Limit Order Book Dynamics with Large Executions*

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Abstract

We provide an endogenous model of limit order book dynamics to be applicable to the execution problem of large investors such as institutional investors. For the ask (bid) side limit order book formed by patient sellers (buyers), we consider the trading information of large investors will be incorporated into the bid (ask) side best price in order to make the limit order book consistent throughout the trading period. By giving the best price exogenously, it is possible to characterize the actions of patient sellers or buyers and identify the shape of the limit order book. As a result, it can be applied to the optimal execution problem.

1 Introduction

As the mainstream of seculity tradings around the world shifts from the quote-driven market to the order-driven market today, many studies considering Limit Order Book (LOB) have been actively conducted. This is especially important when considering the liquidity risk management of large traders such as institutional investors, who have an impact on prices through their own trades. For the optimal execution problem considering the LOB, the dynamics of the LOB is formulated and the execution strategy is derived. The previous studies of these types include [4] and [11]. In addition, as the dynamics of the limit order book, [6] and [12] provided the dynamic model considering to the waiting cost for the liquidity traders. In [10], the optimal execution problem considering the LOB is studied, however the execution strategy is derived assuming that the LOB is given exogenously. In [9], the price model in [10], that is, the block-shaped LOB with exponential resilience is explained from the order book construction process. Many studies of optimal execution problems consider linear impact, that is, the block-shaped LOB, but many empirical studies such as [1] and [3] have reported that price impact is non-linear.

In this research, instead of constructing the exponential decay from the block-shaped LOB, we form the system equation so that it becomes the exponential decay using the boundary condition moving over time. Especially in [9], the range of limit order book that liquidity investors can enter was fixed to a certain level, but we constructed a model in which the range of LOBs narrows due to the large execution of an institutional investor. This makes it possible to build a model in which trading information of large traders increases liquidity.

The remainder of this paper is organized as follows. Section 2 gives an overview of the LOB formation process based on [9] and [12]. We consider that a patient seller submits one unit sell limit order and waits for a buy market order. Then we give the mean and the standard deviation

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of bid-ask spread and the price impact. In Section 3, we consider a model in which execution information of the large trader is incorporated into prices from the formation of LOBs. Section 4 concludes the paper.

2 Dynamics of LOB

In this section, we give an overview of the one-sided (ask-side) endogenous LOB dynamics model based on Kuno (2020) and Roşu (2009).

2.1 Model setup

Firstly we consider two types of economic agents. One is a single risk averse large trader who submits large market buy orders at each predetermined equidistant trading time $t(=1,2,\cdots,T)$. Others are many risk-neutral liquidity traders who trade continuously one unit at a time. Since we consider only an ask side LOB, we can divide the liquidity traders into two types. One is the patient sellers and the other is the impatient buyers. The patient seller or the impatient buyer enters into the book one by one at random. The patient sellers enters into the book with Poisson arrival rate λ_{PS} and submit one unit of limit order then wait until it is matched. Moreover they are able to cancel and resubmit their order without a fee. Therefore they resubmit their orders to maximize their utility. We define the expected utility of such a patient seller u_t as

$$u_t := E_t\left[P_\tau - r(\tau - t)\right], \tag{2.1}$$

where r is a patient coefficient which has the same value for all patient sellers and τ is the random time which the limit sell order is executed. On the other hand, the impatient buyers who arrive at the book with Poisson arrival rate λ_{IB} submit one unit of market order. This market order is executed immediately and they leave the order book. Moreover we also assume that the maximum number of limit sellers who can submit their order and wait until their order are executed is M_t, which is fixed for each time interval. In the following, we will find that M_t is determined by the spread between the upper bound and the lower bound at time t. The upper bound A and lower bound B of the book are not fixed because we will extend to a dynamic framework. Under this setting, we get following system equation,

$$\begin{cases} u_1(t) := A_t, \\ u_m(t) = \frac{c}{1+c}u_{m+1}(t) + \frac{1}{1+c}u_{m-1}(t) - \frac{r}{\lambda_{PS}+\lambda_{IB}}, \\ u_{M_t}(t) = u_{M_t-1}(t) - \frac{r}{\lambda_{IB}+\nu}, \\ u_{M_t}(t) := B_t \end{cases} \tag{2.2}$$

where we define $c := \frac{\lambda_{PS}}{\lambda_{IB}}$ and λ_{PS},λ_{IB},r,ν, and ρ are fixed. ν represents the Poisson arrival rate, which makes a change to the market order of the patient seller with a limit order at the bottom of the spread.

Therefore if $\lambda_{PS} \neq \lambda_{IB}$, the expected utility of the patient seller m with the patient coefficient r in equilibrium is

$$u_m(t) = A_t + C_t\left(\left(\frac{1}{c}\right)^m - 1\right) + \frac{r}{\lambda_{PS} - \lambda_{IB}}m, \tag{2.3}$$

where

$$C_t := \frac{r}{\lambda_{PS} - \lambda_{IB}} \cdot \frac{\frac{\lambda_{PS}+\nu}{\lambda_{IB}+\nu}}{\left(\frac{1}{c}\right)^{M_t-1} - \left(\frac{1}{c}\right)^{M_t}}. \tag{2.4}$$

The (expected) maximum state M_t is determined by the following relation,

$$E_t\left[\frac{A_t - B_t}{\frac{r}{\lambda_{PS}-\lambda_{IB}}}\right] = \frac{\lambda_{PS}+\nu}{\lambda_{IB}+\nu}\frac{c^{M_t}-1}{c-1} - M_t. \tag{2.5}$$

On the other hand, if $\lambda_{PS} = \lambda_{IB}$, the expected utility of seller m is,

$$u_m(t) = A - bm + r\frac{1}{\lambda_{PS}+\lambda_{IB}}m^2, \tag{2.6}$$

where

$$b = \frac{2r}{\lambda_{PS} - \lambda_{IB}}\left(M - \frac{\nu - \lambda_{PS}}{2(\nu + \lambda_{PS})}\right). \tag{2.7}$$

Finally the maximum state M_t under $\lambda_{PS} = \lambda_{IB}$ is,

$$M_t = \frac{\nu - \lambda_{PS}}{2(\nu + \lambda_{PS})} + \sqrt{\left(\frac{\nu - \lambda_{PS}}{2(\nu + \lambda_{PS})}\right)^2 + \frac{r}{\lambda_{PS}+\lambda_{IB}}} \tag{2.8}$$

From equation (2.8), when $\lambda_{PS} = \lambda_{IB}$, it can be seen that the maximum state is constant regardless of time.

Remark 1 When $\lambda_{PS} \neq \lambda_{IB}$, u_m depends on the lower bound B as follows,

$$\frac{du_m}{dB} = \frac{1 - \left(\frac{1}{c}\right)^m}{1 - \left(\frac{1}{c}\right)^M}. \tag{2.9}$$

Since M is depends on B under λ_{PS},λ_{IB},r and ν are fixed, u_m depends on C, that is, on M. Then we differentiate u_m by B,

$$\frac{du_m}{dB} = \frac{dC}{dB}\left(\left(\frac{1}{c}\right)^m - 1\right). \tag{2.10}$$

Nextly, when $m = M$,

$$u_M = B = A + C\left(\left(\frac{1}{c}\right)^M - 1\right) + \frac{r}{\lambda_{PS} - \lambda_{IB}}M. \tag{2.11}$$

Therefore,

$$1 = \frac{dC}{dB}\left(\left(\frac{1}{c}\right)^M - 1\right) \Longleftrightarrow \frac{dC}{dB} = \frac{1}{\left(\frac{1}{c}\right)^M - 1}. \tag{2.12}$$

2.2 Empirical Implications

This subsection gives an overview of the mean and standard deviations of the LOB spreads, using the definitions in [5] and [6]. This also shows the empirical properties of the LOB such as price impacts.

In the state m, the bid-ask spread S_m in equilibrium is

$$S_m = a_m - B \tag{2.13}$$

where a_m is the best ask price and when $m < M$, $a_m = u_{m-1}$. Then, the mean μ_S and standard deviation σ_S of S_m are,

$$\mu_S \approx \epsilon \ln\left(\frac{1}{\epsilon}\right) \frac{c(c+1)}{(c-1)\ln c}, \tag{2.14}$$

$$\sigma_S \approx \sqrt{\epsilon(A-B)\left(\frac{(c+1)(c^3+c^2-c)}{(c-1)^3}\right)}, \tag{2.15}$$

where $\epsilon = \frac{r}{\lambda_{PS}+\lambda_{IB}}$ is called granularity and $c = \frac{\lambda_{PS}}{\lambda_{IB}}$ is called competition. λ_{PS} and λ_{IB} are observable in the market, while r needs to be estimated.

Remark 2 When $c > 1$, the patient sellers arrive at the market sooner than the impatient buyers. That is, the consumed LOB gradually recovers.while $c \leq 1$, the consumed LOB do not recover on average and bid-ask spread will expand. Due to reality and consistency with empirical study, we only consider the case of $c > 1$. Moreover, when c takes large number, μ_S becomes small. But the opposite happens with larger c. There are two reasons for the increase in c. First, patient sellers arrive at the market faster than impatient buyers. Second, impatient buyers arrive at the market late enough. In the latter case, the waiting cost becomes large, so the execution price is set high and the bid-ask spread expands.

Remark 3 Both μ_S and σ_S increase with respect to ϵ. Furthermore, as ϵ increases, the LOB becomes rough. That is, the number of orders at the unit price is reduced. This indicates that the patient seller will be more patient or it is difficult for orders to come to LOB.

2.3 Price Impact

The price impact I_m in state m is defined as the change in price per unit execution and

$$I_m = -\frac{du_{m-1}}{dm}. \tag{2.16}$$

Furthermore, based on [9], the impact θ of consumption in discrete x units is

$$\begin{aligned} \theta(t+1,x) &:= u_{m-x}(k) - u_m(k) \\ &= \frac{r}{\lambda_{PS}-\lambda_{IB}}\left\{\frac{c^{-m}(c^x-1)}{c^{-M_t}(c-1)}\frac{\lambda_{PS}+\nu}{\lambda_{IB}+\nu} - x\right\} \\ &= \alpha(t)c^x - \beta x - \gamma(t), \end{aligned} \tag{2.17}$$

where

$$\alpha(t) := C_t c^{-m}, \tag{2.18}$$

$$\beta := \frac{r}{\lambda_{PS}-\lambda_{IB}}, \tag{2.19}$$

$$\gamma(t) := C_t c^{-m}. \tag{2.20}$$

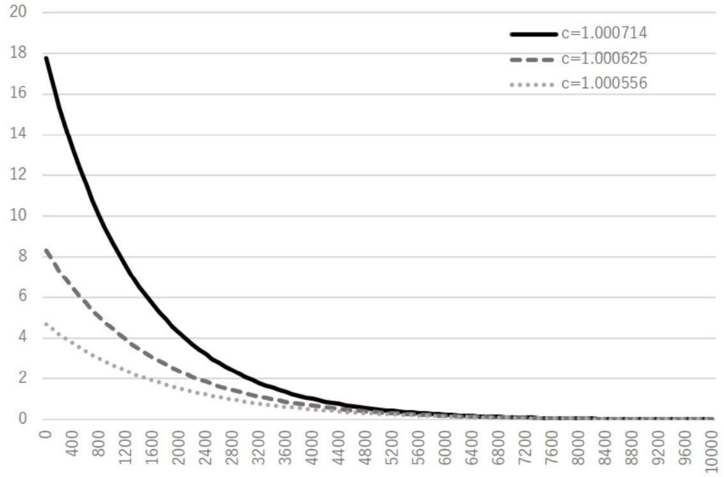

Figure 1: Values of α and γ due to differences in competition c in the market impact function

Figure 1 shows the value of $\alpha(t) = \gamma(t) = C_t c^{-m}$ due to the difference in competition in the market impact function based on the value of m. It can be seen that α and γ are decreasing functions of m. Here, we set the maximum state $M = 10000$, the patient coefficient $r = 0.00005$, and $\nu = 100$. The solid line represents $c = 1.000714$, the broken line represents $c = 1.000625$, and the dotted line represents $c = 1.000556$. Moreover, $\lambda_{PS} = 7000, 8000, 9000$ and $\lambda_{IB} = 6995, 7995, 8995$ respectively.

As with the bid-ask spread, the mean μ_I and standard deviation σ_I of the price impact I_m are,

$$\mu_I \approx \epsilon \ln\left(\frac{1}{\epsilon}\right)\frac{c+1}{\ln c}, \tag{2.21}$$

$$\sigma_I \approx \sqrt{\epsilon(A-B)\frac{c+1}{c-1}}. \tag{2.22}$$

Remark 4 The property of the price impact is the same as bid-ask spread, depending on competitive c and granuity ϵ. Then both the bid-ask spread and the price impact will form the market depth, and when the market is deep (there are many order s in a certain price range), the price impact is small because the interval of limit orders in the process of forming the LOB is also narrow.

3 Consideration for Large Execution

In this section, we consider the formation of the LOB when a large trader submits a large number of buy market orders.

3.1 Asymmetric information and fundamental value

In [9], a block-shape LOB and exponential resilience are modeled by setting $\lambda_{IB} = 0 (c \to \infty)$. The price is recovering exponentially however, the change in the fundamental value of assets is not considered. The system equation in [9] is given as follow,

$$\begin{cases} u_1(t) := A_t = p_{t-1} + \epsilon_t + e^{-\rho s}\theta, \\ u_m(t) = u_{m+1}(t) - \frac{r}{\lambda_{PS}}, \\ u_{M_t}(t) = u_{M_t-1}(t) - \frac{r}{\nu}, \\ u_{M_t}(t) := B_t = A_t - L, \end{cases} \tag{3.1}$$

This type of LOB represents a block shape and the exponential decay because,

$$u_{m-1}(t) - u_m(t) = u_m(t) - u_{m-1}(t) = D, \tag{3.2}$$

and

$$B_t = A_t - L = p_{t-1} + \epsilon_t + e^{-\rho\theta} - L. \tag{3.3}$$

However, empirical studies such as [2] and others have shown the covariability of best bid price and best ask price in Euronext and others. Therefore we need to build a model that changes according to the market order of impatient buyers, rather than giving the best bid price exogenously. This is to improve maximum state M and lower bound B. Another way of thinking is to consider the effect of limit buy order on the two side order book model. In the first case(Case 1), system equation in [9] is given as an improvement for B only.

$$\begin{cases} u_1(t) := A_t = p_{t-1} + \epsilon_t + e^{-\rho s}\theta(t-1, q_{t-1}) + S_{t-1}, \\ u_m(t) = \frac{1}{1+c}u_{m+1}(t) + \frac{c}{1+c}u_{m-1} - \frac{r}{\lambda_{PS}+\lambda_{IB}}, \\ u_{M_t}(t) = u_{M_{t-1}}(t) - \frac{r}{\lambda_{IB}+\nu}, \\ u_{M_t}(t) := B_t = A_t - L, \end{cases} \tag{3.4}$$

where

$$S_t := e^{-\rho t}(1 - e^{-\rho}) \sum_{k=1}^{t-1} \theta(k, q_k) e^{\rho k}, \tag{3.5}$$

L is constant, and μ, λ_{PS}, λ_{IB} r, ν, and ρ are fixed. Under this system equation, the expected maximum state M_t is,

$$L = \frac{r}{\lambda_{PS} - \lambda_{IB}} \left\{ \frac{(\lambda_{PS} + \mu)}{(\lambda_{IB} + \mu)} \cdot \frac{(c^{-M_t} - 1)}{(c^{-1} - 1)} - M_t \right\} \tag{3.6}$$

In the second case (Case2), we give the following system equation,

$$\begin{cases} u_1(t) := A_t = p_{t-1} + \theta(t-1, q_{t-1}), \\ u_m(t) = \frac{1}{1+c}u_{m+1}(t) + \frac{c}{1+c}u_{m-1} - \frac{r}{\lambda_{PS}+\lambda_{IB}}, \\ u_{M_t}(t) = u_{M_{t-1}}(t) - \frac{r}{\lambda_{IB}+\nu}, \\ u_{M_t}(t) := B_t == p_{t-1} + \epsilon_t + e^{-\rho s}\theta(t-1, q_{t-1}) + S_{t-1}, \end{cases} \tag{3.7}$$

where

$$S_t := e^{-\rho t}(1 - e^{-\rho}) \sum_{k=1}^{t-1} \theta(k, q_k) e^{\rho k}, \tag{3.8}$$

and μ, λ_{PS}, λ_{IB} r, ν, and ρ are fixed. Under this system equation, the expected maximum state M_t is similarly,

$$\theta(t-1, q_{t-1}) = \frac{1}{1 - e^{-\rho}} \left[\frac{r}{\lambda_{PS} - \lambda_{IB}} \left\{ \frac{(\lambda_{PS} + \mu)}{(\lambda_{IB} + \mu)} \cdot \frac{(c^{-M_t} - 1)}{(c^{-1} - 1)} - M_t \right\} + S_{t-1} \right] \tag{3.9}$$

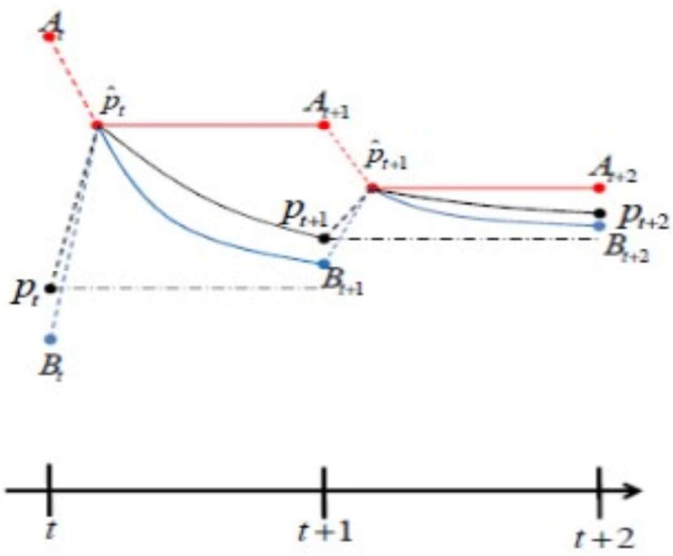

Figure 2: Price process in case2

Remark 5 When considering the application to the optimal execution problem, it is necessary to solve the exponential equation of Equation (3.9) in order to find M, which is the maximum state. If $A - B = constant$, the calculation is not so difficult.

Figure 2 shows the price process in Case2. It can be seen that the lower bound B is gradually increasing. As the range of the upper bound A and the lower bound B narrows over time, the maximum state (maximum number of noise traders who can enter the order book) M decreases. Since the target of the lower bound B is the pre-execution price, the trade information is clearly stated in the price. That is, it reflects the fundamental value.

Moreover, by targeting the pre-execution price as the lower bound, S makes it possible to express a slow boom with a psychological factor.

4 Conclusion

In this paper, we considered the dynamics of the limit order book that can be applied to the optimal execution problem. If impatient buyers do not arrive at the LOB, it is possible to form a block-shaped order book, but not a more realistic non-linear order book. We construct a convex formed order book by considering the impatient buyers. Furthermore, by incorporating the transaction information of a large trader into the best bid price, it is possible to express exponential resiliences and permanent impacts. In particular, if the model the difference between the upper and lower bounds change over time, it is possible to express various liquidity indicators such as bid-ask spread and depth. On the other hand, the complicated LOB dynamics makes it difficult to apply to the optimal execution problem. Derivation of a specific optimal execution strategy in the LOB model, which enables the expression of more liquidity indicators, is our future work.

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