Mann type approximation theorem by using balanced mappings on geodesic spaces
測地距離空間における balanced mapping を用いた Mann 型収束定理

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Abstract

In this paper, we show a fixed point approximation theorem with the form of balanced mapping using Mann's iterative method for nonexpansive mappings on admissible complete CAT(1) spaces and complete CAT(−1) spaces.

1 Introduction

The CAT(κ) space is one of the metric spaces with convex structures, and several researchers study the convex analysis on those spaces. Fixed point approximation is one of the topics of the convex analysis, and has been studied in CAT(κ) spaces as well. Mann's iterative scheme is the method to generate a sequence converging to a fixed point of mappings produced by Mann [10] in 1953. The most basic form of Mann's iterative scheme is expressed by

$$x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) T x_n \quad (n \in \mathbb{N}),$$

where $\{x_n\}_{n\in\mathbb{N}}$ is an iterative sequence, $\{\alpha_n\}_{n\in\mathbb{N}} \subset [0, 1]$ is a real sequence, T is a mapping that has fixed points, and $\oplus$ is a symbol of the convex combination. Mann [10] showed a convergence theorem with Mann's iterative scheme for a nonexpansive mapping in a Hilbert space. Thereafter, many researchers has been studied that scheme in several spaces. For instance, that scheme is studied on Banach spaces, complete CAT(0) spaces, complete CAT(1) spaces, and so on. Henceforth, we call the theorem using Mann's iterative scheme a Mann type approximation theorem.

In 2008, Dhompongsa and Panyanak [1] proved a Mann type approximation theorem in a complete CAT(0) space. Five years later, Kimura, Saejung, and Yotkaew [8] proved the same type theorem in a complete CAT(1) space. Recently, Kimura [5] showed the following Mann type approximation theorem.

Theorem 1.1 ([5]). *Let X be a complete* CAT(0) *space and $\{T_k\}_{k=1}^N$ nonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \varnothing$. Let $\{\alpha_n\}, \{\beta_n^k\}_{k=1}^N \subset [a,b] \subset]0,1[$ be real sequences such that $\sum_{k=1}^N \beta_n^k = 1$ for all $n \in \mathbb{N}$. Let $x_1 \in X$ arbitrarily and define a sequence $\{x_n\} \subset X$ by*

$$x_{n+1} = \operatorname*{argmin}_{y\in X} \left(\alpha_n d(x_n, y)^2 + (1-\alpha_n) \sum_{k=1}^N \beta_n^k d(T_k x_n, y)^2 \right)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to an element of $\bigcap_{k=1}^N F(T_k)$.

In the assumption of Theorem 1.1, put $\gamma_n^1 = \alpha_n$ and $\gamma_n^{k+1} = (1-\alpha_n)\beta_n^k$ for each $n \in \mathbb{N}$ and $k \in \{1, 2, \dots, N\}$, and let $S_1 \colon X \to X$ be the identity mapping and put $S_{k+1} = T_k$ for each $k \in \{1, 2, \dots, N\}$. Then we can describe the iteration of Theorem 1.1 as

$$\text{(i)} \qquad x_{n+1} = \operatorname*{argmin}_{y\in X} \sum_{k=1}^{N+1} \gamma_n^k d(S_k x_n, y)^2,$$

and $\sum_{k=1}^{N+1} \gamma_n^k = 1$ holds. The right-hand side of (i) has the form of the balanced mapping proposed by Hasegawa and Kimura [3].

In the above theorem, the equation on the right-hand side can be described as a balanced mapping. In this paper, we show a fixed point approximation theorem with the form of balanced mapping using Mann's iterative method on admissible complete CAT(1) spaces and complete CAT(-1) spaces.

2 Preliminaries

Let X be a metric space with a metric d. For a mapping $T \colon X \to X$, the set of all fixed points of T is denoted by $F(T)$. A mapping $\gamma \colon [0,1] \to X$ is called a geodesic joining $x \in X$ and $y \in X$ if $\gamma(0) = x$, $\gamma(1) = y$, and $d(\gamma(s), \gamma(t)) = |s-t|\, d(x,y)$ hold for any $s, t \in [0,1]$. For $D \in]0, \infty]$, X is called a uniquely D-geodesic space if there exists a unique geodesic $\gamma \colon [0,1] \to X$ for any $x, y \in X$ with $d(x,y) < D$. In particular, ∞-geodesic space is merely said a geodesic space. For a uniquely D-geodesic space, the image of the geodesic joining x and y is denoted by $[x, y]$.

Let (X, d) be a uniquely D-geodesic space. For two points $x, y \in X$, we define a convex combination $tx \oplus (1-t)y$ by $\gamma(1-t)$ for $t \in [0,1]$, where γ is a geodesic joining x and y. A subset C of X is said to be convex if $[x, y] \subset C$ holds for any $x, y \in C$. For points $x, y, z \in X$, we define a geodesic triangle $\triangle(x, y, z)$ with vertices $x, y, z \in X$ by the union $[x, y] \cup [y, z] \cup [z, x]$.

Let $\kappa \in \{-1, 0, 1\}$ and (X, d) a uniquely D_κ-geodesic space, where $D_{-1} = D_0 = \infty$, and $D_1 = \pi$. Let us denote a 2-dimensional model space with the constant curvature κ by (M_κ, d'), where d' is a metric on M_κ. To be accurate, M_{-1} is the 2-dimensional hyperbolic space, M_0 is the 2-dimensional Euclidean space, and M_1 is the 2-dimensional unit sphere. M_κ is one of the uniquely D_κ-geodesic spaces. Then for each geodesic triangle $\triangle(x, y, z)$ on X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, there exists a geodesic triangle with vertices $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z}) \subset M_\kappa$ such that $d(x, y) = d'(\overline{x}, \overline{y})$, $d(y, z) = d'(\overline{y}, \overline{z})$, and $d(z, x) = d'(\overline{z}, \overline{x})$. That triangle on M_κ is called a comparison triangle of $\triangle(x, y, z)$. For a triangle $\triangle(x, y, z) \subset X$ with its comparison triangle $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z}) \subset M_\kappa$, and two vertices $s_1, s_2 \in \{x, y, z\}$ and these corresponding vertices $\overline{s_1}$ and $\overline{s_2}$, there exists a point $\overline{p} \in [\overline{s_1}, \overline{s_2}]$ for each $p \in [s_1, s_2]$ such that $d(s_1, p) = d'(\overline{s_1}, \overline{p})$. Such a point $\overline{p} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$ is called a comparison point of p. X is called a CAT(κ) space if, for any $\triangle(x, y, z) \subset X$ with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ and $\overline{\triangle}(\overline{x}, \overline{y}, \overline{z}) \subset M_\kappa$, the inequality $d(p, q) \leq d'(\overline{p}, \overline{q})$ always holds for $p, q \in \triangle(x, y, z)$ and their comparison points $\overline{p}, \overline{q} \in \overline{\triangle}(\overline{x}, \overline{y}, \overline{z})$. A CAT($\kappa$) space is said to be admissible if the distance between any two points is less than $D_\kappa/2$. Every CAT(-1) space is admissible obviously.

Let $\kappa \in \{-1, 1\}$ and X a CAT(κ) space. Then the following inequalities hold for any $x, y, z \in X$ and $t \in]0, 1[$:

- if $\kappa = -1$, then

$$\begin{aligned}&\cosh d(tx \oplus (1-t)y, z) \sinh d(x, y)\\ &\qquad \leq \cosh d(x, z) \sinh t d(x, y) + \cosh d(y, z) \sinh((1-t)d(x, y)),\end{aligned}$$

- if $\kappa = 1$, then

$$\begin{aligned}&\cos d(tx \oplus (1-t)y, z) \sin d(x, y)\\ &\qquad \geq \cos d(x, z) \sin t d(x, y) + \cos d(y, z) \sin((1-t)d(x, y)).\end{aligned}$$

Considering the convexity or concavity of sinh and sin, we easily obtain the following:

- if $\kappa = -1$, then $\cosh d(tx \oplus (1-t)y, z) \leq t \cosh d(x, z) + (1-t) \cosh d(y, z)$,
- if $\kappa = 1$, then $\cos d(tx \oplus (1-t)y, z) \geq t \cos d(x, z) + (1-t) \cos d(y, z)$

for any $x, y, z \in X$ and $t \in]0, 1[$.

Let X be a set and f a function from X into $\mathbb{R}$. We write the set of all minimizers (resp. maximizers) of f as $\operatorname{argmin}_{x \in X} f(x)$ (resp. $\operatorname{argmax}_{x \in X} f(x)$). For a set X and a mapping $T \colon X \to X$, let us denote the set of all fixed points of T by $F(T)$.

Let X be a metric space with a metric d. For $\{x_n\} \subset X$, define an asymptotic center of $\{x_n\}$ by $\operatorname{argmin}_{x \in X} \limsup_{n \to \infty} d(x, x_n)$, and denote it by $AC(\{x_n\})$. The sequence $\{x_n\}$ is said to Δ-converge to x_0 if $AC(\{x_{n_i}\}) = \{x_0\}$ holds for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, and then refer to x_0 as the Δ-limit of $\{x_n\}$. The Δ-convergence of $\{x_n\}$ to x_0 is expressed by the notation $x_n \xrightarrow{\Delta} x_0$.

Let X be a complete CAT(1) space. Then a sequence $\{x_n\} \subset X$ is said to be spherically bounded if $\inf_{y\in X} \limsup_{n\to\infty} d(x_n, y) < \pi/2$ holds. Then the asymptotic center of any spherically bounded sequence $\{x_n\}$ is exactly one point, and $\{x_n\}$ has Δ-convergent subsequences. See [2] for details. Incidentally, suppose X is a complete CAT(0) space, then the asymptotic center of any bounded sequence $\{x_n\}$ is exactly one point, and $\{x_n\}$ has Δ-convergent subsequences [9]. These natures contribute to prove theorems for Mann's iterative schemes on CAT(κ) spaces.

3 Main results

In this section, we prove approximation theorems using a Mann's iterative scheme with balanced mappings on a CAT(1) space and a CAT(-1) space.

Lemma 3.1 ([4]). *Let X be an admissible complete* CAT(1) *space and C a nonempty closed convex subset of X. For $u_1, u_2, \dots, u_N \in X$ and $\alpha^1, \alpha^2, \dots, \alpha^N \in [0,1]$ with $\sum_{k=1}^N \alpha^k = 1$, define a function $g\colon X \to]0,1]$ by $g(x) = \sum_{k=1}^N \alpha^k \cos d(u_k, x)$ for each $x \in X$. Then g has a unique maximizer on C.*

Let X be an admissible complete CAT(1) space and $\{\alpha^k\}_{k=1}^N$ a real sequence on $[0,1]$ satisfying $\sum_{k=1}^N \alpha^k = 1$, and let $\{T_k\}_{k=1}^N$ be mappings from X into itself. Then a mapping $U\colon X \to X$ defined by

$$Ux = \operatorname*{argmax}_{y\in X} \sum_{k=1}^N \alpha^k \cos d(T_k x, y) \quad (x \in X)$$

is well-defined as a single-valued mapping from Lemma 3.1. This mapping U is called a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$ on an admissible complete CAT(1) space.

Lemma 3.2 ([7]). *Let X be an admissible complete* CAT(1) *space and let $\{T_k\}_{k=1}^N$ be mappings from X into itself. Let $\{\alpha^k\}_{k=1}^N \subset]0,1[$ such that $\sum_{k=1}^N \alpha^k = 1$ and $U\colon X \to X$ a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then the following inequality holds for any $x, z \in X$:*

$$\sum_{k=1}^N \alpha^k \cos d(T_k x, Ux) \cos d(Ux, z) \geq \sum_{k=1}^N \alpha^k \cos d(T_k x, z).$$

Lemma 3.3 ([4]). *Let X be an admissible complete* CAT(1) *space and let $\{T_k\}_{k=1}^N$ be quasinonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha^k\}_{k=1}^N \subset]0,1[$ such that $\sum_{k=1}^N \alpha^k = 1$ and $U\colon X \to X$ a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then $F(U) = \bigcap_{k=1}^N F(T_k)$ holds, and U is also quasinonexpansive.*

Using above lemmas, we prove the following convergence theorem on an admissible complete CAT(1) space.

Theorem 3.4. *Let X be an admissible complete* CAT(1) *space and let $\{T_k\}_{k=1}^N$ be nonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n^k\}_{k=1}^N \subset [a,b] \subset]0,1[$ be real sequences such that $\sum_{k=1}^N \beta_n^k = 1$ for all $n \in \mathbb{N}$. Let $x_1 \in X$ arbitrarily and define a sequence $\{x_n\} \subset X$ by*

$$x_{n+1} = \operatorname*{argmax}_{y \in X} \left(\alpha_n \cos d(x_n, y) + (1-\alpha_n) \sum_{k=1}^N \beta_n^k \cos d(T_k x_n, y) \right)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to an element of $\bigcap_{k=1}^N F(T_k)$.

Proof. Define a mapping $U_n \colon X \to X$ by $U_n x = \operatorname{argmax}_{y \in X} \big(\alpha_n \cos d(x_n, y) + (1-\alpha_n) \sum_{k=1}^N \beta_n^k \cos d(T_k x_n, y)\big)$ for each $x \in X$ and $n \in \mathbb{N}$. The well-definedness of the mapping U_n is guaranteed by Lemma 3.1. Then $x_{n+1} = U_n x_n$ holds for any $n \in \mathbb{N}$. From Lemma 3.3, we have $F(U_n) = \bigcap_{k=1}^N F(T_k)$ for all $n \in \mathbb{N}$, and we also get the quasinonexpansiveness of U_n.

Let $p \in \bigcap_{n=1}^\infty F(T_k)$. Then we have $d(x_{n+1}, p) = d(U_n x_n, p) \le d(x_n, p)$ and hence $\{d(x_n, p)\}$ is decreasing. It means that the real sequence $\{d(x_n, p)\}$ has the limit value for any $p \in \bigcap_{k=1}^N F(T_k)$.

We show $\lim_{n\to\infty} d(T_k x_n, x_n) = 0$ for any $k = 1, 2, \ldots, N$. Let $t \in]0,1[$. Using Lemma 3.2, we have

$$\begin{aligned} &\left(\alpha_n \cos d(x_n, U_n x_n) + (1-\alpha_n) \sum_{k=1}^N \beta^k \cos d(T_k x_n, U_n x_n) \right) \cos d(U_n x_n, p) \\ &\ge \alpha_n \cos d(x_n, p) + (1-\alpha_n) \sum_{k=1}^N \beta^k \cos d(T_k x_n, p) \\ &\ge \cos d(x_n, p) \end{aligned}$$

and thus

$$\text{(ii)} \qquad \alpha_n \cos d(x_n, U_n x_n) + (1-\alpha_n) \sum_{k=1}^N \beta^k \cos d(T_k x_n, U_n x_n) \ge \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}.$$

From (ii), we obtain

$$\alpha_n \cos d(x_n, U_n x_n) + 1 - \alpha_n \ge \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}.$$

Since $\alpha_n \ge a$, we get

$$-a(1 - \cos d(x_n, U_n x_n)) + 1 \ge \frac{\cos d(x_n, p)}{\cos d(U_n x_n, p)}$$

and thus

$$1-\cos d(x_n,U_nx_n)\le \frac{1}{a}\left(1-\frac{\cos d(x_n,p)}{\cos d(x_{n+1},p)}\right)\to 0 \quad (n\to\infty).$$

Therefore we have $\lim_{n\to\infty} d(x_n,U_nx_n)=0$.

Moreover, from (ii), we get

$$\alpha_n+(1-\alpha_n)\sum_{k=1}^{N}\beta^k\cos d(T_kx_n,U_nx_n)\ge \frac{\cos d(x_n,p)}{\cos d(U_nx_n,p)}$$

and it implies that

$$(1-\alpha_n)\sum_{k=1}^{N}\beta^k(1-\cos d(T_kx_n,U_nx_n))\le 1-\frac{\cos d(x_n,p)}{\cos d(U_nx_n,p)}.$$

Fix $k\in\{1,\,2,\,\ldots,\,N\}$. Then we obtain

$$(1-\alpha_n)\beta^k(1-\cos d(T_kx_n,U_nx_n))\le 1-\frac{\cos d(x_n,p)}{\cos d(U_nx_n,p)}$$

and hence

$$1-\cos d(T_kx_n,U_nx_n)\le \frac{1}{(1-b)a}\left(1-\frac{\cos d(x_n,p)}{\cos d(U_nx_n,p)}\right)\to 0 \quad (n\to\infty).$$

Thus we have $\lim_{k\to\infty} d(T_kx_n,U_nx_n)=0$ for any $k=1,2,\ldots,N$ and hence we obtain $\lim_{n\to\infty} d(T_kx_n,x_n)=0$ for any $k=1,2,\ldots,N$.

Let $x_0\in AC(\{x_n\})$ and $\{x_{n_i}\}$ a subsequence of $\{x_n\}$, and let $u\in AC(\{x_{n_i}\})$. For $k=1,2,\ldots,N$, we get

$$\begin{aligned}\limsup_{i\to\infty} d(x_{n_i},u) &\le \limsup_{i\to\infty} d(x_{n_i},T_ku)\\ &\le \limsup_{i\to\infty}\,(d(x_{n_i},T_kx_{n_i})+d(T_kx_{n_i},T_ku))\\ &\le \limsup_{i\to\infty} d(x_{n_i},u)\end{aligned}$$

and thus $\limsup_{i\to\infty} d(x_{n_i},u)=\limsup_{i\to\infty} d(x_{n_i},T_ku)$. By the uniqueness of the asymptotic center of $\{x_{n_i}\}$, we have $u\in\bigcap_{k=1}^{N}F(T_k)$. Then we obtain

$$\begin{aligned}\limsup_{n\to\infty} d(x_n,x_0)\le \lim_{n\to\infty} d(x_n,u)&=\lim_{i\to\infty} d(x_{n_i},u)\\ &\le \limsup_{i\to\infty} d(x_{n_i},x_0)\le \limsup_{n\to\infty} d(x_n,x_0).\end{aligned}$$

Therefore we get $\limsup_{n\to\infty} d(x_n,x_0)=\lim_{n\to\infty} d(x_n,u)$. By the uniqueness of the asymptotic center of $\{x_n\}$, we have $x_0=u$. From the definition of the Δ-convergence, we get $x_n\stackrel{\Delta}{\rightharpoonup}x_0\in\bigcap_{k=1}^{N}F(T_k)$, which is the desired result. □

Next, we consider the balanced mapping on complete CAT(-1) spaces. Let X be a complete CAT(-1) space and $\{\alpha^k\}_{k=1}^N$ a real sequence on $[0,1]$ satisfying $\sum_{k=1}^N \alpha^k = 1$, and let $\{T_k\}_{k=1}^N$ be mappings from X into itself. Then a mapping $U\colon X \to X$ defined by

$$Ux = \operatorname*{argmin}_{y \in X} \sum_{k=1}^N \alpha^k \cosh d(T_k x, y) \quad (x \in X)$$

is well-defined as a single-valued mapping from the following Lemma 3.5. This mapping U is called a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$ on a complete CAT(-1) space.

Lemma 3.5. *Let X be a complete* CAT(-1) *space and C a nonempty closed convex subset of X. For $u_1, u_2, \dots, u_N \in X$ and $\alpha^1, \alpha^2, \dots, \alpha^N \in [0,1]$ with $\sum_{k=1}^N \alpha^k = 1$, define a function $g\colon X \to [1, \infty[$ by $g(x) = \sum_{k=1}^N \alpha^k \cosh d(u_k, x)$ for all $x \in X$. Then g has a unique minimizer on C.*

The above theorem can be proved in the same way as Lemma 3.1.

Definition 3.6 ([6])**.** Let X be a uniquely geodesic space. For each $u, v \in X$ and $t \in [0,1]$, we put $tu \overset{-1}{\oplus} (1-t)v = \operatorname{argmin}_{x \in X}(t \cosh d(u,x) + (1-t)\cosh d(v,x))$. If $u \neq v$, then $tu \overset{-1}{\oplus} (1-t)v$ coincides with

$$\left(\frac{1}{D}\tanh^{-1}\frac{t \sinh D}{1 - t + t\cosh D}\right)u \oplus \left(\frac{1}{D}\tanh^{-1}\frac{(1-t)\sinh D}{t + (1-t)\cosh D}\right)v,$$

where $D = d(u,v)$. It obviously holds that $tu \overset{-1}{\oplus} (1-t)v = tu \oplus (1-t)v$ if $u = v$.

Lemma 3.7 ([6])**.** *Let X be a* CAT(-1) *space and let $x, y, z \in X$. Then*

$$\cosh d(\alpha x \overset{-1}{\oplus} (1-\alpha)y, z) \le \frac{\alpha \cosh d(x,z) + (1-\alpha)\cosh d(y,z)}{\sqrt{\alpha^2 + 2\alpha(1-\alpha)\cosh d(x,y) + (1-\alpha)^2}}$$

holds for any $\alpha \in [0,1]$.

Lemma 3.8. *Let X be a complete* CAT(-1) *space and $\{T_k\}_{k=1}^N$ mappings from X into itself. Let $\{\alpha^k\}_{k=1}^N \subset [0,1]$ such that $\sum_{k=1}^N \alpha^k = 1$ and let U be a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then the following inequality holds for any $x, z \in X$:*

$$\sum_{k=1}^N \alpha^k \cosh d(T_k x, Ux) \cosh d(Ux, z) \le \sum_{k=1}^N \alpha^k \cosh d(T_k x, z).$$

Proof. Let $t \in]0,1[$ and put $D = d(Ux, z)$. By Lemma 3.7 we have

$$\begin{aligned}\sum_{k=1}^{N} \alpha^k \cosh d(T_k x, Ux) &\le \sum_{k=1}^{N} \alpha^k \cosh d(T_k x, tUx \overset{-1}{\oplus} (1-t)z) \\ &\le \sum_{k=1}^{N} \frac{\alpha^k t \cosh d(T_k x, Ux) + \alpha^k (1-t) \cosh d(T_k x, z)}{\sqrt{t^2 + 2t(1-t)\cosh D + (1-t)^2}}.\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\left(1 - \frac{t}{\sqrt{t^2 + 2t(1-t)\cosh D + (1-t)^2}}\right) \sum_{k=1}^{N} \alpha^k \cosh d(T_k x, Ux) \\ &\qquad \le \frac{1-t}{\sqrt{t^2 + 2t(1-t)\cosh D + (1-t)^2}} \sum_{k=1}^{N} \alpha^k \cosh d(T_k x, z)\end{aligned}$$

and it implies that

$$\begin{aligned}&\frac{2t(1-t)\cosh D + (1-t)^2}{t + \sqrt{t^2 + 2t(1-t)\cosh D + (1-t)^2}} \sum_{k=1}^{N} \alpha^k \cosh d(T_k x, Ux) \\ &\qquad \le (1-t) \sum_{k=1}^{N} \alpha^k \cosh d(T_k x, z).\end{aligned}$$

Dividing both sides by $1-t$ and letting $t \to 1$, we obtain the desired result. □

The proof of above theorem used the different type of convex combination $\overset{-1}{\oplus}$, but actually it can be proven by using ordinal convex combination. The reason we used that convex combination is that it is simpler to calculate than the ordinal one.

Lemma 3.9. *Let X be a complete* CAT(-1) *space and $\{T_k\}_{k=1}^N$ quasinonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha^k\}_{k=1}^N \subset]0,1[$ such that $\sum_{k=1}^N \alpha^k = 1$ and $U\colon X \to X$ a balanced mapping for $\{\alpha^k\}$ and $\{T_k\}$. Then $F(U) = \bigcap_{k=1}^N F(T_k)$ holds, and U is also quasinonexpansive.*

Proof. Let $z \in \bigcap_{k=1}^N F(T_k)$. Then we have

$$Uz = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha^k \cosh d(T_k z, y) = \operatorname*{argmin}_{y \in X} \sum_{k=1}^{N} \alpha^k \cosh d(z, y) = z$$

and thus $F(U) \supset \bigcap_{k=1}^N F(T_k)$ holds.

We show $F(U) \subset \bigcap_{k=1}^N F(T_k)$. Let $z \in F(U)$ and $w \in \bigcap_{k=1}^N F(T_k)$. By Lemma 3.8, we have

$$\begin{aligned}\sum_{k=1}^N \alpha^k \cosh d(T_k z, z) \cosh d(z, w) &= \sum_{k=1}^N \alpha^k \cosh d(T_k z, Uz) \cosh d(Uz, w) \\ &\le \sum_{k=1}^N \alpha^k \cosh d(T_k z, w) \le \cosh d(z, w).\end{aligned}$$

Hence we get $\sum_{k=1}^N \alpha^k \cosh d(T_k z, z) \le 1$ and it implies $z \in \bigcap_{k=1}^N F(T_k)$.

Finally we show the quasinonexpansiveness of U. Let $x \in X$ and $z \in F(U)$. Then we have $z \in \bigcap_{k=1}^N F(T_k)$. From Lemma 3.8, we get

$$\begin{aligned}\cosh d(Ux, z) &\le \sum_{k=1}^N \alpha^k \cosh d(T_k x, Ux) \cosh d(Ux, z) \\ &\le \sum_{k=1}^N \alpha^k \cosh d(T_k x, z) \le \cosh d(x, z)\end{aligned}$$

and it implies $d(Ux, z) \le d(x, z)$. Therefore we get the conclusion. □

Using above lemmas, we get the following result.

Theorem 3.10. *Let X be a complete* CAT(-1) *space and $\{T_k\}_{k=1}^N$ nonexpansive mappings from X into itself such that $\bigcap_{k=1}^N F(T_k) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n^k\}_{k=1}^N \subset [a, b] \subset]0, 1[$ be real sequences such that $\sum_{k=1}^N \beta_n^k = 1$ for all $n \in \mathbb{N}$. Let $x_1 \in X$ arbitrarily and define a sequence $\{x_n\} \subset X$ by*

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left(\alpha_n \cosh d(x_n, y) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cosh d(T_k x_n, y) \right)$$

for each $n \in \mathbb{N}$. Then $\{x_n\}$ Δ-converges to an element of $\bigcap_{k=1}^N F(T_k)$.

Proof. Define a mapping $U_n \colon X \to X$ by $U_n x = \operatorname{argmin}_{y \in X} \big(\alpha_n \cosh d(x_n, y) + (1 - \alpha_n) \sum_{k=1}^N \beta_n^k \cosh d(T_k x_n, y) \big)$ for each $x \in X$ and $n \in \mathbb{N}$. The well-definedness of the mapping U_n is guaranteed by Lemma 3.5. Then $x_{n+1} = U_n x_n$ holds for any $n \in \mathbb{N}$. From Lemma 3.9, we have $F(U_n) = \bigcap_{k=1}^N F(T_k)$ for all $n \in \mathbb{N}$, and we also get the quasinonexpansiveness of U_n.

Let $p \in \bigcap_{n=1}^\infty F(T_k)$. Then we have $d(x_{n+1}, p) = d(U_n x_n, p) \le d(x_n, p)$ and hence $\{d(x_n, p)\}$ is decreasing. It means that the real sequence $\{d(x_n, p)\}$ has the limit value for any $p \in \bigcap_{k=1}^N F(T_k)$.

We show $\lim_{n\to\infty} d(T_k x_n, x_n) = 0$ for any $k = 1, 2, \ldots, N$. Let $t \in]0,1[$. Using Lemma 3.8, we have

$$\left(\alpha_n \cosh d(x_n, U_n x_n) + (1-\alpha_n)\sum_{k=1}^{N} \beta^k \cosh d(T_k x_n, U_n x_n)\right) \cosh d(U_n x_n, p)$$
$$\le \alpha_n \cosh d(x_n, p) + (1-\alpha_n)\sum_{k=1}^{N} \beta^k \cosh d(T_k x_n, p)$$
$$\le \cosh d(x_n, p)$$

and thus

$$\text{(iii)} \quad \alpha_n \cosh d(x_n, U_n x_n) + (1-\alpha_n)\sum_{k=1}^{N} \beta^k \cosh d(T_k x_n, U_n x_n) \le \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}.$$

From (iii), we obtain

$$\alpha_n \cosh d(x_n, U_n x_n) + 1 - \alpha_n \le \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}.$$

Since $\alpha_n \ge a$, we get

$$a(\cosh d(x_n, U_n x_n) - 1) + 1 \le \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}$$

and thus

$$\cosh d(x_n, U_n x_n) - 1 \le \frac{1}{a}\left(\frac{\cosh d(x_n, p)}{\cosh d(x_{n+1}, p)} - 1\right) \to 0 \quad (n \to \infty).$$

Therefore we have $\lim_{n\to\infty} d(x_n, U_n x_n) = 0$.

Moreover, from (iii), we get

$$\alpha_n + (1-\alpha_n)\sum_{k=1}^{N} \beta^k \cosh d(T_k x_n, U_n x_n) \le \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)}$$

and it implies that

$$(1-\alpha_n)\sum_{k=1}^{N} \beta^k (\cosh d(T_k x_n, U_n x_n) - 1) \le \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)} - 1.$$

Fix $k \in \{1, 2, \ldots, N\}$. Then we obtain

$$(1-\alpha_n)\beta^k (\cosh d(T_k x_n, U_n x_n) - 1) \le \frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)} - 1$$

and hence

$$\cosh d(T_k x_n, U_n x_n) - 1 \leq \frac{1}{(1-b)a}\left(\frac{\cosh d(x_n, p)}{\cosh d(U_n x_n, p)} - 1\right) \to 0 \quad (n \to \infty).$$

Thus we have $\lim_{k\to\infty} d(T_k x_n, U_n x_n) = 0$ for any $k = 1, 2, \dots, N$ and hence we obtain $\lim_{n\to\infty} d(T_k x_n, x_n) = 0$ for any $k = 1, 2, \dots, N$.

Let $x_0 \in AC(\{x_n\})$ and $\{x_{n_i}\}$ a subsequence of $\{x_n\}$, and let $u \in AC(\{x_{n_i}\})$. For $k = 1, 2, \dots, N$, we get

$$\begin{aligned}
\limsup_{i\to\infty} d(x_{n_i}, u) &\leq \limsup_{i\to\infty} d(x_{n_i}, T_k u) \\
&\leq \limsup_{i\to\infty} \left(d(x_{n_i}, T_k x_{n_i}) + d(T_k x_{n_i}, T_k u)\right) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, u)
\end{aligned}$$

and thus $\limsup_{i\to\infty} d(x_{n_i}, u) = \limsup_{i\to\infty} d(x_{n_i}, T_k u)$. By the uniqueness of the asymptotic center of $\{x_{n_i}\}$, we have $u \in \bigcap_{k=1}^{N} F(T_k)$. Then we obtain

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, x_0) \leq \lim_{n\to\infty} d(x_n, u) = \lim_{i\to\infty} d(x_{n_i}, u)& \\
\leq \limsup_{i\to\infty} d(x_{n_i}, x_0) \leq \limsup_{n\to\infty} d(x_n, x_0)&.
\end{aligned}$$

Therefore we get $\limsup_{n\to\infty} d(x_n, x_0) = \lim_{n\to\infty} d(x_n, u)$. By the uniqueness of the asymptotic center of $\{x_n\}$, we have $x_0 = u$. From the definition of the Δ-convergence, we get $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \bigcap_{k=1}^{N} F(T_k)$, which is the desired result. □

Theorem 3.4 is a convergence theorem on admissible complete CAT(1) spaces, and it targets at nonexpamsive mappings. Since we have not found many examples of nonexpansive mappings in CAT(1) spaces, we consider Theorem 3.4 to be less useful. On the other hand, there exists many examples of nonexpansive mappings in CAT(−1) space and thus we consider Theorem 3.10 to be useful.

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