A procedure of listing KKT points for a quadratic fractional programming problem

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Abstract

In this paper, we consider a quadratic fractional programming problem (QFP) whose feasible set is defined by quadratic convex functions. It is known that such a problem can be transformed into a quadratic dc programming problem(QDP). By incorporating a procedure for listing KKT (Karush-Kuhn-Tucker) points of (QDP) into a branch-and-bound procedure, we propose a global optimization algorithm for (QFP).

1 Introduction

In this paper, we consider a quadratic fractional programming problem (QFP) to minimize the ratio of two quadratic convex functions over a convex set defined by quadratic convex functions. It is known that fractional programming is one of the typical problems in Global Optimization. Several types of iterative methods for solving (QFP) have been proposed by many researchers. However, such algorithms are not effective in the case where the dimension of variables is so large. One of the difficulties in solving (QFP) is the complexity of the objective function. Hence, in order to overcome this drawback, we transform (QFP) into a parametric quadratic dc programing problem (QDP) minimizing a quadratic dc function over a convex set. Moreover, to find an approximate solution of (QDP), we introduce an algorithm for listing KKT (Karush-Kuhn-Tucker) points of (QDP). Since every locally optimal solution of (QDP) satisfies KKT conditions, we can calculate most of locally optimal solutions contained in the intersection of the boundaries of convex sets defining the feasible set by utilizing our algorithm. Furthermore, to improve calculation efficiency, we incorporate our algorithm into a branch-and-bound procedure for Lagrange multipliers of constraint functions. The proposed algorithm can calculate an approximate solution of large scale (QFP). The effectiveness of the proposed algorithm has been shown by the result of the computer experiment.

Throughout this paper, we use the following notation: $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of all real numbers and an n-dimensional Euclidean space. The origin of $\mathbb{R}^n$ is denoted by $\mathbf{0}_n$. Given a vector $\boldsymbol{a} \in \mathbb{R}^n$, $\boldsymbol{a}^\top$ denotes the transposed vector of $\boldsymbol{a}$. For given real numbers α and β ($\alpha < \beta$), we set $[\alpha, \beta] := \{x \in \mathbb{R} : \alpha \le x \le \beta\}$, $]\alpha, \beta[:= \{x \in \mathbb{R} : \alpha < x < \beta\}$, $]\alpha, \beta] := \{x \in \mathbb{R} : \alpha < x \le \beta\}$ and $[\alpha, \beta[:= \{x \in \mathbb{R} : \alpha \le x < \beta\}$. The sets of all nonnegative real numbers, all positive real numbers and all nonnegative vectors are denoted by $\mathbb{R}_+$, $\mathbb{R}_{++}$ and $\mathbb{R}^n_+$ respectively, that is, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \ge 0\}$, $\mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}^n_+ := \{\boldsymbol{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n : x_i \ge 0 \ i = 1, \dots, n\}$. Moreover, $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Given a vector $\boldsymbol{a} \in \mathbb{R}^n$, $\|\boldsymbol{a}\|$ denotes the Euclidean norm of $\boldsymbol{a}$, that is, $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a}^\top \boldsymbol{a}}$. Given a vector $\boldsymbol{a} \in \mathbb{R}^n$ and a positive real number

$r \in \mathbb{R}_{++}$, $B^n_<(\boldsymbol{a}, r) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| < r\}$, $B^n_\leq(\boldsymbol{a}, r) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| \leq r\}$ and $B^n_=(\boldsymbol{a}, r) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x} - \boldsymbol{a}\| = r\}$. For a subset $X \subset \mathbb{R}^n$, $\operatorname{int} X$, $\operatorname{ri} X$ $\operatorname{cl} X$, $\operatorname{bd} X$ and $\operatorname{co} X$ denote the interior, the relative interior, the closure, the boundary and the convex hull of X, respectively. Given a nonempty subset $X \subset \mathbb{R}^n$, $\operatorname{span} X$ denotes the subspace spanned by X. For a subset $X \subset \mathbb{R}^n$, $\operatorname{diam} X$ denotes the diameter of X, that is, $\operatorname{diam} X := \max_{x', x'' \in X} \|\boldsymbol{x}' - \boldsymbol{x}''\|$. The $n \times n$ unit matrix is denoted by I_n. Given real numbers $a_1, \ldots, a_n$, $\operatorname{diag}\{a_1, \ldots, a_n\}$ denotes the $n \times n$ diagonal matrix whose diagonal elements are $a_1, \ldots, a_n$. For a given differentiable function $f : \mathbb{R} \to \mathbb{R}$, $\dfrac{d}{dx} f(\bar{x})$ and $\dfrac{d^2}{dx^2} f(\bar{x})$ denote the differential and the second order differential of f at $\bar{x} \in \mathbb{R}$, respectively. For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(\boldsymbol{x})$ denotes the gradient vector of f at $\boldsymbol{x} \in \mathbb{R}^n$. Given a subset $A \subset \mathbb{N}$, $|A|$ denotes the number of elements contained in A.

2 A quadratic fractional programming problem

Let us consider the following quadratic fractional programming problem:

$$
\text{(QFP)} \begin{cases} \text{Minimize} & \dfrac{f_{m+1}(\boldsymbol{x})}{f_{m+2}(\boldsymbol{x})} \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0, \; i = 1, \ldots, m, \quad \boldsymbol{x} \in \mathbb{R}^n, \end{cases}
$$

where

- $f_i(\boldsymbol{x}) := \dfrac{1}{2} \boldsymbol{x}^\top A_i \boldsymbol{x} + \left(\boldsymbol{b}^i\right)^\top \boldsymbol{x} + c_i, \quad i = 1, \ldots, m+1,$
- $f_{m+2}(\boldsymbol{x}) := \dfrac{1}{2} \boldsymbol{x}^\top \boldsymbol{x} + c_{m+2},$
- $A_i \in \mathbb{R}^{n \times n}$ $(i = 1, \ldots, m+1)$ are positive definite symmetric matrices,
- $\boldsymbol{b}^i \in \mathbb{R}^n$ $(i = 1, \ldots, m+1)$ and $c_i \in \mathbb{R}$ $(i = 1, \ldots, m+2)$.

From the definition of A_i, f_i $(i = 1, \ldots, m+2)$ are strictly convex functions. For (QFP), we assume the followings.

(A1) $c_{m+1} > \dfrac{1}{2} \left(\boldsymbol{b}^{m+1}\right)^\top A_{m+1}^{-1} \boldsymbol{b}^{m+1}$

(A2) $c_{m+2} > 0$

(A3) $c_i < 0$ for each $i = 1, \ldots, m$.

By assumption (A3) and the definition of f_i,

$$
f_i(\boldsymbol{0}_n) < 0, \quad i = 1, \ldots, m,
$$

that is, the feasible set of (QFP) is a nonempty compact convex set. Moreover, from assumptions (A1) and (A2), for any $\boldsymbol{x} \in \mathbb{R}^n$,

$$
f_i(\boldsymbol{x}) > 0, \quad i = m+1, m+2.
$$

This implies that the objective function of (QFP) is continuous over $\mathbb{R}^n$ and that $\dfrac{f_{m+1}(\boldsymbol{x})}{f_{m+2}(\boldsymbol{x})} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Therefore,

$$\min(\text{QFP}) > 0, \tag{1}$$

where min(QFP) denotes the optimal value of (QFP).

Now, we consider the following parametric quadratic programming problem with respect to $\alpha \in \mathbb{R}$.

$$(\text{QDP}(\alpha)) \begin{cases} \text{Minimize} & f_{m+1}(\boldsymbol{x}) - \alpha f_{m+2}(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0, \ i = 1, \ldots, m, \quad x \in \mathbb{R}^n. \end{cases}$$

Theorem 2.1 (Jagannathan [3], Theorem 5) *Let $\bar{\alpha} \in \mathbb{R}$ and $x(\bar{\alpha})$ a globally optimal solution of* ((QDP(α)). *Then,* $\min(\text{QDP}(\bar{\alpha})) = 0$ *if and only if $\bar{\alpha}$ and $x(\bar{\alpha})$ are the globally optimal value and a globally optimal solution of* (QFP), *respectively.*

Corollary 2.1 *The following statements hold.*

(i) $\min(\text{QDP}(\alpha)) < 0$ *for each* $\alpha > \min(\text{QFP})$.

(ii) $\min(\text{QDP}(\alpha)) > 0$ *for each* $\alpha < \min(\text{QFP})$.

From Theorem 2.1 and Corollary 2.1, we notice that (QFP) can be solved by finding α satisfying

$$\min(\text{QDP}(\alpha)) = 0. \tag{2}$$

When (2) holds, it is true that $\alpha = \min(\text{QFP})$. Hence, by (1), we can restrict the search range of α as follows.

(A4) $\alpha > 0$

Moreover, we note that the objective function of (QDP(α)) is a dc(difference of two convex functions) function for each $\alpha > 0$. Therefore, under Assumption (A4), (QDP(α)) is a quadratic dc programming problem.

3 Optimality conditions

In this section, we propose optimality conditions for (QDP(α)) under Assumption (A1),...,(A4).

From KKT(Karush-Kuhn-Tucker) conditions for nonlinear programming problem, if $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is a globally optimal solution of (QDP(α)), then there exist Lagrange multipliers $\tilde{s}_1, \ldots, \tilde{s}_m \in \mathbb{R}$ satisfying the following conditions.

(KKT1) $\nabla f_{m+1}(\bar{\boldsymbol{x}}) - \alpha \nabla f_{m+2}(\bar{\boldsymbol{x}}) + \sum_{i=1}^{m} \tilde{s}_i \nabla f_i(\bar{\boldsymbol{x}}) = \mathbf{0}_n$, i.e.,

$$A_{m+1}\bar{\boldsymbol{x}} + \boldsymbol{b}^{m+1} - \alpha \bar{\boldsymbol{x}} + \sum_{i=1}^{m} \tilde{s}_i \left(A_i \bar{\boldsymbol{x}} + \boldsymbol{b}^i \right) = \mathbf{0}_n \tag{3}$$

(KKT2) $f_i(\bar{\boldsymbol{x}}) \leq 0 \; i = 1, \dots, m$

(KKT3) $\tilde{s}_i f_i(\bar{\boldsymbol{x}}) = 0$, $\tilde{s}_i \geq 0$, $i = 1, \dots, m$

We set $\bar{s}_1, \dots, \bar{s}_{m+1} \in \mathbb{R}$ as follows.

$$\bar{s}_i := \frac{\tilde{s}_i}{1 + \sum_{j=1}^m \tilde{s}_j}, \; i = 1, \dots, m$$
$$\bar{s}_{m+1} := \frac{1}{1 + \sum_{j=1}^m \tilde{s}_j}$$

Then,

$$\sum_{i=1}^{m+1} \bar{s}_i = 1, \; \bar{s}_i \geq 0, \; i = 1, \dots, m, \; s_{m+1} > 0.$$

Hence,

$$\bar{\boldsymbol{s}} = (\bar{s}_1, \dots, \bar{s}_{m+1})^\top \in S, \tag{4}$$

where

$$S := \left\{ \boldsymbol{s} \in \mathbb{R}^{m+1} : \sum_{i=1}^{m+1} s_i = 1, \; s_i \geq 0, \; i = 1, \dots, m \right\}.$$

By dividing both sides of equation (3) by $1 + \sum_{i=1}^{m} \tilde{s}_i$, (KKT1) can be rewritten as follows.

(KKT1) $A(\bar{\boldsymbol{s}})\bar{\boldsymbol{x}} + \boldsymbol{b}(\bar{\boldsymbol{s}}) - \alpha \bar{s}_{m+1} \bar{\boldsymbol{x}} = \boldsymbol{0}_n$

Here,

$$A(\bar{\boldsymbol{s}}) := \sum_{i=1}^{m+1} \bar{s}_i A_i,$$
$$\boldsymbol{b}(\bar{\boldsymbol{s}}) := \sum_{i=1}^{m+1} \bar{s}_i \boldsymbol{b}^i.$$

From (2), if $\bar{\alpha}$ and $\bar{\boldsymbol{x}}$ are the globally optimal value and a globally optimal solution of (QFP) respectively, the following condition holds.

$$f_{m+1}(\bar{\boldsymbol{x}}) - \bar{\alpha} f_{m+2}(\bar{\boldsymbol{x}}) = 0 \tag{5}$$

Hence, we rewrite (KKT3) by adding (5) as follows.

(KKT3) $\bar{s}_i f_i(\bar{\boldsymbol{x}}) = 0 \; (i = 1, \dots, m)$, $\bar{s}_{m+1} \left(f_{m+1}(\bar{\boldsymbol{x}}) - \bar{\alpha} f_{m+2}(\bar{\boldsymbol{x}}) \right) = 0$ and $\bar{\boldsymbol{s}} \in S$

Moreover, if $(\bar{\boldsymbol{x}}, \bar{\alpha}, \bar{\boldsymbol{s}})$ satisfies (KKT1) and (KKT 3), then $\bar{\boldsymbol{s}} \in S$ and the following condition holds.

$$\begin{aligned}
&\sum_{i=1}^{m+1} \bar{s}_i f_i(\bar{\boldsymbol{x}}) - \bar{\alpha}\bar{s}_{m+1} f_{m+2}(\bar{\boldsymbol{x}}) \\
&= \frac{1}{2}\bar{\boldsymbol{x}}^\top A(\bar{\boldsymbol{s}})\bar{\boldsymbol{x}} + \boldsymbol{b}(\bar{\boldsymbol{s}})^\top \bar{\boldsymbol{x}} + c(\bar{\boldsymbol{s}}) - \frac{1}{2}\bar{\alpha}\bar{s}_{m+1}\bar{\boldsymbol{x}}^\top \bar{\boldsymbol{x}} - \bar{\alpha}\bar{s}_{m+1} c_{m+2} \\
&= \frac{1}{2}\left(A(\bar{\boldsymbol{s}})\bar{\boldsymbol{x}} + \boldsymbol{b}(\bar{\boldsymbol{s}}) - \bar{\alpha}\bar{s}_{m+1}\bar{\boldsymbol{x}}\right)^\top \bar{\boldsymbol{x}} + \frac{1}{2}\boldsymbol{b}(\bar{\boldsymbol{s}})^\top \bar{\boldsymbol{x}} + c(\bar{\boldsymbol{s}}) - \bar{\alpha}\bar{s}_{m+1} c_{m+2} \\
&= \frac{1}{2}\boldsymbol{b}(\bar{\boldsymbol{s}})^\top \bar{\boldsymbol{x}} + c(\bar{\boldsymbol{s}}) - \bar{\alpha}\bar{s}_{m+1} c_{m+2} = 0
\end{aligned} \tag{6}$$

Here, $c(\boldsymbol{s}) = \sum_{i=1}^{m+1} s_i c_i$. Since it is difficult to find $(\boldsymbol{x}, \alpha, \boldsymbol{s})$ satisfying (KKT3), we relax (KKT3) as follows.

(KKT3)' $\bar{\boldsymbol{s}} \in S$ and

$$\frac{1}{2}\boldsymbol{b}(\bar{\boldsymbol{s}})^\top \bar{\boldsymbol{x}} + c(\bar{\boldsymbol{s}}) - \bar{\alpha}\bar{s}_{m+1} c_{m+2} = 0$$

Let $\alpha > 0$ and $\boldsymbol{s} \in S$. Then, $A(\boldsymbol{s})$ is a positive definite symmetric matrix. Hence, there exists an orthogonal matrix $P(\boldsymbol{s})$ satisfying

$$P(\boldsymbol{s})^\top A(\boldsymbol{s}) P(\boldsymbol{s}) = \mathrm{diag}\left\{\lambda_1(\boldsymbol{s}), \ldots, \lambda_n(\boldsymbol{s})\right\} =: \Lambda(\boldsymbol{s})$$

where $\lambda_1(\boldsymbol{s}), \ldots, \lambda_n(\boldsymbol{s})$ are eigen values of $A(\boldsymbol{s})$ such that

$$0 < \lambda_1(\boldsymbol{s}) \leq \cdots \leq \lambda_n(\boldsymbol{s}).$$

By replacing $\bar{\boldsymbol{x}}$ by $P(\boldsymbol{s})\bar{\boldsymbol{y}}$ ($\bar{\boldsymbol{y}} \in \mathbb{R}^n$), (KKT1), (KKT2) and (KKT3)' can be transformed as follows.

(KKT1) $\left(\Lambda(\boldsymbol{s}) - \alpha s_{m+1} I_n\right) \bar{\boldsymbol{y}} = -\bar{\boldsymbol{b}}(\boldsymbol{s})$

(KKT2) $f_i(P(\boldsymbol{s})\bar{\boldsymbol{y}}) \leq 0$, $i = 1, \ldots, m$

(KKT3)' $\boldsymbol{s} \in S$ and

$$\frac{1}{2}\bar{\boldsymbol{b}}(\boldsymbol{s})^\top \boldsymbol{y} + c(\boldsymbol{s}) - \alpha s_{m+1} c_{m+2} = 0$$

Here, $I_n \in \mathbb{R}^{n \times n}$ is the unit matrix and $\bar{\boldsymbol{b}}(\boldsymbol{s}) := P(\boldsymbol{s})^\top \boldsymbol{b}(\boldsymbol{s})$.

4 Procedures for listing KKT points

In order to find $(\boldsymbol{y}, \alpha, \boldsymbol{s})$ satisfying (KKT1) and (KKT3)', for a given $\boldsymbol{s} \in S$, we set $\bar{\lambda}_1(\boldsymbol{s}), \ldots, \bar{\lambda}_{n(s)}$ satisfying the followings.

- For each $i \in \{1, \ldots, n\}$, there exists $\hat{i} \in \{1, \ldots, n(\boldsymbol{s})\}$ such that $\lambda_i(\boldsymbol{s}) = \bar{\lambda}_{\hat{i}}(\boldsymbol{s})$.
- $0 < \bar{\lambda}_1(\boldsymbol{s}) < \bar{\lambda}_2(\boldsymbol{s}) < \cdots < \bar{\lambda}_{n(s)}(\boldsymbol{s})$.

Moreover, for each $\hat{i} \in \{1, \ldots, n(\boldsymbol{s})\}$, we define $J(\boldsymbol{s}, \hat{i})$ and $\ell(\boldsymbol{s}, \hat{i})$ as follows.

$$\begin{aligned} &J(\boldsymbol{s}, \hat{i}) := \{j : \lambda_j(\boldsymbol{s}) = \bar{\lambda}_{\hat{i}}(\boldsymbol{s}),\ j = 1, \ldots, n\}. \\ &\ell(\boldsymbol{s}, \hat{i}) := |J(\boldsymbol{s}, \hat{i})| \end{aligned}$$

Let $\alpha \neq \dfrac{\lambda_i(\boldsymbol{s})}{s_{m+1}}$ for all $i \in \{1, \ldots, n\}$. Then, $\Lambda(\boldsymbol{s}) - \alpha s_{m+1} I_n$ is a non-singular matrix. We define $\boldsymbol{y}(\alpha; \boldsymbol{s})$ as follows.

$$\begin{aligned} \boldsymbol{y}(\alpha; \boldsymbol{s}) &:= -\left(\Lambda(\boldsymbol{s}) - \alpha s_{m+1} I_n\right)^{-1} \boldsymbol{b}(\boldsymbol{s}) \\ &= \left(\frac{-b_1(\boldsymbol{s})}{\lambda_1(\boldsymbol{s}) - \alpha s_{m+1}}, \ldots, \frac{-b_n(\boldsymbol{s})}{\lambda_n(\boldsymbol{s}) - \alpha s_{m+1}} \right)^\top \end{aligned}$$

IF $(\boldsymbol{y}(\alpha; \boldsymbol{s}), \alpha, \boldsymbol{s})$ satisfies (KKT3)', the following equation holds.

$$\begin{aligned} F(\alpha; \boldsymbol{s}) &= \frac{1}{2} \boldsymbol{b}(\boldsymbol{s})^\top \boldsymbol{y}(\alpha; \boldsymbol{s}) + c(\boldsymbol{s}) - \alpha s_{m+1} c_{m+2} \\ &= -\sum_{i=1}^{n} \frac{b_i(\boldsymbol{s})^2}{2(\lambda_i(\boldsymbol{s}) - \alpha s_{m+1})} + c(\boldsymbol{s}) - \alpha s_{m+1} c_{m+2} = 0 \end{aligned} \tag{7}$$

For every $i \in \{1, \ldots, n+1\}$, we define $L_i(\boldsymbol{s}) \subset \mathbb{R}$ as follows.

$$\begin{aligned} L_1(\boldsymbol{s}) &:= \left] 0, \frac{\lambda_1(\boldsymbol{s})}{s_{m+1}} \right[, \\ L_i(\boldsymbol{s}) &:= \left] \frac{\lambda_{i-1}(\boldsymbol{s})}{s_{m+1}}, \frac{\lambda_i(\boldsymbol{s})}{s_{m+1}} \right[, \quad i = 2, \ldots, n, \\ L_{n+1}(\boldsymbol{s}) &:= \left] \frac{\lambda_n(\boldsymbol{s})}{s_{m+1}}, +\infty \right[. \end{aligned}$$

On each $L_i(\boldsymbol{s})$ $(i = 1, \ldots, n+1)$, the following statements hold.

- $\displaystyle\lim_{\alpha \to \frac{\lambda_{i-1}(s)}{s_{m+1}}} F(\alpha; \boldsymbol{s}) = \infty$ if $b_{i-1}(\boldsymbol{s}) \neq 0$
- $\displaystyle\lim_{\alpha \to \frac{\lambda_i(s)}{s_{m+1}}} F(\alpha; \boldsymbol{s}) = -\infty$ if $b_i(\boldsymbol{s}) \neq 0$
- $\displaystyle\frac{d}{d\alpha} F(\alpha; \boldsymbol{s}) = -\sum_{i=1}^{n} \frac{s_{m+1} b_i(\boldsymbol{s})^2}{2(\lambda_i(\boldsymbol{s}) - \alpha s_{m+1})^2} - s_{m+1} c_{m+2} < 0$ for each $\alpha \in L_i(\boldsymbol{s})$

Therefore, $F(\alpha; \boldsymbol{s})$ is a decreasing function on $L_i(\boldsymbol{s})$. Moreover, we define $F_1^i(\alpha; \boldsymbol{s}), F_2^i(\alpha; \boldsymbol{s}) : L_i(\boldsymbol{s}) \to \mathbb{R}$ respectively as follows.

$$\begin{aligned} &F_1^i(\alpha; \boldsymbol{s}) = -\sum_{j=i}^{n} \frac{b_j(\boldsymbol{s})^2}{2(\lambda_j(\boldsymbol{s}) - \alpha s_{m+1})} + c(\boldsymbol{s}) - \alpha s_{m+1} c_{m+2},\ i = 1, \ldots, n, \\ &F_1^{n+1}(\alpha; \boldsymbol{s}) = 0, \\ &F_2^1(\alpha; \boldsymbol{s}) = 0, \\ &F_2^i(\alpha; \boldsymbol{s}) = \sum_{j=1}^{i-1} \frac{b_j(\boldsymbol{s})^2}{2(\lambda_j(\boldsymbol{s}) - \alpha s_{m+1})},\ i = 2, \ldots, n+1. \end{aligned}$$

Then, $F(\alpha;\boldsymbol{s})$ can be written as follows on every $L_i(\boldsymbol{s})$.

$$F(\alpha;\boldsymbol{s}) = F_1^i(\alpha;\boldsymbol{s}) - F_2^i(\alpha;\boldsymbol{s})$$

On each $L_i(\boldsymbol{s})$ $(i = 1,\ldots,n+1)$, we obtain the second derivative of $F_1^i(\alpha;\boldsymbol{s})$ and $F_2^i(\alpha;\boldsymbol{s})$ respectively as follows.

$$\frac{d^2}{d\,\alpha^2}F_1^i(\alpha;\boldsymbol{s}) = -\sum_{j=i}^{n}\frac{b_j(\boldsymbol{s})^2 s_{m+1}^2}{(\lambda_j(\boldsymbol{s}) - \alpha s_{m+1})^3} > 0,\ i = 1,\ldots,n$$

$$\frac{d^2}{d\,\alpha^2}F_1^{n+1}(\alpha;\boldsymbol{s}) = 0$$

$$\frac{d^2}{d\,\alpha^2}F_2^1(\alpha;\boldsymbol{s}) = 0$$

$$\frac{d^2}{d\,\alpha^2}F_2^i(\alpha;\boldsymbol{s}) = \sum_{j=1}^{i-1}\frac{b_j(\boldsymbol{s})^2 s_{m+1}^2}{(\lambda_j(\boldsymbol{s}) - \alpha s_{m+1})^3} > 0,\ i = 2,\ldots,n+1$$

This implies that $F_1^i(\alpha;\boldsymbol{s})$ and $F_2^i(\alpha;\boldsymbol{s})$ are convex on each $L_i(\boldsymbol{s})$. Therefore, $F(\alpha;\boldsymbol{s})$ is a decreasing dc function on each $L_i(\boldsymbol{s})$.

Let $\alpha' \in L_i(\boldsymbol{s})$. Then, we consider the following two cases.

Case I: $F(\alpha';\boldsymbol{s}) < 0$

Case II: $F(\alpha';\boldsymbol{s}) > 0$

In Case I, we define $G_1(\alpha;\alpha',i,\boldsymbol{s}) : L_i(\boldsymbol{s}) \to \mathbb{R}$ as follows.

$$G_1(\alpha;\alpha',i,\boldsymbol{s}) := F_1^i(\alpha;\boldsymbol{s}) - F_2^i(\alpha';\boldsymbol{s}) - \frac{d}{d\,\alpha}F_2^i(\alpha';\boldsymbol{s})\cdot(\alpha - \alpha')$$

We note that $G_1(\alpha;\alpha',i,\boldsymbol{s})$ is convex on $L_i(\boldsymbol{s})$. Moreover, from the convexity of $F_2^i(\alpha;\boldsymbol{s})$ on $L_i(\boldsymbol{s})$, the following inequalities holds.

$$G_1(\alpha;\alpha',i,\boldsymbol{s}) \geq F(\alpha;\boldsymbol{s}) \quad \text{for each } \alpha \in L_i(\boldsymbol{s})$$

In Case II, we define $G_2(\alpha;\alpha',i,\boldsymbol{s}) : L_i(\boldsymbol{s}) \to \mathbb{R}$ as follows.

$$G_2(\alpha;\alpha',i,\boldsymbol{s}) := F_1^i(\alpha';\boldsymbol{s}) + \frac{d}{d\,\alpha}F_1^i(\alpha';\boldsymbol{s})\cdot(\alpha - \alpha') - F_2^i(\alpha;\boldsymbol{s})$$

We note that $G_2(\alpha;\alpha',i,\boldsymbol{s})$ is concave on $L_i(\boldsymbol{s})$. Moreover, from the convexity of $F_1^i(\alpha;\boldsymbol{s})$ on $L_i(\boldsymbol{s})$, the following inequalities holds.

$$G_2(\alpha;\alpha',i,\boldsymbol{s}) \leq F(\alpha;\boldsymbol{s}) \quad \text{for each } \alpha \in L_i(\boldsymbol{s})$$

For given $\boldsymbol{s} \in S$ and $i \in \{1,\ldots,n+1\}$, to find a solution of $F(\alpha;\boldsymbol{s}) = 0$ on $L_i(\boldsymbol{s})$, we propose Algorithm LKKT as follows.

Algorithm LKKT

Step 0: Choose $\alpha' \in L_i(\boldsymbol{s})$, and go to Step 1.

Step 1: If $F(\alpha'; \boldsymbol{s}) = 0$, then stop. Otherwise go to Step 2.

Step 2: If $F(\alpha'; \boldsymbol{s}) < 0$, calculate α_L satisfying $\alpha_L < \alpha'$ and $G_1(\alpha_L; \alpha', i, \boldsymbol{s}) = 0$, replace α' by α_L and return to Step 1. Otherwise, calculate α_R satisfying $\alpha_R > \alpha'$ and $G_2(\alpha_R; \alpha', i, \boldsymbol{s}) = 0$, replace α' by α_R and return to Step 1.

For given $\alpha' \in L_i(\boldsymbol{s})$ by executing Algorithm LKKT, $P(\boldsymbol{s})\boldsymbol{y}(\alpha'; \boldsymbol{s})$ is a globally optimal solution of (QFP), if $((\boldsymbol{y}(\alpha'; \boldsymbol{s}), \alpha', \boldsymbol{s})$ satisfies (KKT2) and (KKT3).

5 Branch and Bound Procedure

In this section, we propose a branch and bound procedure for calculating a globally optimal solution of (QFP).

In order to execute Algorithm LKKT throughout S, we propose a branch and bound procedure as follows.

Algorithm BBP

Step 0: Set tolerances $\tau, \rho \geq 0$, $\mathcal{S}_1 = \{S\}$, $\boldsymbol{x}^1 = \boldsymbol{0}_n$, $k = 1$, Go to Step 1.

Step 1: If $\mathcal{S}_k = \emptyset$, then stop and output

$$\bar{\boldsymbol{x}} \in \arg\min \left\{ \frac{f_{m+1}(\boldsymbol{x})}{f_{m+2}(\boldsymbol{x})} : \boldsymbol{x} \in \bigcup_{i=1}^{k} \Omega(\boldsymbol{s}^k) \right\}$$

as an approximate solution of (QFP). Otherwise, go to Step 2.

Step 2. Choose $S_k \in \mathcal{S}_k$ satisfying $\operatorname{diam} S_k = \max_{S \in \mathcal{S}_k} \operatorname{diam} S$. Set $\boldsymbol{s}_k$ as follows.

$$\boldsymbol{s}_k := \frac{1}{m} \sum_{i=1,\ldots,m} \boldsymbol{\kappa}^i,$$

where $\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m$ are all vertices of S_k. Go to Step 3.

Step 3: Set $\Omega(\boldsymbol{s}^k)$ of all solutions calculated by Algorithm LKKT for $\boldsymbol{s}^k$ selected at Step 2. Go to Step 4.

Step 4: Choose $\boldsymbol{\kappa}', \boldsymbol{\kappa}'' \in \{\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m\}$ satisfying $\|\boldsymbol{\kappa}' - \boldsymbol{\kappa}''\| = \operatorname{diam} S_k$. Update $\mathcal{S}_{k+1}$ as follows.

$$\mathcal{S}_{k+1} := \begin{cases} (\mathcal{S}_k \cup \{S', S''\}) \setminus \{S_k\}, & \text{if } \operatorname{diam} S' \geq \rho \text{ and } \operatorname{diam} S'' \geq \rho, \\ (\mathcal{S}_k \cup \{S'\}) \setminus \{S_k\}, & \text{if } \operatorname{diam} S' \geq \rho \text{ and } \operatorname{diam} S'' < \rho, \\ (\mathcal{S}_k \cup \{S''\}) \setminus \{S_k\}, & \text{if } \operatorname{diam} S' < \rho \text{ and } \operatorname{diam} S'' \geq \rho, \\ \mathcal{S}_k \setminus \{S_k\}, & \text{if } \operatorname{diam} S' < \rho \text{ and } \operatorname{diam} S'' < \rho, \end{cases}$$

where $S' := \operatorname{co}\left(\{\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m, \check{\boldsymbol{\kappa}}\} \setminus \{\boldsymbol{\kappa}''\}\right)$, $S'' := \operatorname{co}\left(\{\boldsymbol{\kappa}^1, \ldots, \boldsymbol{\kappa}^m, \check{\boldsymbol{\kappa}}\} \setminus \{\boldsymbol{\kappa}'\}\right)$, and $\check{\boldsymbol{\kappa}} := \dfrac{\boldsymbol{\kappa}' - \boldsymbol{\kappa}''}{2}$. Set $k \leftarrow k + 1$ and return to Step 1.

Since S_k is bisected at Step 4 of Algorithm BBP, by setting a tolerance ρ to a positive number, Algorithm BBP is terminates within a finite number of iterations (see, e.g., Theorem IV.1 and Proposition IV.2 in [2]).

6 Conclusions

In this paper, we propose a global optimization algorithm for (QFP). By combining a parametric optimization method, a procedure for listing KKT points and a branch-and-bound procedure, the proposed algorithm can found an approximate solution of (QFP).

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References

[1] M.S. Bazaraa, H.D. Sherali and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*, 3rd ed., John Wiley Sons Inc, United States, 2006.

[2] R. Horst and H. Tuy, *Global Optimization: Deterministic Approaches, Third, Revised and Enlarged Edition*, Springer-Verlag, Berlin, 1996.

[3] R.Jagannathan, "On some properties of programming problems in parametric form pertaining to fractional programming," Management Science, Vol.12, pp.609–615, 1966

[4] S.Yamada, T.Tanaka, T.Tanino, Outer approximation method incorporating a quadratic approximation for a dc programming problem, Journal of Optimization Theory and Applications, Vol.144(1), pp.156–183 (2010)