TWISTED ENDOSCOPIC CHARACTER RELATION FOR TORAL REGULAR SUPERCUSPIDAL L-PACKETS: ANNOUNCEMENT

MASAO OI
DEPARTMENT OF MATHEMATICS (HAKUBI CENTER),
KYOTO UNIVERSITY

This note is an announcement of a recent result of the author. The details will be presented in a forthcoming article which is in preparation.

Let F be a p-adic field. For a connected reductive group $\mathbf{G}$ over F, it is generally expected that there exists a natural map (*local Langlands correspondence*)

$$\mathrm{LLC}_{\mathbf{G}} \colon \Pi(\mathbf{G}) \rightarrow \Phi(\mathbf{G})$$

from the set $\Pi(\mathbf{G})$ of equivalence classes of irreducible admissible representations of $\mathbf{G}(F)$ to the set $\Phi(\mathbf{G})$ of equivalence classes of L-parameters of $\mathbf{G}$.

Recently, Kaletha proposed an explicit way of constructing the map $\mathrm{LLC}_{\mathbf{G}}$ for a wide class of supercuspidal representations called "regular supercuspidal representations" of tamely ramified connected reductive groups ([Kal19]). He checked that the correspondence for those representations indeed satisfies various properties which are usually expected. Especially, he proved that the *standard endoscopic character relation* holds when the supercuspidal representations satisfy the *torality*, which is a certain kind of extremal regularity.

To be more precise, let $\mathbf{H}$ be an endoscopic group of $\mathbf{G}$ and we assume that both $\mathbf{G}$ and $\mathbf{H}$ are tamely ramified. Suppose that an L-parameter ϕ of $\mathbf{G}$ factors through the L-embedding of ${}^{L}\mathbf{H}$ into ${}^{L}\mathbf{G}$ and is toral in the sense of Kaletha (see [Kal19, Definition 6.1.1], here we assume that the torality as both an L-parameter of $\mathbf{G}$ and $\mathbf{H}$ is satisfied). Then, according to [Kal19], we obtain the L-packets $\Pi_{\phi}^{\mathbf{G}}$ and $\Pi_{\phi}^{\mathbf{H}}$ (namely, the finite sets which should be the fibers of $\mathrm{LLC}_{\mathbf{G}}$ and $\mathrm{LLC}_{\mathbf{H}}$ at ϕ), which consist of toral supercuspidal representations of $\mathbf{G}(F)$ and $\mathbf{H}(F)$, respectively. Then the standard endoscopic character relation, which was proved by Kaletha, is as follows:

Theorem 1 ([Kal19, Theorem 6.3.4], standard endoscopic character relation)**.** *Assume the residual characteristic p is sufficiently large. For each $\pi \in \Pi_{\phi}^{\mathbf{G}}$, there exists a constant $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi) \in \mathbb{C}$ such that the following identity holds for any strongly regular semisimple elliptic element $\delta \in \mathbf{G}(F)$:*

$$\sum_{\pi \in \Pi_{\phi}^{\mathbf{G}}} \Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi) \Theta_{\pi}(\delta) = \sum_{\gamma \in H/\mathrm{st}} \frac{D_{\mathbf{H}}(\gamma)^2}{D_{\mathbf{G}}(\delta)^2} \Delta_{\mathbf{H},\mathbf{G}}(\gamma, \delta) \sum_{\pi_{\mathbf{H}} \in \Pi_{\phi}^{\mathbf{H}}} \Theta_{\pi_{\mathbf{H}}}(\gamma),$$

where

- *Θ_{π} (resp. $\Theta_{\pi_{\mathbf{H}}}$) is the Harish-Chandra character of π (resp. $\pi_{\mathbf{H}}$),*
- *γ runs over the stable conjugacy classes of norms of δ in $H = \mathbf{H}(F)$,*
- *$D_{\mathbf{G}}$ and $D_{\mathbf{H}}$ are the Weyl discriminants, and*

- $\Delta_{\mathbf{H},\mathbf{G}}$ *is the Langlands–Shelstad transfer factor.*

Remark 2. In [Kal19, Theorem 6.3.4], it is stated that the identity holds for a strongly regular semisimple element of $\mathbf{G}(F)$ which have a "normal r-approximation". In fact, any elliptic regular semisimple element satisfies this condition. Thus we can state the theorem as above.

Our main result is a generalization of the above theorem to the case of twisted endoscopy. More precisely, we assume that $\mathbf{G}$ is quasi-split and consider an F-rational automorphism θ of $\mathbf{G}$ which preserves an (fixed) F-pinning of $\mathbf{G}$. Then we obtain a twisted space $\tilde{\mathbf{G}} = \mathbf{G} \rtimes \theta$, which is a bi-$\mathbf{G}$-torsor over F whose double $\mathbf{G}$-action is given by $g_1(g \rtimes \theta)g_2 = (g_1 g \theta(g_2)) \rtimes \theta$. We suppose that the order of θ is finite. For an endoscopic group $\mathbf{H}$ for the pair $(\mathbf{G}, \theta)$, we consider the same situation as before; let $\Pi_\phi^{\mathbf{G}}$ and $\Pi_\phi^{\mathbf{H}}$ be toral L-packets corresponding to a toral L-parameter ϕ.

Theorem 3 (Main result, twisted endoscopic character relation)**.** *Assume the residual characteristic p is sufficiently large. For each $\pi \in \Pi_\phi^{\mathbf{G}}$, there exists a constant $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi) \in \mathbb{C}$ such that the following identity holds for any strongly regular semisimple elliptic element $\delta \in \tilde{\mathbf{G}}(F)$:*

$$\sum_{\pi \in \Pi_\phi^{\mathbf{G}}} \Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi) \Theta_{\tilde{\pi}}(\delta) = \sum_{\gamma \in H/\mathrm{st}} \frac{D_{\mathbf{H}}(\gamma)^2}{D_{\tilde{\mathbf{G}}}(\delta)^2} \Delta_{\mathbf{H},\mathbf{G}}(\gamma, \delta) \sum_{\pi_{\mathbf{H}} \in \Pi_\phi^{\mathbf{H}}} \Theta_{\pi_{\mathbf{H}}}(\gamma).$$

Here,

- $\Theta_{\tilde{\pi}}$ *is the θ-twisted character of π, which is a function on (the regular semisimple locus of) $\tilde{\mathbf{G}}(F)$ and can be defined when π is θ-stable,*
- $D_{\tilde{\mathbf{G}}}$ *is the twisted Weyl discriminant, and*
- $\Delta_{\mathbf{H},\mathbf{G}}$ *is the Kottwitz–Shelstad transfer factor.*

Remark 4. By linear independence of θ-twisted characters, a family $\{\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi)\}_{\pi \in \Pi_\phi^{\mathbf{G}}}$ of constants as in Theorem 3 is unique if exists. When π is not θ-stable, we put $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi)$ to be zero. Hence, although the symbol $\Theta_{\tilde{\pi}}$ does not make sense in such a case, we may understand that the summand $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi)\Theta_{\tilde{\pi}}(\delta)$ is zero. On the other hand, the definition of $\Delta_{\mathbf{H},\mathbf{G}}^{\mathrm{spec}}(\phi, \pi)$ for a θ-stable π is very subtle and related to the computation of the twisted characters $\Theta_{\tilde{\pi}}$ and the transfer factors $\Delta_{\mathbf{H},\mathbf{G}}(\gamma, \delta)$ deeply.

Remark 5. According to [Kal19], each regular supercuspidal L-packet or L-parameter is naturally associated to a pair $(\mathbf{S}, \vartheta)$ of

- a tamely ramified torus $\mathbf{S}$ and
- a "regular" character $\vartheta\colon \mathbf{S}(F) \to \mathbb{C}^\times$

equipped with certain auxiliary data (called "regular supercuspidal L-packet datum", see [Kal19, Definition 5.2.4]). By Kaletha's construction, members of a regular supercuspidal L-packet $\Pi_\phi^{\mathbf{G}}$ are parametrized by the set (say $\mathcal{J}_G^{\mathbf{G}}$) of the G-conjugacy classes (where $G := \mathbf{G}(F)$) in a stable $\mathbf{G}$-conjugacy class of embeddings of $\mathbf{S}$ into $\mathbf{G}$. In fact, under the assumption that the toral (or, more generally, regular supercuspidal) L-parameter ϕ of $\mathbf{G}$ factors through the L-group ${}^L\mathbf{H}$ of $\mathbf{H}$, we can show that the corresponding L-packet $\Pi_\phi^{\mathbf{G}}$ is θ-stable, that is, stable under the θ-twist as set ($\Pi_\phi^{\mathbf{G}} \circ \theta = \Pi_\phi^{\mathbf{G}}$). Furthermore, we can describe explicitly the θ-stable members of $\Pi_\phi^{\mathbf{G}}$ in terms of the parametrizing set $\mathcal{J}_G^{\mathbf{G}}$.

Theorem 3 is proved by imitating Kaletha's proof of Theorem 1 in the framework of twisted spaces. In the following, we would like to give a few comments on our proof of Theorem 3. In the following, for an algebraic variety $\mathbf{J}$ defined over F, we let J denote the set $\mathbf{J}(F)$ of its F-valued points.

The first step of the proof is to establish an explicit formula of the twisted characters of toral supercuspidal representations. On the endoscopic side, where the group $\mathbf{H}$ is not twisted, each character $\Theta_{\pi_{\mathbf{H}}}$ is described by the Adler–DeBacker–Spice formula ([AS09, DS18]). To explain it, let us take a member $\pi_{\mathbf{H}}$ of $\Pi_{\phi}^{\mathbf{H}}$, which is a toral supercuspidal representation of H, and put r to be the depth of $\pi_{\mathbf{H}}$. Let $(\mathbf{S}_{\mathbf{H}}, \vartheta_{\mathbf{H}})$ be the pair (a part of the regular supercuspidal L-packet datum) associated to the toral L-packet $\Pi_{\phi}^{\mathbf{H}}$. When $\pi_{\mathbf{H}}$ is associated to an embedding $j_{\mathbf{H}}\colon \mathbf{S}_{\mathbf{H}} \hookrightarrow \mathbf{H}$ under the parametrization in Remark 5 ($j_{\mathbf{H}} \in \mathcal{J}_{H}^{\mathbf{H}}$), we get a maximal torus $\mathbf{S}_{\mathbf{H},j_{\mathbf{H}}} := j_{\mathbf{H}}(\mathbf{S}_{\mathbf{H}})$ of $\mathbf{H}$. According to Yu's construction, $\pi_{\mathbf{H}}$ is obtained by the compact induction of an irreducible smooth representation $\sigma_{\mathbf{H}}$ of a certain open compact subgroup $K_{\sigma_{\mathbf{H}}}$ (see [AS09, Section 2] for the details):

$$\pi_{\mathbf{H}} = \text{c-Ind}_{K_{\sigma_{\mathbf{H}}}}^{H} \sigma_{\mathbf{H}}.$$

Proposition 6 ([AS09, DS18])**.** *When a strongly regular semisimple element $\gamma \in \mathbf{H}(F)$ has a normal r-approximation $\gamma = \gamma_{<r}\gamma_{\geq r}$, we have*

$$\Theta_{\pi_{\mathbf{H}}}(\gamma) = \sum_{\substack{h \in S_{\mathbf{H},j_{\mathbf{H}}}\backslash H/H_{\gamma_{<r}} \\ h\gamma_{<r}h^{-1} \in S_{\mathbf{H},j_{\mathbf{H}}}}} \Theta_{\sigma_{\mathbf{H}}}(h\gamma_{<r}h^{-1}) \hat{\mu}^{\mathbf{H}_{\gamma_{<r}}}_{h^{-1}X^{*}_{\mathbf{H}}h}(\exp^{-1}(\gamma_{\geq r})),$$

where $X^{*}_{\mathbf{H}}$ is an element of the dual of the Lie algebra of $\mathbf{H}_{\gamma_{<r}}$ (the connected centralizer of $\gamma_{<r}$ in $\mathbf{H}$) determined by $\vartheta_{\mathbf{H}}$ (see [AS09, Section 2] *for the details) and $\hat{\mu}$ denotes the Fourier transform of the orbital integral.*

Roughly speaking, a *normal r-approximation to γ* is a nice product decomposition of γ into two parts: one is $\gamma_{<r}$, whose root-values are p-adically shallower than r; the other one is $\gamma_{\geq r}$, whose root-values are p-adically deeper than or equal to r (see [AS08, Definition 6.8] for the precise definition). We note that the deeper part $\gamma_{\geq r}$ belongs to the connected centralizer $\mathbf{H}_{\gamma_{<r}}$ of $\gamma_{<r}$ in $\mathbf{H}$. The important feature of the above character formula is that these two parts $\gamma_{<r}$ and $\gamma_{\geq r}$ contribute to summands separately. Especially, the contribution of $\gamma_{\geq r}$ is expressed by the quantity on the group $\mathbf{H}_{\gamma_{<r}}$.

In fact, by formulating the notion of an r-approximation in the twisted space $\tilde{G}$ appropriately, we can reproduce the arguments of Adler–DeBacker–Spice for the twisted characters. If we introduce the similar notations to above ($\pi \in \Pi_{\phi}^{\mathbf{G}}$, $(\mathbf{S}, \vartheta)$, $j \in \mathcal{J}_{G}^{\mathbf{H}}$, σ, X^{*}), then we obtain the following formula:

Proposition 7. *When a strongly regular semisimple element $\delta \in \tilde{G}$ has a normal r-approximation $\delta = \delta_{<r}\delta_{\geq r}$, we have*

$$\Theta_{\tilde{\pi}}(\delta) = \sum_{\substack{g \in S_{j}\backslash G/G_{\delta_{<r}} \\ g\delta_{<r}g^{-1} \in \tilde{S}}} \Theta_{\tilde{\sigma}}(g\delta_{<r}g^{-1}) \hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^{*}g}(\exp^{-1}(\delta_{\geq r})).$$

We note that when $\delta \in \tilde{G}$ has a normal r-approximation $\delta = \delta_{<r}\delta_{\geq r}$, the shallower part $\delta_{<r}$ (resp. the deeper part $\delta_{\geq r}$) belongs to $\tilde{G}$ (resp. G). We also remark that $\mathbf{S}$ extends to a "twisted maximal torus" $\tilde{\mathbf{S}}$ of $\tilde{\mathbf{G}}$ in the sense of Waldspurger

(see, e.g., [MW16, Section I.1.3]) when π is θ-stable. For these reasons, we may discuss whether $g\delta_{<r}g^{-1}$ belongs to $\tilde{S}$ or not for $g \in G$ and thus the index set of the above formula makes sense.

By Propositions 6 and 7, in order to get the identity of Theorem 3, we are reduced to compare the $\mathbf{G}$-side of the desired identity

$$\sum_{\pi\in\Pi_\phi^{\mathbf{G}}} \Delta^{\mathrm{spec}}_{\mathbf{H},\mathbf{G}}(\phi,\pi) \sum_{\substack{g\in S_j\backslash G/G_{\delta_{<r}} \\ g\delta_{<r}g^{-1}\in\tilde{S}}} \Theta_{\tilde{\sigma}}(g\delta_{<r}g^{-1})\hat{\mu}^{\mathbf{G}_{\delta_{<r}}}_{g^{-1}X^*g}(\exp^{-1}(\delta_{\geq r}))$$

with the $\mathbf{H}$-side

$$\sum_{\gamma\in H/\mathrm{st}} \frac{D_{\mathbf{H}}(\gamma)^2}{D_{\tilde{\mathbf{G}}}(\delta)^2}\Delta_{\mathbf{H},\mathbf{G}}(\gamma,\delta) \sum_{\pi_{\mathbf{H}}\in\Pi_\phi^{\mathbf{H}}} \sum_{\substack{h\in S_{\mathbf{H},j_{\mathbf{H}}}\backslash H/H_{\gamma_{<r}} \\ h\gamma_{<r}h^{-1}\in S_{\mathbf{H},j_{\mathbf{H}}}}} \Theta_{\sigma_{\mathbf{H}}}(h\gamma_{<r}h^{-1})\hat{\mu}^{\mathbf{H}_{\gamma_{<r}}}_{h^{-1}X^*_{\mathbf{H}}h}(\exp^{-1}(\gamma_{\geq r})).$$

This can be done by utilizing the result of Waldspurger and Ngô ([Wal97, Wal06, Wal08] and [Ngô10]) on the Lie algebra transfer, which basically enables us to compare the Fourier transforms of orbital integrals on the Lie algebras of a connected reductive group and its endoscopic group. However, for this, we have at least two difficulties to overcome:

(1) Fourier transforms of orbital integrals themselves cannot be compared directly. They must be stabilized (summed up within its stable conjugacy class) to be compared.
(2) Furthermore, the stabilized sums must be weighted by the transfer factors.

On (1), we consider rearranging the double sums on the $\mathbf{G}$-side or $\mathbf{H}$-side of the endoscopic character relation. By combining the sum over $\Pi_\phi^{\mathbf{H}}$ with the sum over $\{h \in S_{\mathbf{H},j_{\mathbf{H}}}\backslash H/H_{\gamma_{<r}} \mid h\gamma_{<r}h^{-1} \in S_{\mathbf{H},j_{\mathbf{H}}}\}$ and dividing it again, we can create the sum over rational $H_{\gamma_{<r}} = \mathbf{H}_{\gamma_{<r}}(F)$-conjugacy classes within a stable $\mathbf{H}_{\gamma_{<r}}$-conjugacy class of $X^*_{\mathbf{H}}$. Doing the same process on the $\mathbf{G}$-side, we also get the sum over an index set needed for the comparison via Waldspurger–Ngô transfer.

On the other hand, the key to (2) is computing the contribution of the shallower part in the character formula ($\Theta_{\tilde{\sigma}}(g\delta_{<r}g^{-1})$ on the $\mathbf{G}$-side and $\Theta_{\sigma_{\mathbf{H}}}(h\gamma_{<r}h^{-1})$ on the $\mathbf{H}$-side) explicitly. For the untwisted characters ($\Theta_{\sigma_{\mathbf{H}}}(h\gamma_{<r}h^{-1})$), this part was firstly computed in [AS09] and rewritten in [DS18] and [Kal19] via several endoscopic quantities such as the transfer factors or Kottwitz signs. The computation for the twisted side ($\Theta_{\tilde{\sigma}}(g\delta_{<r}g^{-1})$) can be done in basically the same way as the untwisted case. The twisting process of the computation in [AS09] is eventually reduced to establish a twisted version of Gérardin's character formula for the Heisenberg–Weil representations over finite fields. Once we establish this, we next have to relate it to the transfer factors. This can be done by utilizing Waldspurger's machinery on the descent in the twisted endoscopy [Wal08], instead of the Langlands–Shelstad descent [LS90] in the untwisted case.

Acknowledgment. The author would like to thank Professor Takuya Yamauchi and Professor Kazuki Morimoto for giving him an opportunity to talk in the conference. This work is supported by (i) the Program for Leading Graduate Schools, MEXT, Japan and (ii) JSPS KAKENHI Grant Number 17J05451 (DC2)/19J00846 (PD).

References

[AS08] J. D. Adler and L. Spice, *Good product expansions for tame elements of p-adic groups*, Int. Math. Res. Pap. IMRP (2008), no. 1, Art. ID rp. 003, 95.

[AS09] ———, *Supercuspidal characters of reductive p-adic groups*, Amer. J. Math. **131** (2009), no. 4, 1137–1210.

[DS18] S. DeBacker and L. Spice, *Stability of character sums for positive-depth, supercuspidal representations*, J. Reine Angew. Math. **742** (2018), 47–78.

[Kal19] T. Kaletha, *Regular supercuspidal representations*, J. Amer. Math. Soc. **32** (2019), no. 4, 1071–1170.

[LS90] R. Langlands and D. Shelstad, *Descent for transfer factors*, The Grothendieck Festschrift, Vol. II, Progr. Math., vol. 87, Birkhäuser Boston, Boston, MA, 1990, pp. 485–563.

[MW16] C. Mœglin and J.-L. Waldspurger, *Stabilisation de la formule des traces tordue. Vol. 1*, Progress in Mathematics, vol. 316, Birkhäuser/Springer, Cham, 2016.

[Ngô10] B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. (2010), no. 111, 1–169.

[Wal97] J.-L. Waldspurger, *Le lemme fondamental implique le transfert*, Compositio Math. **105** (1997), no. 2, 153–236.

[Wal06] ———, *Endoscopie et changement de caractéristique*, J. Inst. Math. Jussieu **5** (2006), no. 3, 423–525.

[Wal08] ———, *L'endoscopie tordue n'est pas si tordue*, Mem. Amer. Math. Soc. **194** (2008), no. 908, x+261.

Department of Mathematics (Hakubi center), Kyoto University, Kitashirakawa, Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan.

Email address: `masaooi@math.kyoto-u.ac.jp`