{AX + XB}{2}.$$

Theorem 2.1. *Let $A, B \in \mathcal{M}_m$ be positive invertible. Then for any operator mean $\mathfrak{M}$, there exists a positive probability measure $d\mu$ on $[0, 1]$ such that*

$$\mathfrak{M}(\mathbb{L}_A, \mathbb{R}_B)(X) = \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)A^{-1}} X e^{-x\lambda B^{-1}} dx \right) d\mu(\lambda)$$

for all $X \in \mathcal{M}_m$.

To prove Theorem 2.1, we shall use the following result.

Theorem B ([12], [4, Theorem VII.2.3]). *Let A and B be operators whose spectra are contained in the open right half-plane and left half-plane, respectively. Then the solution of the equation $AX - XB = Y$ can be expressed as*

$$X = \int_0^\infty e^{-xA} Y e^{xB} dx.$$

Proof of Theorem 2.1. Firstly, we shall show the case of λ-weighted harmonic mean $\mathfrak{M}_f$ for $\lambda \in [0, 1]$, i.e., a representing function f of $\mathfrak{M}_f$ is

$$f(x) = [1 - \lambda + \lambda x^{-1}]^{-1}.$$

In this case, the harmonic mean of $\mathbb{L}_A$ and $\mathbb{R}_B$ on $\mathcal{M}_m$ is

$$\mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X) = [(1 - \lambda)\mathbb{L}_A^{-1} + \lambda \mathbb{R}_B^{-1}]^{-1}(X).$$

We notice that if A and B are positive invertible matrices, then $\mathbb{L}_A$ and $\mathbb{R}_B$ are positive invertible, and hence the above formula is well-defined. Put $Y := \mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X)$. Then it is a solution of a matrix equation

$$[(1 - \lambda)\mathbb{L}_A^{-1} + \lambda \mathbb{R}_B^{-1}](Y) = X.$$

Thus for $X \in \mathcal{M}_m$, we just have to give a solution Y of the following matrix equation

$$(1 - \lambda)A^{-1}Y + \lambda Y B^{-1} = X,$$

and it is equivalent to

$$\{(1 - \lambda)A^{-1}\}Y - Y(-\lambda B^{-1}) = X.$$

By Theorem B, we have $Y = \mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X)$ as follows.

(2.1)
$$\mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X) = \int_0^\infty e^{-x(1-\lambda)A^{-1}} X e^{-x\lambda B^{-1}} dx$$
$$= \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)A^{-1}} X e^{-x\lambda B^{-1}} dx \right) d\mu(\lambda),$$

where $\mu([0,1]) := \delta_{\{\lambda\}}([0,1])$, the Dirac delta supported on $\{\lambda\}$.

Next we shall show an arbitrary operator mean case. Let $\mathfrak{M}_f$ be an operator mean. Then it is known (cf. [11]) that there exists a positive probability measure $d\mu$ on $[0,1]$ such that

$$(2.2) \qquad f(x) = \int_0^1 [1 - \lambda + \lambda x^{-1}]^{-1} d\mu(\lambda).$$

Hence we have

$$\mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X) = \mathbb{L}_A f(\mathbb{L}_A^{-1} \mathbb{R}_B)(X)$$

$$= \int_0^1 [(1 - \lambda)\mathbb{L}_A^{-1} + \lambda \mathbb{R}_B^{-1}]^{-1} d\mu(\lambda)(X)$$

$$= \int_0^1 [(1 - \lambda)\mathbb{L}_A^{-1} + \lambda \mathbb{R}_B^{-1}]^{-1}(X) d\mu(\lambda)$$

$$= \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)A^{-1}} X e^{-x\lambda B^{-1}} dx \right) d\mu(\lambda).$$

$\square$

Now we shall give another formula of $\mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X)$.

Theorem 2.2. *Let $A, B \in \mathcal{M}_m$ be positive invertible with the spectral decompositions $A = \sum_{i=1}^m s_i P_i$ and $B = \sum_{j=1}^m t_j Q_j$, respectively. Then for each operator mean $\mathfrak{M}_f$,*

$$(2.3) \qquad \mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X) = \sum_{i,j=1}^m \mathcal{P}_f(s_i, t_j) P_i X Q_j,$$

where the perspective $\mathcal{P}_f$ of f is defined by $\mathcal{P}_f(s, t) := s f\left(\frac{t}{s}\right)$.

Proof. Let $A = \sum_{i=1}^m s_i P_i$ and $B = \sum_{j=1}^m t_j Q_j$ be spectral decompositions of A and B, respectively. For a representing function f on $(0, \infty)$ of an operator mean $\mathfrak{M}_f$, by (2.2), the perspective $\mathcal{P}_f$ of f is given by

$$\mathcal{P}_f(s, t) = \int_0^1 [(1 - \lambda)s^{-1} + \lambda t^{-1}]^{-1} d\mu(\lambda)$$

for $s, t > 0$. Then by Theorem 2.1, we have

$$\mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X)$$

$$= \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)A^{-1}} X e^{-x\lambda B^{-1}} dx \right) d\mu(\lambda)$$

$$= \int_0^1 \left\{ \int_0^\infty \left(\sum_{i=1}^m e^{-x(1-\lambda)s_i^{-1}} P_i \right) X \left(\sum_{j=1}^m e^{-x\lambda t_j^{-1}} Q_j \right) dx \right\} d\mu(\lambda)$$

$$= \sum_{i,j=1}^m \int_0^1 \left(\int_0^\infty e^{-\{(1-\lambda)s_i^{-1} + \lambda t_j^{-1}\}x} dx \right) d\mu(\lambda) P_i X Q_j$$

$$= \sum_{i,j=1}^m \int_0^1 [(1 - \lambda)s_i^{-1} + \lambda t_j^{-1}]^{-1} d\mu(\lambda) P_i X Q_j = \sum_{i,j=1}^m \mathcal{P}_f(s_i, t_j) P_i X Q_j.$$

$\square$

If $A, B \in B(\mathcal{H})^+$, then for each $\varepsilon > 0$, $A_\varepsilon := A + \varepsilon I$ and $B_\varepsilon := B + \varepsilon I$ are both positive invertible. Then we can define $\mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X)$ by

$$\mathfrak{M}_f(\mathbb{L}_A, \mathbb{R}_B)(X) = \lim_{\varepsilon \searrow 0} \mathfrak{M}_f(\mathbb{L}_{A_\varepsilon}, \mathbb{R}_{B_\varepsilon})(X).$$

Definition 2 (Generalization of the Aluthge transformation). *Let $T = U|T| \in \mathcal{M}_m$ be the polar decomposition with the spectral decomposition $|T| = \sum_{i=1}^n s_i P_i$ of $|T|$. For an operator mean $\mathfrak{M}_f$, a **generalization of the Aluthge transformation** $\Delta_{\mathfrak{M}_f}(T)$ of T respect to an operator mean $\mathfrak{M}_f$ is defined by*

$$\Delta_{\mathfrak{M}_f}(T) := \sum_{i,j=1}^m \mathcal{P}_f(s_i, s_j) P_i U P_j.$$

By using Theorem 2.2, we have another formula of a generalized Aluthge transformation.

Corollary 2.3. *Let $T = U|T| \in \mathcal{M}_m$ be the polar decomposition such that $|T|$ is invertible. For an operator mean $\mathfrak{M}$, there exists a positive probability measure $d\mu$ on $[0,1]$ such that*

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)|T|^{-1}} U e^{-x\lambda|T|^{-1}} dx \right) d\mu(\lambda).$$

Example 1. *Let $T \in \mathcal{M}_m$ with the polar decomposition $T = U|T|$ and the spectral decomposition $|T| = \sum_{i=1}^m s_i P_i$, and let $\mathfrak{M}$ be an operator mean. Then we have the following examples of $\Delta_{\mathfrak{M}}(T)$. Note that $\sum_{i=1}^m P_i = U^*U$.*

(1) *Mean transformation [18].*
   Let $\mathfrak{M}_f$ be the λ- weighted arithmetic mean, i.e., the representing function of $\mathfrak{M}_f$ is $f(t) = 1 - \lambda + \lambda t$. Then

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j=1}^m [(1-\lambda)s_i + \lambda s_j] P_i U P_j$$

$$= (1-\lambda) \sum_{i,j=1}^m s_i P_i U P_j + \lambda \sum_{i,j=1}^m s_j P_i U P_j$$

$$= (1-\lambda) \left(\sum_{i,=1}^m s_i P_i \right) U \left(\sum_{j=1}^m P_j \right)$$

$$+ \lambda \left(\sum_{i=1}^m P_i \right) U \left(\sum_{j=1}^m s_j P_j \right)$$

$$= (1-\lambda)|T|U + \lambda U^*U|T|.$$

   Especially, if U is an isometry, then $\Delta_{\mathfrak{M}_f}(T) = \hat{T}$ (i.e., the weighted mean transform).

(2) *Generalized Aluthge transformation [14].*
   Let $\mathfrak{M}_f$ be the λ-weighted geometric mean, i.e., the representing function of $\mathfrak{M}_f$

is $f(t) = t^\lambda$. *Then*

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j=1}^{m} s_i^{1-\lambda} s_j^\lambda P_i U P_j$$

$$= \left(\sum_{i=1}^{m} s_i^{1-\lambda} P_i\right) U \left(\sum_{j=1}^{m} t_j^\lambda P_j\right) = |T|^{1-\lambda} U |T|^\lambda.$$

From Corollary 2.3, we have a basic property of a generalization of the Aluthge transformation.

Theorem 2.4. *Let* $T \in \mathcal{M}_m$. *Let* $\mathfrak{M}_f$ *be an operator mean satisfying* $f'(1) \in (0,1)$. *Then* $\Delta_{\mathfrak{M}_f}(T) = T$ *if and only if* T *is normal.*

Proof. For $\varepsilon > 0$, $|T|_\varepsilon := |T| + \varepsilon I$ is positive invertible, and $|T|_\varepsilon \searrow |T|$ as $\varepsilon \searrow 0$. By considering this fact, we may assume that $|T|$ is invertible. Let $T = U|T|$ be the polar decomposition of T. Then it is known that T is normal if and only if $U|T| = |T|U$. Hence if T is normal, then by Corollary 2.3,

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)|T|^{-1}} U e^{-x\lambda|T|^{-1}} dx\right) d\mu(\lambda)$$

$$= U \int_0^1 \left(\int_0^\infty e^{-x|T|^{-1}} dx\right) d\mu(\lambda) = T.$$

Let $|T| = \sum_{i=1}^{m} s_i P_i$ be the spectral decomposition. Assume that $\Delta_{\mathfrak{M}_f}(T) = T$ holds. Then we have

$$\sum_{i,j=1}^{m} \mathcal{P}_f(s_i, s_j) P_i U P_j = \Delta_{\mathfrak{M}_f}(T) = T = \sum_{j=1}^{m} s_j U P_j.$$

By multiplying P_i and P_j from the left and right sides, respectively,

$$\mathcal{P}_f(s_i, s_j) P_i U P_j = s_j P_i U P_j,$$

since P_i are orthogonal projections. By $f'(1) \in (0,1)$ and an inequality (1.2), $\mathcal{P}_f(s_i, s_j) \neq s_j$ holds for $i \neq j$. Then $P_i U P_j = 0$ holds for $i \neq j$. Hence we have

$$\Delta_{\mathfrak{M}_f}(T) = \sum_{i,j=1}^{n} \mathcal{P}_f(s_i, s_j) P_i U P_j$$

$$= \sum_{i=1}^{n} \mathcal{P}_f(s_i, s_i) P_i U P_i$$

$$= \sum_{i=1}^{n} s_i P_i U P_i$$

$$= \sum_{i,j=1}^{n} \sqrt{s_i s_j} P_i U P_j = \Delta(T) \quad \text{(Aluthge transformation)}.$$

Therefore T is normal since $T = \Delta(T)$ [15, Proposition 1.10]. $\square$

3. Double operator integrals

Although $\mathbb{L}_A$ and $\mathbb{R}_B$ can be defined on $B(\mathcal{H})$, we cannot consider operator means of $\mathbb{L}_A$ and $\mathbb{R}_B$, since $B(\mathcal{H})$ is not a Hilbert space. To discuss similar argument in $B(\mathcal{H})$, we shall use the double operator integrals. The double operator integrals was first appeared in [9]. Then it was developed by Birman and Solomyak in [5] and Peller in [21, 22] (nice surveys are [6, 13]). Let $A, B \in \mathcal{B}(\mathcal{H})^+$ with the spectral decompositions

$$A = \int_{\sigma(A)} s dE_s \text{ and } B = \int_{\sigma(B)} t dF_t.$$

Let λ (resp. μ) be a finite positive measure on an interval $\sigma(A)$ (resp. $\sigma(B)$) equivalent (in the absolute continuity sense) to dE_s (resp. dF_t). Let $\varphi \in L^\infty(\sigma(A) \times \sigma(B); \lambda \times \mu)$. For $X \in \mathcal{B}(\mathcal{H})$, the **double operator integrals** is given by

$$\Phi_{A,B,\varphi}(X) := \int_{\sigma(A)} \int_{\sigma(B)} \varphi(s, t) dE_s X dF_t.$$

If $X \in \mathcal{C}_2(\mathcal{H})$ (Hilbert-Schmidt class), then $\Phi_{A,B,\varphi}(X) \in \mathcal{C}_2(\mathcal{H})$ because $\mathcal{C}_2(\mathcal{H})$ is a Hilbert space, and $\Phi_{A,B,\varphi}$ can be defined by the similar way to Theorem 2.2. To extend this into $X \in B(\mathcal{H})$, we shall consider Schur multiplier as follows:

Definition 3 (Schur multiplier, (cf. [13])). *When $\Phi_{A,B,\varphi}(= \Phi_{A,B,\varphi}|_{\mathcal{C}_1(\mathcal{H})}) : X \mapsto \Phi_{A,B,\varphi}(X)$ gives rise to a bounded transformation on the ideal $\mathcal{C}_1(\mathcal{H})(\subset \mathcal{C}_2(\mathcal{H}))$ of trace class operators, $\varphi(s, t)$ is called a **Schur multiplier** (relative to the pair (A, B)).*

The double operator integrals can be extended to $\mathcal{B}(\mathcal{H})$ by making use of the duality $\mathcal{B}(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$ via

$$(X, Y) \in \mathcal{C}_1(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \mapsto \text{trace}(XY^*) \in \mathbb{C}.$$

This proof is introduced in [13].

Theorem C ([13, 21, 22]). *For $\varphi \in L^\infty(\sigma(A) \times \sigma(B); \lambda \times \mu)$, the following conditions are all equivalent:*

(i) *φ is a Schur multiplier;*

(ii) *whenever a measurable function $k : \sigma(A) \times \sigma(B) \to \mathbb{C}$ is the kernel of a trace class operator $L^2(\sigma(A); \lambda) \to L^2(\sigma(B); \mu)$, so is the product $\varphi(s, t)k(s, t)$;*

(iii) *one can find a finite measure space (Ω, σ') and functions $\alpha \in L^\infty(\sigma(A) \times \Omega; \lambda \times \sigma')$, $\beta \in L^\infty(\sigma(B) \times \Omega; \mu \times \sigma')$ such that*

(3.1) $$\varphi(s, t) = \int_\Omega \alpha(s, x)\beta(t, x) d\sigma'(x)$$

for all $s \in \sigma(A)$, $t \in \sigma(B)$;

(iv) *one can find a measure space (Ω, σ') and measurable functions α, β on $\sigma(A) \times \Omega$, $\sigma(B) \times \Omega$ respectively such that the above (3.1) holds and*

$$\left\| \int_\Omega |\alpha(\cdot, x)|^2 d\sigma'(x) \right\|_{L^\infty(\lambda)} \left\| \int_\Omega |\beta(\cdot, x)|^2 d\sigma'(x) \right\|_{L^\infty(\lambda)} < \infty.$$

An important result of this report is to give a guarantee the perspective of representing functions f of all operator means are Schur multiplier.

Theorem 3.1. *Let f be a representing function of an operator mean. Then the perspective $\mathcal{P}_f$ of f is a Schur multiplier.*

7

Proof. By (2.2), a perspective $\mathcal{P}_f$ of f is given as follows:

$$\mathcal{P}_f(s,t) = sf\left(\frac{t}{s}\right) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1} d\mu(\lambda).$$

By elementary computation, we have

$$\mathcal{P}_f(s,t) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1} d\mu(\lambda)$$
$$= \int_0^1 \left(\int_0^\infty e^{-(1-\lambda)s^{-1}x} e^{-\lambda t^{-1}x} dx\right) d\mu(\lambda).$$

Putting $\alpha(s,x,\lambda) = e^{-(1-\lambda)s^{-1}x}$ and $\beta(t,x,\lambda) = e^{-\lambda t^{-1}x}$. Then $\mathcal{P}_f$ can be represented as the form of (3.1), and it is a Schur multiplier. $\square$

4. A generalization of the Aluthge transformation in the operators case

In this section, we shall give a definition of a generalized Aluthge transformation by using the double operator integrals which is introduced in the previous section.

Definition 4 (Generalization of the Aluthge transformation). *Let $T = U|T| \in B(\mathcal{H})$ with the spectral decomposition $|T| = \int_{\sigma(|T|)} s dE_s$. For an operator mean $\mathfrak{M}_f$, a **generalization of the Aluthge transformation** $\Delta_{\mathfrak{M}_f}(T)$ of T respect to an operator mean $\mathfrak{M}_f$ is defined by*

$$\Delta_{\mathfrak{M}_f}(T) := \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s U dE_t.$$

By Theorem 3.1, $\mathcal{P}_f$ is the Schur multiplier. Hence the above double operator integrals is well-defined. As in the similar discussion of Example 1, we obtain concrete forms of generalizations of the Aluthge transformation. The following theorem is an extension of Theorem 2.1.

Theorem 4.1. *Let $T = U|T| \in B(\mathcal{H})$, s.t., $|T| \in B(\mathcal{H})^{++}$. For each operator mean $\mathfrak{M}$, there exists a positive probability measure $d\mu$ on $[0,1]$ such that*

$$\Delta_{\mathfrak{M}}(T) = \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)|T|^{-1}} U e^{-x\lambda|T|^{-1}} dx\right) d\mu(\lambda).$$

Proof. For an operator mean $\mathfrak{M}_f$, there exists a positive probability measure $d\mu$ on $[0,1]$ such that

$$\mathcal{P}_f(s,t) = \int_0^1 [(1-\lambda)s^{-1} + \lambda t^{-1}]^{-1} d\mu(\lambda)$$
$$= \int_0^1 \left(\int_0^\infty e^{\{-x(1-\lambda)s^{-1}\}} e^{\{-x\lambda t^{-1}\}} dx\right) d\mu(\lambda).$$

Then by using Fubini-Tonelli's theorem, we have

$$\Delta_{\mathfrak{M}_f}(T) = \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s U dE_t$$

$$= \int_{\sigma(|T|)} \int_{\sigma(|T|)} \left\{ \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)s^{-1}} e^{-x\lambda t^{-1}} dx \right) d\mu(\lambda) \right\} dE_s U dE_t$$

$$= \int_0^1 \left\{ \int_0^\infty \left(\int_{\sigma(|T|)} e^{-x(1-\lambda)s^{-1}} dE_s \right) U \left(\int_{\sigma(|T|)} e^{-x\lambda t^{-1}} dE_t \right) dx \right\} d\mu(\lambda)$$

$$= \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)|T|^{-1}} U e^{-x\lambda|T|^{-1}} dx \right) d\mu(\lambda).$$

Hence the proof is completed. $\qquad\square$

Proposition 4.2. *Let $T \in B(\mathcal{H})$ with the polar decomposition $T = U|T|$. Then for any operator mean $\mathfrak{M}_f$ and $\alpha \in \mathbb{C}$, the following statements hold.*

(1) $\Delta_{\mathfrak{M}_f}(\alpha T) = \alpha \Delta_{\mathfrak{M}_f}(T)$,
(2) $\Delta_{\mathfrak{M}_f}(V^*TV) = V^* \Delta_{\mathfrak{M}_f}(T)V$ *for all unitary V,*
(3) $\Delta_{\mathfrak{M}_f}(T) - \alpha I = \Phi_{|T|,|T|,\mathcal{P}_f}(U - \alpha|T|^{-1})$ *if $|T| \in B(\mathcal{H})^{++}$,*
(4) $\|\Delta_{\mathfrak{M}_f}(T)\| \le \|T\|$, *where $\|\cdot\|$ means the spectral norm on $B(\mathcal{H})$.*

Proof. Let $|T| = \int_{\sigma(|T|)} s dE_s$ be the spectral decomposition.

(1) Let $\alpha = re^{i\theta}$ $(r \ge 0)$ be a polar form of $\alpha \in \mathbb{C}$. Then $\alpha T = (e^{i\theta}U)(\alpha|T|)$ is a polar decomposition of αT moreover, $r|T| = \int_{\sigma(|T|)} rs dE_s$. Hence we have

$$\Delta_{\mathfrak{M}_f}(\alpha T) = \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(rs, rt) dE_s(e^{i\theta}U) dE_t$$

$$= \int_{\sigma(|T|)} \int_{\sigma(|T|)} re^{i\theta} \mathcal{P}_f(s,t) dE_s U dE_t = \alpha \Delta_{\mathfrak{M}_f}(T).$$

(2) Let V be unitary. Then $V^*TV = V^*UV \cdot V^*|T|V$ is the polar decomposition. Then we have

$$\Delta_{\mathfrak{M}_f}(V^*TV) = \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) d(V^*E_s V) V^*UV d(V^*E_t V)$$

$$= V^* \left(\int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s U dE_t \right) V = V^* \Delta_{\mathfrak{M}_f}(T)V.$$

9

(3) Since dE_s is an orthogonal projection measure,

$$\Phi_{|T|,|T|,\mathcal{P}_f}(U - \alpha|T|^{-1}) = \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s (U - \alpha|T|^{-1}) dE_t$$

$$= \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s U dE_t$$

$$- \alpha \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s |T|^{-1} dE_t$$

$$= \Delta_{\mathfrak{M}_f}(T) - \alpha \int_{\sigma(|T|)} \int_{\sigma(|T|)} \mathcal{P}_f(s,t) dE_s dE_t |T|^{-1}$$

$$= \Delta_{\mathfrak{M}_f}(T) - \alpha \int_{\sigma(|T|)} \mathcal{P}_f(s,s) dE_s |T|^{-1}$$

$$= \Delta_{\mathfrak{M}_f}(T) - \alpha \int_{\sigma(|T|)} s dE_s |T|^{-1} = \Delta_{\mathfrak{M}_f}(T) - \alpha I.$$

(4) By Theorem 4.1, there exists a positive probability measure $d\mu$ on $[0,1]$ such that

$$\Delta_{\mathfrak{M}_f}(T) = \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)|T|^{-1}} U e^{-x\lambda|T|^{-1}} dx \right) d\mu(\lambda).$$

Since $g(x) = e^{-cx^{-1}}$ is an increasing function on $x > 0$ and for $c > 0$, we have $e^{-cA^{-1}} \leq e^{-c\|A\|^{-1}} I$. Hence we have

$$\|\Delta_{\mathfrak{M}_f}(T)\| \leq \int_0^1 \left(\int_0^\infty \|e^{-x(1-\lambda)|T|^{-1}} U e^{-x\lambda|T|^{-1}}\| dx \right) d\mu(\lambda)$$

$$\leq \int_0^1 \left(\int_0^\infty e^{-x(1-\lambda)\|T\|^{-1}} e^{-x\lambda\|T\|^{-1}} dx \right) d\mu(\lambda) = \|T\|,$$

where the last equality follows from $\int_0^1 d\mu(\lambda) = 1$. $\qquad\square$

Proposition 4.3. *Let $T \in B(\mathcal{H})$, and let $\mathfrak{M}$ be an operator mean. If $T \in C_1(\mathcal{H})$, then*

$$\mathrm{trace}(\Delta_{\mathfrak{M}}(T)) = \mathrm{trace}(T).$$

Proof. It follows from Theorem 4.1. $\qquad\square$

We notice that it is known that if $\mathfrak{M}$ is a weighted geometric mean (i.e., $\Delta_{\mathfrak{M}}(T)$ is the generalized Aluthge transformation), then $\sigma(\Delta_{\mathfrak{M}}(T)) = \sigma(T)$ holds for all $T \in B(\mathcal{H})$ [14]. However, there is a counterexample for this equation when $\mathfrak{M}$ is an arithmetic mean [18].

5. Iteration

In this section, we shall consider iteration of $\Delta_{\mathfrak{M}}$. For each natural number n, define $\Delta_{\mathfrak{M}}^n(T) := \Delta_{\mathfrak{M}}(\Delta_{\mathfrak{M}}^{n-1}(T))$ and $\Delta_{\mathfrak{M}}^0(T) := T$ for $T \in B(\mathcal{H})$. It is known that iteration of the Aluthge transformation (i.e., $\mathfrak{M}$ is a geometric mean) has been considered by many authors, for example, (i) a sequence $\{\Delta_{\mathfrak{M}}^n(T)\}_{n=0}^\infty$ converges to a normal matrix if $T \in \mathcal{M}_m$ [2, 3], (ii) there exists an operator $T \in B(\mathcal{H})$ such that a sequence $\{\Delta_{\mathfrak{M}}^n(T)\}_{n=0}^\infty$ does not converge in the week operator topology [8], (iii) for each $T \in B(\mathcal{H})$, $\lim_{n\to\infty} \|\Delta_{\mathfrak{M}}^n(T)\| = r(T)$, where $r(T)$ is the spectral radius of T [23].

Theorem 5.1. *Let $T \in \mathcal{M}_m$ be invertible with the polar decomposition $T = U|T|$. Let $\mathfrak{A}$ be a non-weighted arithmetic mean. Then a sequence $\{\Delta_{\mathfrak{A}}^n(T)\}$ converges to a normal matrix N such that $\mathrm{trace}(T) = \mathrm{trace}(N)$ and $\mathrm{trace}(|T|) = \mathrm{trace}(|N|)$.*

The iteration of mean transformation has been considered in [7]. However, in [7], the authors have considered only rank one operators or they required some conditions. We notice that if T is invertible, then $\Delta_{\mathfrak{A}}(T)$ coincides with the mean transformation $\hat{T}$ of T (see Example 1).

To prove Theorem 5.1, we use a useful formula which is shown in [7].

Theorem D ([7]). *Let* $T \in \mathcal{B}(\mathcal{H})$ *and suppose that* $\ker(T^*) \subseteq \ker(T)$. *Let* $T = U|T|$ *be the canonical polar decomposition of* T, *and let* $n \in \mathbb{N}$. *Then the polar decomposition of* $\hat{T}^{(n)}$ *is*

$$\hat{T}^{(n)} = U \cdot \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (U^*)^k |T| U^k,$$

where $\hat{T}^{(n)} := (T^{(\hat{n-1})})$ *and* $\hat{T}^{(1)} := \hat{T}$.

Proof of Theorem 5.1. Since T is invertible, $\Delta_{\mathfrak{A}}(T)$ coincides with the mean transformation $\hat{T}$. Moreover $\Delta_{\mathfrak{A}}(T) = U \cdot \frac{1}{2}(|T| + U^*|T|U)$ is the polar decomposition by Theorem D. We notice that since T is invertible, U and $|T|$ are invertible, and hence $\Delta_{\mathfrak{A}}(T)$ is invertible. Therefore for each natural number n, $\Delta_{\mathfrak{A}}^{(n)}(T) = \hat{T}^{(n)}$ holds, and it is invertible. By Theorem D, we only prove that $\{|\Delta_{\mathfrak{A}}^n(T)|\}$ converges to a positive matrix. Since U is unitary, we can diagonalize U as $U = V^*DV$ with a unitary matrix V and $D = \mathrm{diag}(e^{\theta_1\sqrt{-1}}, ..., e^{\theta_m\sqrt{-1}})$. Then by Proposition 4.2 (2), $\Delta_{\mathfrak{A}}^n(T) = V^*\Delta_{\mathfrak{A}}^n(DV|T|V^*)V$, and

$$|\Delta_{\mathfrak{A}}^n(T)| = V^*|\Delta_{\mathfrak{A}}^n(DV|T|V^*)|V$$

$$= V^* \left\{ \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} (D^*)^k V|T|V^*D^k \right\} V.$$

Let $V|T|V^* = P$. Then $D^{*k}V|T|V^*D^k = [e^{k(\theta_j-\theta_i)\sqrt{-1}}] \circ P$, where $[e^{k(\theta_j-\theta_i)\sqrt{-1}}] \in \mathcal{M}_m$ with the (i,j)-entry $e^{k(\theta_j-\theta_i)\sqrt{-1}}$, and $\circ$ means the Hadamard product. Hence

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} D^{*k}V|T|V^*D^k = \frac{1}{2^n} \left[\sum_{k=0}^{n} \binom{n}{k} e^{k(\theta_j-\theta_i)\sqrt{-1}} \right] \circ P$$

$$= \frac{1}{2^n} \left[(1 + e^{(\theta_j-\theta_i)\sqrt{-1}})^n \right] \circ P$$

$$\text{(by the binomial expansion)}$$

$$= \left[\left(\frac{1 + e^{(\theta_j-\theta_i)\sqrt{-1}}}{2} \right)^n \right] \circ P.$$

We notice that $\left| \frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2} \right| < 1$ if $\theta_j \neq \theta_i + 2k\pi$ for all integers k, and $\frac{1+e^{(\theta_j-\theta_i)\sqrt{-1}}}{2} = 1$ if $\theta_j = \theta_i + 2k\pi$ for some integers k. Hence there exists a matrix P_0 such that

$$\lim_{n\to\infty} |\Delta_{\mathfrak{A}}^n(T)| = \lim_{n\to\infty} V^* \left(\left[\left(\frac{1 + e^{(\theta_j-\theta_i)\sqrt{-1}}}{2} \right)^n \right] \circ P \right) V = P_0,$$

and we have $\lim_{n\to\infty} \Delta_{\mathfrak{A}}^n(T) = \lim_{n\to\infty} U|\Delta_{\mathfrak{A}}^n(T)| = UP_0 = N$. Moreover since $UP_0 = \Delta_{\mathfrak{A}}(UP_0) = \frac{UP_0+P_0U}{2}$, $UP_0 = P_0U$ holds, i.e., $N = UP_0$ is a normal matrix.

By Proposition 4.3 and the above, we have $\mathrm{trace}(T) = \mathrm{trace}(\Delta_{\mathfrak{A}}^n(T)) = \mathrm{trace}(N)$ and $\mathrm{trace}(|T|) = \mathrm{trace}(|\Delta_{\mathfrak{A}}^n(T))| = \mathrm{trace}(|N|)$ for all $n = 0, 1, 2, ...$ $\qquad\square$

Theorem 5.2. *Let $\mathfrak{M}_f$ be an operator mean whose representing function satisfies $\lambda = f'(1) \in (0,1)$. Then there exists an operator $T \in B(\mathcal{H})$ such that $\{\Delta^n_{\mathfrak{M}_f}(T)\}$ does not converge in the week operator topology.*

To prove Theorem 5.2, we prepare the following discussion. It is a modification of [8, Corollary 3.3]: Let $\alpha := (\alpha_0, \alpha_1, \alpha_2, ...)$ be a sequence in ℓ^∞, and let W_α be a weighted unilateral shift on ℓ^2 with a weight sequence α, i.e.,

$$W_\alpha e_n = \alpha_n e_{n+1},$$

where $\{e_n\}$ be the canonical basis of ℓ^2. In what follows, $\alpha_n > 0$ for all $n = 0, 1, 2, ...$

Lemma 5.3. *Let f, g be representing functions of weighted arithmetic and harmonic means with a weight $\lambda \in [0,1]$, respectively, and let $\alpha = (\alpha_0, \alpha_1, \alpha_2, ...)$ be a sequence in ℓ^∞. Then*

$$\mathcal{P}_f(\alpha_1^{(n)}, \alpha_0^{(n)}) = \sum_{j=0}^{n+1} \binom{n+1}{j} \lambda^{n+1-j}(1-\lambda)^j \alpha_j,$$

$$\mathcal{P}_g(\beta_1^{(n)}, \beta_0^{(n)}) = \left[\sum_{j=0}^{n+1} \binom{n+1}{j} \lambda^{n+1-j}(1-\lambda)^j \alpha_j^{-1} \right]^{-1},$$

where $\alpha^{(n)} := (\alpha_0^{(n)}, \alpha_1^{(n)}, ...)$ and $\beta^{(n)} := (\beta_0^{(n)}, \beta_1^{(n)}, ...)$ are

$$\alpha_k^{(n)} = \mathcal{P}_f(\alpha_{k+1}^{(n-1)}, \alpha_k^{(n-1)}),$$
$$\beta_k^{(n)} = \mathcal{P}_g(\beta_{k+1}^{(n-1)}, \beta_k^{(n-1)})$$

and $\alpha_k^{(0)} = \beta_k^{(0)} = \alpha_k$.

Proof. The proof follows from the mathematical induction. □

Lemma 5.4. *Let $\mathfrak{M}_f$ be an operator mean, and let W_α be a weighted unilateral shift with a weight sequence α. Then $\Delta^n_{\mathfrak{M}}(W_\alpha)$ is also a weighted unilateral shift $W_{\alpha'}$ with a weight sequence $\alpha' = (\alpha'_0, \alpha'_1, ...)$, where $\alpha'_n = \mathcal{P}_f(\alpha_{n+1}, \alpha_n)$. ($n = 0, 1, 2, ...$).*

Proof. Let $\{e_n\}$ be the canonical basis of ℓ^2. Then W_a can be represented to

$$W_\alpha = \begin{pmatrix} 0 & & & & \\ \alpha_0 & 0 & & & \\ & \alpha_1 & 0 & & \\ & & \alpha_2 & 0 & \\ & & & \ddots & \ddots \end{pmatrix}$$

respect to $\{e_n\}$, and we have the spectral decomposition $|W_\alpha| = \sum_{n=0}^\infty \alpha_n P_n$, where every $P_n = (p_{ij})$ is a projection satisfying

$$p_{ij} = \begin{cases} 1 & (i = j = n) \\ 0 & (\text{otherwise}) \end{cases}.$$

12

Also assume that $W_\alpha = U|W_\alpha|$ is the polar decomposition. Then U is a unilateral shift. Thus for $n = 0, 1, ...$,

$$\Delta_{\mathfrak{M}_f}(W_\alpha)e_n = \sum_{i,j=0}^{\infty} \mathcal{P}_f(\alpha_i, \alpha_j)P_i U P_j e_n$$

$$= \sum_{i=0}^{\infty} \mathcal{P}_f(\alpha_i, \alpha_n)P_i U P_n e_n$$

$$= \sum_{i=0}^{\infty} \mathcal{P}_f(\alpha_i, \alpha_n)P_i U e_n$$

$$= \sum_{i=0}^{\infty} \mathcal{P}_f(\alpha_i, \alpha_n)P_i e_{n+1}$$

$$= \mathcal{P}_f(\alpha_{n+1}, \alpha_n)P_{n+1} e_{n+1} = \mathcal{P}_f(\alpha_{n+1}, \alpha_n)e_{n+1},$$

i.e., $\Delta_{\mathfrak{M}_f}(W_\alpha)$ is also a weighted unilateral shift with a weight sequence α'. $\qquad\square$

Proposition 5.5. *Let $\mathfrak{A}, \mathfrak{H}$ be λ- weighted arithmetic and harmonic means with a weight $\lambda \in (0,1)$, respectively. Suppose that a and b are any distinct positive real numbers. Let W_α be a unilateral weighted shift whose weights are either a or b. Suppose that only finitely many weights of W_α are equal to a. Then the sequences of the first weights of $\Delta_{\mathfrak{A}}^n(T)$ and $\Delta_{\mathfrak{H}}^n(T)$ converge to b.*

Proof. By Lemmas 5.3 and 5.4, the first weights of $\Delta_{\mathfrak{A}}^n(W_\alpha)$ and $\Delta_{\mathfrak{H}}^n(W_\alpha)$ are

$$\alpha_0^{(n)} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j \alpha_j,$$

$$\beta_0^{(n)} = \left[\sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j \alpha_j^{-1} \right]^{-1},$$

respectively. Next, let p be the largest number satisfying $\alpha_p = a$. Then for $n > p$, we have

$$\alpha_0^{(n)} = \sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j \alpha_j$$

$$= \sum_{j=0}^{p} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j a_j + b \sum_{j=p+1}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j.$$

Here we shall show

$$\lim_{n \to \infty} \sum_{j=0}^{p} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j a_j = 0.$$

Let $m = \max\{a, b\}$ and $M = \max_{0 \leq j \leq p} \frac{1}{j!}(\frac{1-\lambda}{\lambda})^j$. Then

$$0 \leq \sum_{j=0}^{p} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j a_j$$

$$\leq m \sum_{j=0}^{p} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j$$

$$\leq m \sum_{j=0}^{p} n^j \lambda^n \frac{1}{j!} \left(\frac{1-\lambda}{\lambda}\right)^j$$

$$\leq mM \sum_{j=0}^{p} n^p \lambda^n = mM(p+1)n^p \lambda^n.$$

Since $\lim_{n \to \infty} n^p \lambda^n = 0$ (by l'Hospital's rule), we have

$$(5.1) \qquad \lim_{n \to \infty} \sum_{j=0}^{p} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j a_j = 0.$$

Hence we have

$$\lim_{n \to \infty} \alpha_0^{(n)} = \lim_{n \to \infty} \sum_{j=0}^{p} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j a_j$$

$$+ \lim_{n \to \infty} b \sum_{j=p+1}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j = b,$$

since $\sum_{j=0}^{n} \binom{n}{j} \lambda^{n-j}(1-\lambda)^j = 1$ and (5.1). $\lim_{n \to \infty} \beta_0^{(n)} = b$ can be proven by the same way, too. $\qquad \square$

Proposition 5.6. *Suppose a and b are any distinct positive real numbers. Then there is a unilateral weighted shift W_α with a weight sequence α such that both sequences of the first weights of $\Delta_{\mathfrak{A}}^n(W_\alpha)$ and $\Delta_{\mathfrak{H}}^n(W_\alpha)$ have subsequences which are converging to a and b.*

Proof. Suppose that $\alpha = (a, b, b, ...)$. By Proposition 5.5, there exists a natural number n_1 such that

$$|\alpha_0^{(n_1)} - b| < \frac{1}{2} \quad \text{and} \quad |\beta_0^{(n_1)} - b| < \frac{1}{2}.$$

Suppose that

$$\alpha = (\overbrace{a, b, b, ..., b}^{n_1}, a, a, ...).$$

By Proposition 5.5, there exists a natural number n_2 such that $n_1 < n_2$,

$$|\alpha_0^{(n_2)} - a| < \frac{1}{2^2} \quad \text{and} \quad |\beta_0^{(n_2)} - a| < \frac{1}{2^2}.$$

Suppose that

$$\alpha = (\underbrace{\overbrace{a, b, b, ..., b}^{n_1}, a, a, ..., a}_{n_2}, b, b, ...).$$

By Proposition 5.5, there exists a natural number n_3 such that $n_1 < n_2 < n_3$,

$$|\alpha_0^{(n_3)} - b| < \frac{1}{2^3} \quad \text{and} \quad |\beta_0^{(n_3)} - b| < \frac{1}{2^3}.$$

Repeating this process, for each natural number k, there exist natural numbers n_k and n_k' such that

$$|\alpha_0^{(n_k)} - a| < \frac{1}{2^k}, \quad |\beta_0^{(n_k)} - a| < \frac{1}{2^k}$$
$$|\alpha_0^{(n_k')} - b| < \frac{1}{2^k}, \quad |\beta_0^{(n_k')} - b| < \frac{1}{2^k}.$$

Hence there are subsequences of the first weights of $\Delta_{\mathfrak{A}}^n(W_\alpha)$ and $\Delta_{\mathfrak{H}}^n(W_\alpha)$ which converge to a and b. $\qquad\square$

Proof of Theorem 5.2. Let $T := W_\alpha$ be a weighted unilateral shift defined in Proposition 5.6. Define a sequence of the first weight of $\Delta_{\mathfrak{M}_f}^n(T)$ by $\{\gamma_0^{(n)}\}$. By $\lambda = f'(1) \in (0,1)$ and an inequality (1.2), we have

$$\beta_0^{(n)} \leq \gamma_0^{(n)} \leq \alpha_0^{(n)} \quad \text{for } n = 0, 1, 2, \ldots$$

By Proposition 5.6, there exist subsequences of the first weights of $\Delta_{\mathfrak{A}}^n(T)$ and $\Delta_{\mathfrak{H}}^n(T)$ which converge to a in the same choice. Hence under the same choice, a subsequence of $\{\gamma_0^{(n)}\}$ converges to a. On the other hand, we can choose a subsequence of $\{\gamma_0^{(n)}\}$ which converges to b. Hence $\{\Delta_{\mathfrak{M}}^n(T)\}$ does not converge in the week operator topology. $\qquad\square$

References

[1] A. Aluthge, *On p-hyponormal operators for* $0 < p < 1$, Integral Equations Operator Theory, **13** (1990), 307–315.

[2] T. Ando and T. Yamazaki, *The iterated Aluthge transforms of a 2–by–2 matrix converge*, Linear Algebra Appl., **375** (2003), 299–309.

[3] J. Antezana, E.R. Pujals, D. Stojanoff, *The iterated Aluthge transforms of a matrix converge*, Adv. Math., **226** (2011), 1591–1620.

[4] R. Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp.

[5] M.S. Birman and M. Solomjak, *On double Stieltjes operator integrals*, Dokl. Akad. Nauk SSSR, **165** (1965) 1223–1226.

[6] M.S. Birman and M. Solomyak, *Double operator integrals in a Hilbert space*, Integral Equations Operator Theory, **47** (2003), 131–168.

[7] F. Chabbabi, R.E. Curto and M. Mbekhta, *The mean transform and the mean limit of an operator*, Proc. Amer. Math. Soc., **147** (2019), 1119–1133.

[8] M. Chō, I.B. Jung and W.Y. Lee, *On Aluthge transforms of p-hyponormal operators*, Integral Equations Operator Theory, **53** (2005), 321–329.

[9] Yu.L. Daletskii and S.G. Krein, *Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations,* (Russian) Voronezh. Gos. Univ. Trudy Sem. Funkcional. Anal., **1** (1956), 81–105.

[10] C. Foiaş, I.B. Jung, E. Ko and C. Pearcy, *Complete contractivity of maps associated with the Aluthge and Duggal transforms*, Pacific J. Math., **209** (2003), 249–259.

[11] F. Hansen, *The fast track to Loewner's theorem*, Linear Algebra Appl., 438 (2013), 4557–4571.

[12] E. Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung,* Math. Ann., **123**, (1951). 415–438.

[13] F. Hiai and H. Kosaki, *Means of Hilbert space operators*, Lecture Notes in Mathematics, 1820. Springer-Verlag, Berlin, 2003. vii+148 pp.

[14] T. Huruya, *A note on p-hyponormal operators*, Proc. Amer. Math. Soc., **125** (1997), 3617–3624.

[15] I.B. Jung, E. Ko and C. Pearcy, *Aluthge transforms of operators*, Integral Equations Operator Theory, **37** (2000), 437–448.

[16] H. Kosaki, *Strong monotonicity for various means*, J. Funct. Anal., **267** (2014), 1917–1958.

[17] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246** (1979/80), 205–224.

[18] S.H. Lee, W.Y. Lee and J. Yoon, *The mean transform of bounded linear operators*, J. Math. Anal. Appl., 410 (2014), 70–81.

[19] S.H. Lee, *The transforms of Pythaforean and quadratic means of weighted shifts*, J. Chung Cheong Math. Soc., 29 (2016) 123–135.

[20] M. Pálfia, Operator means of probably measures and generalized Karcher equations, Adv. Math., **289** (2016) 951–1007.

[21] V.V. Peller, *Hankel operators and differentiability properties of functions of self-adjoint (unitary) operators*, LOMI Preprints E-1-84, USSR Academy of Sciences Steklov Mathematical Institute Leningrad Department, 1984.

[22] V.V. Peller, *Hankel operators in the perturbation theory of unitary and self-adjoint operators*, Funct. Anal. Appl., **19** (1985), 111-123.

[23] T. Yamazaki, *An expression of spectral radius via Aluthge transformation*, Proc. Amer. Math. Soc., **130** (2002), 1131–1137.

Department of Electrical, Electronic and Computer Engineering, Toyo University, Kawagoe-Shi, Saitama, 350-8585, Japan.

E-mail address: t-yamazaki@toyo.jp