A mathematical analysis of the Kakinuma model for interfacial gravity waves

Tatsuo Iguchi
Department of Mathematics, Keio University

1 Introduction

This article is based on joint researches [4, 5] with Vincent Duchêne (Université de Rennes 1, France). We will consider the motion of the interfacial gravity waves at the interface between two layers of immiscible waters in $(n+1)$-dimensional Euclidean space. Let t be the time, $\boldsymbol{x} = (x_1, \ldots, x_n)$ the horizontal spatial coordinates, and z the vertical spatial coordinate. We assume that the layers are infinite in the horizontal directions, bounded from above by a flat rigid-lid, and from below by a time-independent variable topography, and that the interface, the rigid-lid, and the bottom are represented as $z = \zeta(\boldsymbol{x}, t)$, $z = h_1$, and $z = -h_2 + b(\boldsymbol{x})$, respectively, where $\zeta(\boldsymbol{x}, t)$ is the elevation of the interface, h_1 and h_2 are mean thicknesses of the upper and lower layers, and $b(\boldsymbol{x})$ represents the bottom topography. Therefore, the upper layer $\Omega_1(t)$ and the lower layer $\Omega_2(t)$ of the waters have the form

$$\Omega_1(t) = \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, \zeta(\boldsymbol{x}, t) < z < h_1\},$$
$$\Omega_2(t) = \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, -h_2 + b(\boldsymbol{x}) < z < \zeta(\boldsymbol{x}, t)\}.$$

We denote the interface, the rigid-lid, and the bottom by $\Gamma(t)$, Σ_1, and Σ_2, respectively. We assume that the waters in the upper and the lower layers are both incompressible and inviscid fluids with constant densities ρ_1 and ρ_2, respectively, and that the flows are both irrotational. Then, the motion of the waters is described by the velocity potentials $\Phi_1(\boldsymbol{x}, z, t)$ and $\Phi_2(\boldsymbol{x}, z, t)$ and the pressures $P_1(\boldsymbol{x}, z, t)$ and $P_2(\boldsymbol{x}, z, t)$ in the upper and the lower layers. These velocity potentials should be harmonic and satisfy Bernoulli's equation

$$\rho_\ell \left(\partial_t \Phi_\ell + \frac{1}{2} |\nabla_X \Phi_\ell|^2 + gz \right) + P_\ell = 0 \quad \text{in} \quad \Omega_\ell(t) \qquad (\ell = 1, 2),$$

where $\nabla_X = (\nabla, \partial_z) = (\partial_{x_1}, \ldots, \partial_{x_n}, \partial_z)$ and g is the acceleration due to gravity. We impose the Neumann boundary condition for the velocity potentials on the rigid-lid of the upper layer and the bottom of the lower layer as a kinematical boundary condition. Moreover, on the interface we impose the kinematical boundary conditions

$$\partial_t\Phi_\ell + \nabla\zeta\cdot\nabla\Phi_\ell - \partial_z\Phi_\ell = 0 \quad \text{on} \quad \Gamma(t) \qquad (\ell = 1, 2)$$

and the dynamical boundary condition $P_1 = P_2$ on $\Gamma(t)$. These are basic equations for the interfacial gravity waves and referred as the full model in the following. Throughout this article, we assume Rayleigh's stability condition

$$(\rho_2 - \rho_1)g > 0.$$

As in the case of surface gravity waves (water waves), the full model for the interfacial gravity waves have a variational structure and a Lagrangian is given in terms of velocity potentials Φ_1 and Φ_2 in the upper and the lower layers and the elevation of the interface ζ. We denote the Lagrangian density by $\mathscr{L}(\Phi_1, \Phi_2, \zeta)$. T. Kakinuma [9, 10, 11] approximated the velocity potentials Φ_1 and Φ_2 in the Lagrangian by

$$\Phi_\ell^{\mathrm{app}}(\boldsymbol{x}, z, t) = \sum_{i=0}^{N_\ell} Z_{\ell,i}(z; \tilde{h}_\ell)\phi_{\ell,i}(\boldsymbol{x}, t)$$

for $\ell = 1, 2$, where $\{Z_{1,i}\}$ and $\{Z_{2,i}\}$ are appropriate function systems in the vertical coordinate z and may depend on $\tilde{h}_1(\boldsymbol{x})$ and $\tilde{h}_2(\boldsymbol{x})$, respectively, which are thicknesses of the upper and the lower layers in the rest state, whereas $\boldsymbol{\phi}_\ell = (\phi_{\ell,0}, \phi_{\ell,1}, \dots, \phi_{\ell,N_\ell})^{\mathrm{T}}$, $\ell = 1, 2$, are unknown variables. The Kakinuma model is the Euler–Lagrange equations for an approximated Lagrangian with a density $\mathscr{L}^{\mathrm{app}}(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \zeta) = \mathscr{L}(\Phi_1^{\mathrm{app}}, \Phi_2^{\mathrm{app}}, \zeta)$. According to the analysis of the Isobe–Kakinuma model for surface gravity waves given by Y. Murakami and T. Iguchi [14], R. Nemoto and T. Iguchi [15], and T. Iguchi [6, 7], we will choose the approximated velocity potentials as

$$\begin{cases} \Phi_1^{\mathrm{app}}(\boldsymbol{x}, z, t) = \displaystyle\sum_{i=0}^{N} (-z + h_1)^{2i}\phi_{1,i}(\boldsymbol{x}, t), \\ \Phi_2^{\mathrm{app}}(\boldsymbol{x}, z, t) = \displaystyle\sum_{i=0}^{N^*} (z + h_2 - b(\boldsymbol{x}))^{p_i}\phi_{2,i}(\boldsymbol{x}, t), \end{cases} \tag{1}$$

where N, N^*, and $p_0, p_1, \dots, p_{N^*}$ are nonnegative integers satisfying the conditions

(H1) $N^* = N$ and $p_i = 2i$ $(i = 0, 1, \dots, N)$ in the case of the flat bottom $b(\boldsymbol{x}) \equiv 0$,

(H2) $N^* = 2N$ and $p_i = i$ $(i = 0, 1, \dots, N^*)$ in the case of a general bottom $b(\boldsymbol{x})$.

In this article, we report that the Kakinuma model has a nontrivial stability regime, which can be represented as

$$-\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}}) - \frac{\rho_1\rho_2}{\rho_1 H_2\alpha_2 + \rho_2 H_1\alpha_1}|\nabla\Phi_2^{\mathrm{app}} - \nabla\Phi_1^{\mathrm{app}}|^2 \geq c_0 \quad \text{on} \quad \Gamma(t) \tag{2}$$

with a positive constant c_0, where the approximate pressures $P_\ell^{\rm app}$ ($\ell = 1, 2$) are defined through Bernoulli's equation, $H_1 = h_1 - \zeta$ and $H_2 = h_2 + \zeta - b$ are thicknesses of the upper and the lower layers, and α_1 and α_2 are positive constants depending only on N and converge to 0 as $N \to \infty$. Under this stability condition and compatibility conditions on the initial data, we show that the initial value problem for the Kakinuma model is well-posed in Sobolev spaces. This is not consistent with the fact that the initial value problem for the full model is ill-posed in Sobolev spaces as was shown by T. Iguchi, N. Tanaka, and A. Tani [8]. See also V. Kamotski and G. Lebeau [12] and D. Lannes [13]. Nevertheless, we can show that the Kakinuma model is a higher order shallow water approximation of the full model with an error of order $O(\delta_1^{4N+2} + \delta_2^{4N+2})$, where δ_1 and δ_2 are nondimensional parameters which represent the shallowness of the upper layer and the lower layer, respectively. More precisely, these parameters are defined by $\delta_\ell = \frac{h_\ell}{\lambda}$ ($\ell = 1, 2$), where λ is a typical wavelength in the horizontal direction. Therefore, the Kakinuma model is a desirable model to simulate the interfacial gravity waves in the shallow water regime.

As is well-known that the full model for interfacial gravity waves has a conserved energy which is the sum of the kinetic energies of the waters in the upper and the lower layers and the potential energy due to gravity. Moreover, T. B. Benjamin and T. J. Bridges [1] found that the full model can be written in Hamilton's canonical equations

$$\partial_t \zeta = \frac{\delta \mathscr{H}^{\rm IW}}{\delta \phi}, \qquad \partial_t \phi = -\frac{\delta \mathscr{H}^{\rm IW}}{\delta \zeta},$$

where the canonical variable ϕ is defined by

$$\phi(\boldsymbol{x}, t) = \rho_2 \Phi_2(\boldsymbol{x}, \zeta(\boldsymbol{x}, t), t) - \rho_1 \Phi_1(\boldsymbol{x}, \zeta(\boldsymbol{x}, t), t) \tag{3}$$

and the Hamiltonian $\mathscr{H}^{\rm IW}$ is the total energy. We report that the Kakinuma model has also a Hamiltonian structure with a Hamiltonian $\mathscr{H}^{\rm K}$. Moreover, under an appropriate assumption on the canonical variables (ζ, ϕ), we show the error estimate

$$|\mathscr{H}^{\rm K}(\zeta, \phi) - \mathscr{H}^{\rm IW}(\zeta, \phi)| \lesssim \delta_1^{4N+2} + \delta_2^{4N+2}.$$

2 The basic equations and the Kakinuma model

We first recall the equations governing potential flows for two layers of immiscible, incompressible, homogeneous, and inviscid fluids. The motion of the waters is described by the velocity potentials Φ_1 and Φ_2 and the pressures P_1 and P_2 in the upper and the lower layers satisfying the equations

$$\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 \quad \text{in} \quad \Omega_1(t), \tag{4}$$

$$\Delta\Phi_2 + \partial_z^2\Phi_2 = 0 \quad \text{in} \quad \Omega_2(t), \tag{5}$$

where $\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_n}^2$ is the Laplacian with respect to the horizontal space variables $\boldsymbol{x} = (x_1, \ldots, x_n)$. Bernoulli's equations in each layers have the form

$$\rho_1\left(\partial_t\Phi_1 + \frac{1}{2}|\nabla_X\Phi_1|^2 + gz\right) + P_1 = 0 \quad \text{in} \quad \Omega_1(t), \tag{6}$$

$$\rho_2\left(\partial_t\Phi_2 + \frac{1}{2}|\nabla_X\Phi_2|^2 + gz\right) + P_2 = 0 \quad \text{in} \quad \Omega_2(t). \tag{7}$$

The dynamical boundary condition on the interface is given by

$$P_1 = P_2 \quad \text{on} \quad \Gamma(t). \tag{8}$$

The kinematic boundary conditions on the interface, on the rigid-lid, and on the bottom are given by

$$\partial_t\zeta + \nabla\Phi_1\cdot\nabla\zeta - \partial_z\Phi_1 = 0 \quad \text{on} \quad \Gamma(t), \tag{9}$$

$$\partial_t\zeta + \nabla\Phi_2\cdot\nabla\zeta - \partial_z\Phi_2 = 0 \quad \text{on} \quad \Gamma(t), \tag{10}$$

$$\partial_z\Phi_1 = 0 \quad \text{on} \quad \Sigma_1, \tag{11}$$

$$\nabla\Phi_2\cdot\nabla b - \partial_z\Phi_2 = 0 \quad \text{on} \quad \Sigma_2. \tag{12}$$

These are the basic equations for the internal gravity waves. It follows form Bernoulli's equations (6)–(7) and the dynamical boundary condition (8) that

$$\rho_1\left(\partial_t\Phi_1 + \frac{1}{2}|\nabla_X\Phi_1|^2 + g\zeta\right) - \rho_2\left(\partial_t\Phi_2 + \frac{1}{2}|\nabla_X\Phi_2|^2 + g\zeta\right) = 0 \quad \text{on} \quad \Gamma(t). \tag{13}$$

It is easy to see that the basic equations (4)–(12) for unknowns $(\zeta, \Phi_1, \Phi_2, P_1, P_2)$ are equivalent to (4)–(5) and (9)–(13) for unknowns (ζ, Φ_1, Φ_2), which will be referred as the full model for the interfacial gravity waves in the following.

As in the case of the surface gravity waves, the full model has a variational structure and the corresponding Luke's Lagrangian is given essentially by the vertical integral of the pressure in the water regions. In fact, we first define $\mathscr{L}_{\rm pre}$ by

$$\mathscr{L}_{\rm pre} = \int_{-h_2+b(\boldsymbol{x})}^{\zeta(\boldsymbol{x},t)} P_2(\boldsymbol{x},z,t)\mathrm{d}z + \int_{\zeta(\boldsymbol{x},t)}^{h_1} P_1(\boldsymbol{x},z,t)\mathrm{d}z.$$

By using Bernoulli's equations (6)–(7) to remove the pressures P_1 and P_2, we see that

$$\begin{aligned}\mathscr{L}_{\rm pre} = -\rho_2\int_{-h_2+b}^{\zeta}\left(\partial_t\Phi_2 + \frac{1}{2}|\nabla_X\Phi_2|^2\right)\mathrm{d}z - \rho_1\int_{\zeta}^{h_1}\left(\partial_t\Phi_1 + \frac{1}{2}|\nabla_X\Phi_1|^2\right)\mathrm{d}z \\ - \frac{1}{2}(\rho_2-\rho_1)g\zeta^2 + \frac{1}{2}\left(\rho_2 g(-h_2+b)^2 - \rho_1 g h_1^2\right).\end{aligned}$$

The last term does not contribute the variation of the Lagrangian, so that we define a Lagrangian density $\mathscr{L}(\Phi_1,\Phi_2,\zeta)$ by

$$\mathscr{L}(\Phi_1,\Phi_2,\zeta) = -\rho_2\int_{-h_2+b}^{\zeta}\left(\partial_t\Phi_2+\frac{1}{2}|\nabla_X\Phi_2|^2\right)\mathrm{d}z - \rho_1\int_{\zeta}^{h_1}\left(\partial_t\Phi_1+\frac{1}{2}|\nabla_X\Phi_1|^2\right)\mathrm{d}z$$
$$-\frac{1}{2}(\rho_2-\rho_1)g\zeta^2, \tag{14}$$

and the action function $\mathscr{J}(\Phi_1,\Phi_2,\zeta)$ by

$$\mathscr{J}(\Phi_1,\Phi_2,\zeta) = \int_{t_0}^{t_1}\int_{\mathbf{R}^n}\mathscr{L}(\Phi_1,\Phi_2,\zeta)\mathrm{d}\boldsymbol{x}\mathrm{d}t.$$

The corresponding Euler–Lagrange equation is exactly the same as the full model (4)–(5) and (9)–(13).

Plugging the approximation (1) into the Lagrangian density (14), we obtain an approximate Lagrangian density $\mathscr{L}^{\mathrm{app}}(\boldsymbol{\phi}_1,\boldsymbol{\phi}_2,\zeta) := \mathscr{L}(\Phi_1^{\mathrm{app}},\Phi_2^{\mathrm{app}},\zeta)$. The corresponding Euler–Lagrange equation is the Kakinuma model, which has the form

$$\begin{cases} H_1^{2i}\partial_t\zeta - \displaystyle\sum_{j=0}^{N}\left\{\nabla\cdot\left(\frac{1}{2(i+j)+1}H_1^{2(i+j)+1}\nabla\phi_{1,j}\right) - \frac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,j}\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i=0,1,\dots,N, \\ H_2^{p_i}\partial_t\zeta + \displaystyle\sum_{j=0}^{N^*}\left\{\nabla\cdot\left(\frac{1}{p_i+p_j+1}H_2^{p_i+p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,j}\nabla b\right)\right. \\ \qquad\qquad \left. +\displaystyle\frac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla b\cdot\nabla\phi_{2,j} - \frac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\phi_{2,j}\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i=0,1,\dots,N^*, \\ \rho_1\left\{\displaystyle\sum_{j=0}^{N}H_1^{2j}\partial_t\phi_{1,j} + g\zeta + \frac{1}{2}\left(\left|\sum_{j=0}^{N}H_1^{2j}\nabla\phi_{1,j}\right|^2 + \left(\sum_{j=0}^{N}2jH_1^{2j-1}\phi_{1,j}\right)^2\right)\right\} \\ \quad -\rho_2\left\{\displaystyle\sum_{j=0}^{N^*}H_2^{p_j}\partial_t\phi_{2,j} + g\zeta\right. \\ \qquad \left. +\displaystyle\frac{1}{2}\left(\left|\sum_{j=0}^{N^*}(H_2^{p_j}\nabla\phi_{2,j} - p_jH_2^{p_j-1}\phi_{2,j}\nabla b)\right|^2 + \left(\sum_{j=0}^{N^*}p_jH_2^{p_j-1}\phi_{2,j}\right)^2\right)\right\} = 0, \end{cases} \tag{15}$$

where H_1 and H_2 are thicknesses of the upper and the lower layers, that is,

$$H_1(\boldsymbol{x},t) = h_1 - \zeta(\boldsymbol{x},t), \qquad H_2(\boldsymbol{x},t) = h_2 + \zeta(\boldsymbol{x},t) - b(\boldsymbol{x}).$$

Here and in what follows we use the notational convention $\frac{0}{0}=0$. In this Kakinuma model, we have $(N+N^*+2)$ evolution equations for just one unknown scalar function

ζ whereas we have only one evolution equation for $(N + N^* + 2)$ unknown functions $\boldsymbol{\phi}_1 = (\phi_{1,0}, \phi_{1,1}, \ldots, \phi_{1,N})^{\mathrm{T}}$ and $\boldsymbol{\phi}_2 = (\phi_{2,0}, \phi_{2,1}, \ldots, \phi_{2,N^*})^{\mathrm{T}}$. Therefore, the Kakinuma model is an overdetermined and underdetermined composite system. Anyway the total number of the equations is equal to the total number of unknown functions. We will consider this Kakinuma model (15) in the following.

In the case $N = 0$, that is, if we approximate the velocity potentials in the Lagrangian by functions independent of the vertical spatial variable z as $\Phi_\ell^{\mathrm{app}}(\boldsymbol{x}, z, t) = \phi_\ell(\boldsymbol{x}, t)$ for $\ell = 1, 2$, then the Kakinuma model is reduced to the nonlinear shallow water equations

$$\begin{cases} \partial_t \zeta - \nabla \cdot ((h_1 - \zeta)\nabla\phi_1) = 0, \\ \partial_t \zeta + \nabla \cdot ((h_2 + \zeta - b)\nabla\phi_2) = 0, \\ \rho_1 \left(\partial_t \phi_1 + g\zeta + \dfrac{1}{2}|\nabla\phi_1|^2 \right) - \rho_2 \left(\partial_t \phi_2 + g\zeta + \dfrac{1}{2}|\nabla\phi_2|^2 \right) = 0. \end{cases}$$

The initial value problem to these nonlinear shallow water equations was analyzed by D. Bresch and M. Renardy [2].

3 Stability condition

We linearize the Kakinuma model (15) around an arbitrary flow $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ and denote the variation by $(\dot{\zeta}, \dot{\boldsymbol{\phi}}_1, \dot{\boldsymbol{\phi}}_2)$. By neglecting lower order terms, the linearized equations have the form

$$\begin{cases} \partial_t \dot{\zeta} + \boldsymbol{u}_1 \cdot \nabla\dot{\zeta} - \displaystyle\sum_{j=0}^{N} \frac{1}{2(i+j)+1} H_1^{2j+1} \Delta\dot{\phi}_{1j} = 0 \quad \text{for} \quad i = 0, 1, \ldots, N, \\ \partial_t \dot{\zeta} + \boldsymbol{u}_2 \cdot \nabla\dot{\zeta} + \displaystyle\sum_{j=0}^{N^*} \frac{1}{p_i + p_j + 1} H_2^{p_j+1} \Delta\dot{\phi}_{2j} = 0 \quad \text{for} \quad i = 0, 1, \ldots, N^*, \\ \rho_1 \displaystyle\sum_{j=0}^{N} H_1^{2j} (\partial_t \dot{\phi}_{1j} + \boldsymbol{u}_1 \cdot \nabla\dot{\phi}_{1j}) - \rho_2 \sum_{j=0}^{N^*} H_2^{p_j} (\partial_t \dot{\phi}_{2j} + \boldsymbol{u}_2 \cdot \nabla\dot{\phi}_{2j}) - a\dot{\zeta} = 0, \end{cases} \tag{16}$$

where $\boldsymbol{u}_\ell = (\nabla\Phi_\ell^{\mathrm{app}})|_{z=\zeta}$ for $\ell = 1, 2$ are approximate horizontal velocities in the upper and the lower layers on the interface and $a = -\left(\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}})\right)|_{z=\zeta}$. Here, P_1^{app} and P_2^{app} are approximate pressures in the upper and the lower layers calculated from Bernoulli's equations (6)–(7), that is, $P_\ell^{\mathrm{app}} = -\rho_\ell \left(\partial_t \Phi_1^{\mathrm{app}} + \frac{1}{2}|\nabla_X \Phi_\ell^{\mathrm{app}}|^2 + gz \right)$ for $\ell = 1, 2$.

Now, we freeze the coefficients in the linearized equations (16) and put

$$\begin{cases} \dot{\boldsymbol{\psi}}_1 = (\dot{\phi}_{1,0}, H_1^2 \dot{\phi}_{1,1}, \ldots, H_1^{2N} \dot{\phi}_{1,N})^{\mathrm{T}}, \\ \dot{\boldsymbol{\psi}}_2 = (\dot{\phi}_{2,0}, H_2^{p_1} \dot{\phi}_{2,1}, \ldots, H_2^{p_{N^*}} \dot{\phi}_{2,N^*})^{\mathrm{T}}. \end{cases}$$

Then, the linearized equations (16) can be written in a matrix form

$$\begin{pmatrix} 0 & -\rho_1 \mathbf{1}^{\mathrm{T}} & \rho_2 \mathbf{1}^{\mathrm{T}} \\ H_1 \mathbf{1} & O & O \\ -H_2 \mathbf{1} & O & O \end{pmatrix} \partial_t \begin{pmatrix} \dot{\zeta} \\ \dot{\boldsymbol{\psi}}_1 \\ \dot{\boldsymbol{\psi}}_2 \end{pmatrix} + \begin{pmatrix} a & -\rho_1 \mathbf{1}^{\mathrm{T}}(\boldsymbol{u}_1 \cdot \nabla) & \rho_2 \mathbf{1}^{\mathrm{T}}(\boldsymbol{u}_2 \cdot \nabla) \\ H_1 \mathbf{1}(\boldsymbol{u}_1 \cdot \nabla) & -H_1^2 A_{1,0} \Delta & O \\ -H_2 \mathbf{1}(\boldsymbol{u}_2 \cdot \nabla) & O & -H_2^2 A_{2,0} \Delta \end{pmatrix} \begin{pmatrix} \dot{\zeta} \\ \dot{\boldsymbol{\psi}}_1 \\ \dot{\boldsymbol{\psi}}_2 \end{pmatrix} = \mathbf{0},$$

where $\mathbf{1} = (1, \dots, 1)^{\mathrm{T}}$ and

$$A_{1,0} = \left(\frac{1}{2(i+j)+1} \right)_{0 \le i,j \le N}, \qquad A_{2,0} = \left(\frac{1}{p_i + p_j + 1} \right)_{0 \le i,j \le N^*}.$$

The linearized equations (16) have a nontrivial plane wave solution of the form $\dot{\zeta}(\boldsymbol{x}, t) = \dot{\zeta}_0 \mathrm{e}^{\mathrm{i}(\boldsymbol{\xi} \cdot \boldsymbol{x} - \omega t)}$ and $\dot{\boldsymbol{\psi}}_\ell(\boldsymbol{x}, t) = \dot{\boldsymbol{\psi}}_{\ell,0} \mathrm{e}^{\mathrm{i}(\boldsymbol{\xi} \cdot \boldsymbol{x} - \omega t)}$ for $\ell = 1, 2$ if and only if the wave vector $\boldsymbol{\xi} \in \mathbf{R}^n$ and the angular frequency $\omega \in \mathbf{C}$ satisfy

$$\det \begin{pmatrix} a & \mathrm{i}\rho_1(\omega - \boldsymbol{u}_1 \cdot \boldsymbol{\xi}) \mathbf{1}^{\mathrm{T}} & -\mathrm{i}\rho_2(\omega - \boldsymbol{u}_2 \cdot \boldsymbol{\xi}) \mathbf{1}^{\mathrm{T}} \\ -\mathrm{i} H_1(\omega - \boldsymbol{u}_1 \cdot \boldsymbol{\xi}) \mathbf{1} & (H_1 |\boldsymbol{\xi}|)^2 A_{1,0} & O \\ \mathrm{i} H_2(\omega - \boldsymbol{u}_2 \cdot \boldsymbol{\xi}) \mathbf{1} & O & (H_2 |\boldsymbol{\xi}|)^2 A_{2,0} \end{pmatrix} = 0,$$

which is the linear dispersion relation for the linearized equations (16). We can expand this determinant and the linear dispersion relation can be given simply as

$$\frac{\rho_1}{H_1 \alpha_1} (\omega - \boldsymbol{u}_1 \cdot \boldsymbol{\xi})^2 + \frac{\rho_2}{H_2 \alpha_2} (\omega - \boldsymbol{u}_2 \cdot \boldsymbol{\xi})^2 - a |\boldsymbol{\xi}|^2 = 0, \tag{17}$$

where

$$\alpha_\ell = \frac{\det A_{\ell,0}}{\det \tilde{A}_{\ell,0}}, \qquad \tilde{A}_{\ell,0} = \begin{pmatrix} 0 & \mathbf{1}^{\mathrm{T}} \\ -\mathbf{1} & A_{\ell,0} \end{pmatrix}$$

for $\ell = 1, 2$. It is easy to see that the solutions ω to the dispersion relation (17) are real for any wave vector $\boldsymbol{\xi} \in \mathbf{R}^n$ if and only if

$$a - \frac{\rho_1 \rho_2}{\rho_1 H_2 \alpha_2 + \rho_2 H_1 \alpha_1} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 \ge 0.$$

Otherwise, we have exponentially growing solutions and an instability will appear. As a result, the initial value problem turns out to be ill-posed. This consideration leads us the following stability condition

$$a - \frac{\rho_1 \rho_2}{\rho_1 H_2 a_2 + \rho_2 H_1 a_1} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 \ge c_0 \tag{18}$$

for some positive constant c_0, which is equivalent to (2).

Here, we remark that in the case of surface gravity waves, the corresponding stability condition is given by $-(\partial_z P)|_{z=\zeta} \geq c_0$, where P is the pressure of the water. This condition is also known as a generalized Rayleigh–Taylor sign condition. We remind that the function a in the stability condition (18) can be written as $a = -\left(\partial_z(P_2^{\rm app} - P_1^{\rm app})\right)|_{z=\zeta}$. If we put the density ρ_1 of the upper layer to be 0, then the problem of the interfacial gravity waves is reduced to that of the surface gravity waves and we have $P_1(\boldsymbol{x}, z, t) \equiv 0$ by Bernoulli's equations (6). Therefore, our stability condition is a generalization of this well-known stability condition for the surface gravity waves. We also note that the constants α_1 and α_2 in the stability condition converge to 0 as $N \to \infty$, so that this stability regime diminishes as $N \to \infty$. This fact is consistent with the full problem.

4 Well-posedness of the initial value problem

We proceed to consider the initial value problem for the Kakinuma model (15) under the initial condition

$$(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2) = (\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)}) \quad \text{at} \quad t = 0. \tag{19}$$

Here, we remark that the Kakinuma model has a drawback, that is, the hypersurface $t = 0$ is characteristic for the Kakinuma model, so that the initial value problem for the Kakinuma model (15) and (19) is not solvable in general. In fact, if the problem has a solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$, then by eliminating the time derivative $\partial_t \zeta$ from the equations we see that the solution has to satisfy the relations

$$\begin{cases} H_1^{2i} \sum_{j=0}^{N} \nabla\cdot\left(\dfrac{H_1^{2j+1}}{2j+1}\nabla\phi_{1,j}\right) \\ \quad - \sum_{j=0}^{N}\left\{\nabla\cdot\left(\dfrac{H_1^{2(i+j)+1}}{2(i+j)+1}\nabla\phi_{1,j}\right) - \dfrac{4ij}{2(i+j)-1}H_1^{2(i+j)-1}\phi_{1,j}\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 1, 2, \ldots, N, \\ H_2^{p_i} \sum_{j=0}^{N^*} \nabla\cdot\left(\dfrac{H_2^{p_j+1}}{p_j+1}\nabla\phi_{2,j} - \dfrac{p_j}{p_j}H_2^{p_j}\phi_{2,j}\nabla b\right) \\ \quad - \sum_{j=0}^{N^*}\left\{\nabla\cdot\left(\dfrac{H_2^{p_i+p_j+1}}{p_i+p_j+1}\nabla\phi_{2,j} - \dfrac{p_j}{p_i+p_j}H_2^{p_i+p_j}\phi_{2,j}\nabla b\right)\right. \\ \quad \left. + \dfrac{p_i}{p_i+p_j}H_2^{p_i+p_j}\nabla b\cdot\nabla\phi_{2,j} - \dfrac{p_ip_j}{p_i+p_j-1}H_2^{p_i+p_j-1}(1+|\nabla b|^2)\phi_{2,j}\right\} = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 1, 2, \ldots, N^*, \\ \sum_{j=0}^{N} \nabla\cdot\left(\dfrac{H_1^{2j+1}}{2j+1}\nabla\phi_{1,j}\right) + \sum_{j=0}^{N^*}\nabla\cdot\left(\dfrac{H_2^{p_j+1}}{p_j+1}\nabla\phi_{2,j} - \dfrac{p_j}{p_j}H_2^{p_j}\phi_{2,j}\nabla b\right) = 0. \end{cases} \tag{20}$$

Therefore, as necessary conditions the initial date $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b have to satisfy the relations in (20) for the existence of the solution. These necessary conditions will be referred as the compatibility conditions. In the following, we write $\boldsymbol{\phi}_1 = (\phi_{1,0}, \boldsymbol{\phi}_1')^{\mathrm{T}}$, $\boldsymbol{\phi}_2 = (\phi_{2,0}, \boldsymbol{\phi}_2')^{\mathrm{T}}$, $\boldsymbol{\phi}_{1(0)} = (\phi_{1,0(0)}, \boldsymbol{\phi}_{1(0)}')^{\mathrm{T}}$, and $\boldsymbol{\phi}_{2(0)} = (\phi_{2,0(0)}, \boldsymbol{\phi}_{2(0)}')^{\mathrm{T}}$. We denote by $H^m = H^m(\mathbf{R}^n)$ and $W^{m,\infty} = W^{m,\infty}(\mathbf{R}^n)$ the L^2 and the L^∞ Sobolev spaces of order m, respectively. The following theorem states that the initial value problem for the Kakinuma model is well-posed locally in time in the Sobolev space H^m under the stability condition (18), the compatibility conditions (20), and nondegeneracy of the thicknesses of the upper and lower layers.

Theorem 1 ([4]) *Let $g, \rho_1, \rho_2, h_1, h_2, c_0, M_0$ be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a time $T > 0$ such that if the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b satisfy*

$$\begin{cases} \|(\zeta_{(0)}, \nabla\phi_{1,0(0)}, \nabla\phi_{2,0(0)})\|_{H^m} + \|(\boldsymbol{\phi}_{1(0)}', \boldsymbol{\phi}_{2(0)}')\|_{H^{m+1}} + \|b\|_{W^{m+1,\infty}} \leq M_0, \\ h_1 - \zeta_{(0)}(\boldsymbol{x}) \geq c_0, \quad h_2 + \zeta_{(0)}(\boldsymbol{x}) - b(\boldsymbol{x}) \geq c_0 \quad \textit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

the stability condition (18)*, and the compatibility conditions* (20)*, then the initial value problem* (15) *and* (19) *for the Kakinuma model has a unique solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ satisfying*

$$\zeta, \nabla\phi_{1,0}, \nabla\phi_{2,0} \in C([0,T]; H^m), \quad \boldsymbol{\phi}_1', \boldsymbol{\phi}_2' \in C([0,T]; H^{m+1}).$$

If the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b are suitably small, then the stability condition (18) and the nondegeneracy of the thicknesses of the upper and lower layers are automatically satisfied under Rayleigh's stability condition $(\rho_2 - \rho_1)g > 0$. However, it is not evident how we prepare the initial data satisfying the compatibility conditions (20). By analogy to the canonical variable (3) for interfacial gravity waves introduced by T. B. Benjamin and T. J. Bridges [1], we introduce a canonical variable for the Kakinuma model by

$$\phi = \rho_2 \sum_{j=0}^{N^*} H_2^{p_j} \phi_{2,j} - \rho_1 \sum_{j=0}^{N} H_1^{2j} \phi_{1,j}. \tag{21}$$

Given the initial data $(\zeta_{(0)}, \phi_{(0)})$ for the canonical variables (ζ, ϕ) and the bottom topography b, the compatibility conditions (20) and the relation (21) determine the initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ for the Kakinuma model, which is unique up to an additive constant of the form $(\mathcal{C}\rho_2, \mathcal{C}\rho_1)$ to $(\phi_{1,0(0)}, \phi_{2,0(0)})$. In fact, we have the following proposition.

Proposition 1 ([4]) *Let $\rho_1, \rho_2, h_1, h_2, c, M$ be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exists a positive constant C such that for the canonical variables*

(ζ, ϕ) *and bottom topography* b *satisfying*

$$\begin{cases} \|\zeta\|_{H^m} + \|b\|_{W^{m,\infty}} \leq M, \quad \|\nabla\phi\|_{H^{m-1}} < \infty, \\ h_1 - \zeta(\boldsymbol{x}) \geq c, \quad h_2 + \zeta(\boldsymbol{x}) - b(\boldsymbol{x}) \geq c \quad \text{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

the compatibility conditions (20) *and the relation* (21) *determine the variables* $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ *for the Kakinuma model, uniquely up to an additive constant of the form* $(\mathcal{C}\rho_2, \mathcal{C}\rho_1)$ *to* $(\phi_{1,0}, \phi_{2,0})$*. Moreover, we have*

$$\|(\nabla\phi_{1,0}, \nabla\phi_{2,0})\|_{H^{m-1}} + \|(\boldsymbol{\phi}_1', \boldsymbol{\phi}_2')\|_{H^m} \leq C\|\nabla\phi\|_{H^{m-1}}.$$

5 Equations in a nondimensional form

In order to rigorously validate the Kakinuma model (15) as a higher order shallow water approximation of the full model for interfacial gravity waves (4)–(12), we first introduce nondimensional parameters and then non-dimensionalize the equations, through a convenient rescaling of variables. Let λ be a typical wavelength. Following D. Lannes [13], we introduce a nondimensional parameter δ by

$$\delta = \frac{h}{\lambda} \qquad \text{with} \quad h = \frac{h_1 h_2}{\underline{\rho}_1 h_2 + \underline{\rho}_2 h_1},$$

where $\underline{\rho}_1$ and $\underline{\rho}_2$ are relative densities. We also need to use relative thicknesses $\underline{h}_1$ and $\underline{h}_2$ of the layers. These nondimensional parameters are defined by

$$\underline{\rho}_\ell = \frac{\rho_\ell}{\rho_1 + \rho_2}, \qquad \underline{h}_\ell = \frac{h_\ell}{h} \qquad (\ell = 1, 2),$$

which satisfy the relations

$$\underline{\rho}_1 + \underline{\rho}_2 = 1, \qquad \frac{\underline{\rho}_1}{\underline{h}_1} + \frac{\underline{\rho}_2}{\underline{h}_2} = 1. \tag{22}$$

Note also that $\min\{h_1, h_2\} \leq h \leq \max\{h_1, h_2\}$. Here, we note that the standard shallowness parameters $\delta_1 := \frac{h_1}{\lambda}$ and $\delta_2 := \frac{h_2}{\lambda}$ relative to the upper and the lower layers, respectively, are related to the above parameters by $\delta_\ell = \underline{h}_\ell \delta$ for $\ell = 1, 2$. In this article, we restrict our consideration to the parameter regime

$$\underline{h}_1^{-1}, \underline{h}_2^{-1} \lesssim 1. \tag{23}$$

To understand this restriction, it is convenient to use nondimensional parameters $\gamma := \frac{\rho_1}{\rho_2}$ and $\theta := \frac{h_1}{h_2}$. In terms of these parameters, $\underline{h}_\ell^{-1}$ $(\ell = 1, 2)$ can be represented as

$$\underline{h}_1^{-1} = \frac{\gamma + 1}{\gamma + \theta}, \qquad \underline{h}_2^{-1} = \frac{\gamma^{-1} + 1}{\gamma^{-1} + \theta^{-1}}.$$

Therefore, the only cases that we shall exclude are the case $\gamma, \theta \ll 1$ and the case $\gamma, \theta \gg 1$. In other words, we shall consider the following three regimes concerning the densities and the mean thicknesses of the layers in this article:

(i) $\theta \simeq 1$, i.e., $h_1 \simeq h_2$,

(ii) $\gamma \simeq 1$, i.e., $\rho_1 \simeq \rho_2$,

(iii) $\gamma \ll 1 \ll \theta$, i.e., $\rho_1 \ll \rho_2$ and $h_2 \ll h_1$.

Although the remaining case $\theta \ll 1 \ll \gamma$, i.e., $\rho_2 \ll \rho_1$ and $h_1 \ll h_2$ satisfies (23), this case is not consistent with Rayleigh's stability condition $(\rho_2 - \rho_1)g > 0$, so that we also exclude this case.

Introducing $c_{\mathrm{sw}} := \sqrt{(\underline{\rho}_2 - \underline{\rho}_1)gh}$ the speed of infinitely long and small interfacial gravity waves, we rescale the independent and the dependent variables by

$$\boldsymbol{x} = \lambda\tilde{\boldsymbol{x}}, \quad z = h\tilde{z}, \quad t = \frac{\lambda}{c_{\mathrm{sw}}}\tilde{t}, \quad \zeta = h\tilde{\zeta}, \quad b = h\tilde{b}, \quad \Phi_\ell = \lambda c_{\mathrm{sw}}\tilde{\Phi}_\ell \quad (\ell = 1, 2).$$

Plugging these into the full model (4)–(5) and (9)–(13) and dropping the tilde sign in the notation we obtain

$$\begin{cases} \Delta\Phi_1 + \delta^{-2}\partial_z^2\Phi_1 = 0 & \text{in} \quad \Omega_1(t), \\ \Delta\Phi_2 + \delta^{-2}\partial_z^2\Phi_2 = 0 & \text{in} \quad \Omega_2(t), \\ \partial_t\zeta + \nabla\Phi_1 \cdot \nabla\zeta - \delta^{-2}\partial_z\Phi_1 = 0 & \text{on} \quad \Gamma(t), \\ \partial_t\zeta + \nabla\Phi_2 \cdot \nabla\zeta - \delta^{-2}\partial_z\Phi_2 = 0 & \text{on} \quad \Gamma(t), \\ \partial_z\Phi_1 = 0 & \text{on} \quad \Sigma_1, \\ \nabla\Phi_2 \cdot \nabla b - \delta^{-2}\partial_z\Phi_2 = 0 & \text{on} \quad \Sigma_2, \\ \underline{\rho}_1\Big(\partial_t\Phi_1 + \frac{1}{2}|\nabla\Phi_1|^2 + \frac{1}{2}\delta^{-2}(\partial_z\Phi_1)^2\Big) & \\ \quad -\underline{\rho}_2\Big(\partial_t\Phi_2 + \frac{1}{2}|\nabla\Phi_2|^2 + \frac{1}{2}\delta^{-2}(\partial_z\Phi_2)^2\Big) - \zeta = 0 & \text{on} \quad \Gamma(t), \end{cases} \tag{24}$$

where in this scaling, the interface $\Gamma(t)$, the rigid-lid Σ_1, and the bottom Σ_2 are written as

$$\begin{aligned} \Gamma(t) &= \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, z = \zeta(\boldsymbol{x}, t)\}, \\ \Sigma_1 &= \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, z = \underline{h}_1\}, \\ \Sigma_2 &= \{X = (\boldsymbol{x}, z) \in \mathbf{R}^{n+1}\,;\, z = -\underline{h}_2 + b(\boldsymbol{x})\}. \end{aligned}$$

As for the Kakinuma model, we introduce additionally the rescaled variables

$$\phi_{1,i} = \frac{\lambda c_{\mathrm{sw}}}{h_1^{2i}}\tilde{\phi}_{1,i}, \qquad \phi_{2,i} = \frac{\lambda c_{\mathrm{sw}}}{h_2^{p_i}}\tilde{\phi}_{2,i}.$$

Plugging these and the previous scaling into the Kakinuma model (15) and dropping the tilde sign in the notation we obtain the Kakinuma model in a nondimensional form, which is written as

$$
\begin{cases}
H_1^{2i}\partial_t\zeta - \underline{h}_1 \sum_{j=0}^{N} \biggl\{ \nabla\cdot\biggl(\frac{1}{2(i+j)+1} H_1^{2(i+j)+1} \nabla\phi_{1,j} \biggr) \\
\qquad\qquad -(\underline{h}_1\delta)^{-2} \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1}\phi_{1,j} \biggr\} = 0 \quad \text{for} \quad i = 0,1,\ldots,N, \\
H_2^{p_i}\partial_t\zeta + \underline{h}_2 \sum_{j=0}^{N^*} \biggl\{ \nabla\cdot\biggl(\frac{1}{p_i+p_j+1} H_2^{p_i+p_j+1}\nabla\phi_{2,j} - \frac{p_j}{p_i+p_j} H_2^{p_i+p_j}\phi_{2,j}\underline{h}_2^{-1}\nabla b \biggr) \\
\quad + \frac{p_i}{p_i+p_j} H_2^{p_i+p_j}\underline{h}_2^{-1}\nabla b\cdot\nabla\phi_{2,j} - \frac{p_i p_j}{p_i+p_j-1} H_2^{p_i+p_j-1}((\underline{h}_2\delta)^{-2} + \underline{h}_2^{-2}|\nabla b|^2)\phi_{2,j} \biggr\} = 0 \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad i = 0,1,\ldots,N^*, \\
\underline{\rho}_1 \Biggl\{ \sum_{j=0}^{N} H_1^{2j}\partial_t\phi_{1,j} + \frac{1}{2}\Biggl(\biggl| \sum_{j=0}^{N} H_1^{2j}\nabla\phi_{1,j} \biggr|^2 + (\underline{h}_1\delta)^{-2} \biggl(\sum_{j=0}^{N} 2jH_1^{2j-1}\phi_{1,j} \biggr)^2 \Biggr) \Biggr\} \\
\quad -\underline{\rho}_2 \Biggl\{ \sum_{j=0}^{N^*} H_2^{2j}\partial_t\phi_{2,j} + \frac{1}{2}\Biggl(\biggl| \sum_{j=0}^{N^*} (H_2^{p_j}\nabla\phi_{2,j} - p_j H_2^{p_j-1}\phi_{2,j}\underline{h}_2^{-1}\nabla b) \biggr|^2 \\
\qquad\qquad + (\underline{h}_2\delta)^{-2} \biggl(\sum_{j=0}^{N^*} p_j H_2^{p_j-1}\phi_{2,j} \biggr)^2 \Biggr) \Biggr\} - \zeta = 0,
\end{cases}
\tag{25}
$$

where

$$
H_1(\boldsymbol{x},t) = 1 - \underline{h}_1^{-1}\zeta(\boldsymbol{x},t), \qquad H_2(\boldsymbol{x},t) = 1 + \underline{h}_2^{-1}\zeta(\boldsymbol{x},t) - \underline{h}_2^{-1}b(\boldsymbol{x}). \tag{26}
$$

6 Well-posedness in the shallow water regime

We will revisit the initial value problem for the Kakinuma model (25) in the nondimensional form under the initial conditions (19). In order to rigorously validate the Kakinuma model (25) as a higher order shallow water approximation of the full model for interfacial gravity waves (24), we then need to show the existence of the solution to the initial value problem (25) and (19) on some time interval independent of the shallowness parameters $\delta_1 = \underline{h}_1\delta$ and $\delta_2 = \underline{h}_2\delta$ together with a uniform bound of the solution.

The nondimensional version of the stability condition (18) is given by

$$
a - \frac{\underline{\rho}_1\underline{\rho}_2}{\underline{\rho}_1\underline{h}_2 H_2\alpha_2 + \underline{\rho}_2\underline{h}_1 H_1\alpha_1} |\boldsymbol{u}_1 - \boldsymbol{u}_2|^2 \geq c_0, \tag{27}
$$

where H_1 and H_2 are given by (26), $\boldsymbol{u}_\ell = (\nabla\Phi_\ell^{\mathrm{app}})|_{z=\zeta}$ for $\ell = 1,2$ and $a = -(\partial_z(P_2^{\mathrm{app}} - P_1^{\mathrm{app}}))|_{z=\zeta}$. Here, we note that the nondimensional version of the approximate velocity potentials $\Phi_\ell^{\rm app}$ and pressures $P_\ell^{\rm app}$ for $\ell = 1, 2$ are given by

$$\begin{cases} \Phi_1^{\rm app}(\boldsymbol{x}, z, t) = \displaystyle\sum_{i=0}^{N} \left(1 - \underline{h}_1^{-1} z\right)^{2i} \phi_{1,i}(\boldsymbol{x}, t), \\ \Phi_2^{\rm app}(\boldsymbol{x}, z, t) = \displaystyle\sum_{i=0}^{N^*} \left(1 + \underline{h}_2^{-1} z - \underline{h}_2^{-1} b(\boldsymbol{x})\right)^{p_i} \phi_{2,i}(\boldsymbol{x}, t), \end{cases} \tag{28}$$

and

$$P_\ell^{\rm app}(\boldsymbol{x}, z, t) = -\underline{\rho}_\ell \left(\partial_t \Phi_\ell^{\rm app} + \frac{1}{2} \left(|\nabla \Phi_\ell^{\rm app}|^2 + \delta^{-2} (\partial_z \Phi_\ell^{\rm app})^2 \right) + (\underline{\rho}_2 - \underline{\rho}_1)^{-1} z \right)$$

for $\ell = 1, 2$, respectively.

The nondimensional version of the compatibility conditions (20) is given by

$$\begin{cases} H_1^{2i} \displaystyle\sum_{j=0}^{N} \nabla \cdot \left(\frac{H_1^{2j+1}}{2j+1} \nabla \phi_{1,j} \right) \\ \quad - \displaystyle\sum_{j=0}^{N} \left\{ \nabla \cdot \left(\frac{H_1^{2(i+j)+1}}{2(i+j)+1} \nabla \phi_{1,j} \right) - (\underline{h}_1 \delta)^{-2} \frac{4ij}{2(i+j)-1} H_1^{2(i+j)-1} \phi_{1,j} \right\} = 0 \\ \hfill \text{for} \quad i = 1, 2, \ldots, N, \\ H_2^{p_i} \displaystyle\sum_{j=0}^{N^*} \nabla \cdot \left(\frac{H_2^{p_j+1}}{p_j+1} \nabla \phi_{2,j} - \frac{p_j}{p_j} H_2^{p_j} \phi_{2,j} \underline{h}_2^{-1} \nabla b \right) \\ \quad - \displaystyle\sum_{j=0}^{N^*} \left\{ \nabla \cdot \left(\frac{H_2^{p_i+p_j+1}}{p_i+p_j+1} \nabla \phi_{2,j} - \frac{p_j}{p_i+p_j} H_2^{p_i+p_j} \phi_{2,j} \underline{h}_2^{-1} \nabla b \right) \right. \\ \qquad \left. + \frac{p_i}{p_i+p_j} H_2^{p_i+p_j} \underline{h}_2^{-1} \nabla b \cdot \nabla \phi_{2,j} - \frac{p_i p_j}{p_i+p_j-1} H_2^{p_i+p_j-1} ((\underline{h}_2 \delta)^{-2} + \underline{h}_2^{-2} |\nabla b|^2) \phi_{2,j} \right\} = 0 \\ \hfill \text{for} \quad i = 1, 2, \ldots, N^*, \\ \underline{h}_1 \displaystyle\sum_{j=0}^{N} \nabla \cdot \left(\frac{H_1^{2j+1}}{2j+1} \nabla \phi_{1,j} \right) + \underline{h}_2 \sum_{j=0}^{N^*} \nabla \cdot \left(\frac{H_2^{p_j+1}}{p_j+1} \nabla \phi_{2,j} - \frac{p_j}{p_j} H_2^{p_j} \phi_{2,j} \nabla b \right) = 0. \end{cases} \tag{29}$$

The following theorem states the existence of the solution to the initial value problem (25) and (19) on some time interval independent of nondimensional parameters, especially, the shallowness parameters $\delta_1 = \underline{h}_1 \delta$ and $\delta_2 = \underline{h}_2 \delta$ together with a uniform bound of the solution.

Theorem 2 ([5]) *Let $c_0, M_0, \underline{h}_{\min}$ be positive constants and m an integer such that $m > \frac{n}{2} + 1$. There exist a time $T > 0$ and a constant $M > 0$ such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the natural restrictions* (22)*, $\underline{h}_1 \delta, \underline{h}_2 \delta \leq 1$, and the condition $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$ if the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ and the bottom topography b*

satisfy

$$\begin{cases} \|\zeta_{(0)}\|_{H^m}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \bigl(\|\nabla \boldsymbol{\phi}_{\ell(0)}\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}_{\ell(0)}'\|_{H^m}^2 \bigr) + \underline{h}_2^{-1} \|b\|_{W^{m+1,\infty}} \le M_0, \\ 1 - \underline{h}_1^{-1} \zeta_{(0)}(\boldsymbol{x}) \ge c_0, \quad 1 + \underline{h}_2^{-1} \zeta_{(0)}(\boldsymbol{x}) - \underline{h}_2^{-1} b(\boldsymbol{x}) \ge c_0 \quad \textit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases} \tag{30}$$

the stability condition (27)*, and the compatibility conditions* (29)*, then the initial value problem* (25) *and* (19) *has a unique solution* $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ *on the time interval* $[0, T]$ *satisfying*

$$\zeta, \nabla \phi_{1,0}, \nabla \phi_{2,0} \in C([0,T]; H^m), \qquad \boldsymbol{\phi}_1', \boldsymbol{\phi}_2' \in C([0,T]; H^{m+1}).$$

Moreover, the solution satisfies the uniform bound

$$\|\zeta(t)\|_{H^m}^2 + \sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \bigl(\|\nabla \boldsymbol{\phi}_\ell(t)\|_{H^m}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}_\ell'(t)\|_{H^m}^2 \bigr) \le M$$

for $t \in [0, T]$.

It is not evident how we prepare the initial data $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ satisfying the compatibility conditions (29) together with the uniform bound (30). Again, it is sufficient to specify the initial data for the canonical variables (ζ, ϕ), where ϕ is defined by

$$\phi = \underline{\rho}_2 \sum_{j=0}^{N^*} H_2^{p_j} \phi_{2,j} - \underline{\rho}_1 \sum_{j=0}^{N} H_1^{2j} \phi_{1,j}. \tag{31}$$

In fact, we have the following proposition.

Proposition 2 ([5]) *Let* c, M *be positive constants and* m *an integer such that* $m > \frac{n}{2} + 1$. *There exists a positive constant* C *such that for the canonical variables* (ζ, ϕ) *and bottom topography* b *satisfying*

$$\begin{cases} \underline{h}_1^{-1} \|\zeta\|_{H^m} + \underline{h}_2^{-1} \|\zeta\|_{H^m} + \underline{h}_2^{-1} \|b\|_{W^{m,\infty}} \le M, \quad \|\nabla \phi\|_{H^{m-1}} < \infty, \\ 1 - \underline{h}_1^{-1} \zeta(\boldsymbol{x}) \ge c, \quad 1 + \underline{h}_2^{-1} \zeta(\boldsymbol{x}) - \underline{h}_2^{-1} b(\boldsymbol{x}) \ge c \quad \textit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

the compatibility conditions (29) *and the relation* (31) *determine the variables* $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ *for the Kakinuma model, uniquely up to an additive constant of the form* $(\mathcal{C}\underline{\rho}_2, \mathcal{C}\underline{\rho}_1)$ *to* $(\phi_{1,0}, \phi_{2,0})$*. Moreover, we have*

$$\sum_{\ell=1,2} \underline{\rho}_\ell \underline{h}_\ell \bigl(\|\nabla \boldsymbol{\phi}_\ell\|_{H^{m-1}}^2 + (\underline{h}_\ell \delta)^{-2} \|\boldsymbol{\phi}_\ell'\|_{H^{m-1}}^2 \bigr) \le C \|\nabla \phi\|_{H^{m-1}}^2.$$

Now, we can show that the solution to the Kakinuma model (25) constructed in Theorem 2 satisfies approximately the full problem for the interfacial gravity waves with an error of order $O(\delta_1^{4N+2} + \delta_2^{4N+2})$ for the time interval $[0, T]$. There are several versions of this kind of result on the consistency. The most sophisticated version would be an approximation of the Hamiltonians.

7 Hamiltonian structures

As is well-known that the full model (24) for interfacial gravity waves has a conserved energy

$$\mathscr{E} = \sum_{\ell=1,2} \iint_{\Omega_\ell(t)} \frac{1}{2}\underline{\rho}_\ell \big(|\nabla \Phi_\ell(\boldsymbol{x},z,t)|^2 + \delta^{-2}(\partial_z \Phi_\ell(\boldsymbol{x},z,t))^2 \big) \mathrm{d}\boldsymbol{x}\mathrm{d}z + \int_{\mathbf{R}^n} \frac{1}{2}\zeta(\boldsymbol{x},t)^2 \mathrm{d}\boldsymbol{x},$$

which is the total energy, that is, the sum of the kinetic energies of the waters in the upper and the lower layers and the potential energy due to gravity. Putting $\phi_\ell = \Phi_\ell|_{z=\zeta}$ for $\ell = 1,2$, we can rewrite this total energy as

$$\mathscr{E} = \sum_{\ell=1,2} \frac{1}{2}\underline{\rho}_\ell (\Lambda_\ell(\zeta)\phi_\ell, \phi_\ell)_{L^2} + \frac{1}{2}\|\zeta\|_{L^2}^2,$$

where $\Lambda_1(\zeta) = \Lambda_1(\zeta,\delta,\underline{h}_1)$ and $\Lambda_2(\zeta) = \Lambda_2(\zeta,b,\delta,\underline{h}_2)$ are the Dirichlet-to-Neumann maps for Laplace's equation. More precisely, these are defined by

$$\begin{cases} \Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 = \big(-\delta^{-2}\partial_z\Phi_1 + \nabla\zeta\cdot\nabla\Phi_1\big)\big|_{z=\zeta}, \\ \Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 = \big(\delta^{-2}\partial_z\Phi_2 - \nabla\zeta\cdot\nabla\Phi_2\big)\big|_{z=\zeta}, \end{cases} \tag{32}$$

where Φ_1 and Φ_2 are unique solutions to the boundary value problems

$$\begin{cases} \Delta\Phi_1 + \delta^{-2}\partial_z^2\Phi_1 = 0 & \text{in} \quad \Omega_1(t), \\ \Phi_1 = \phi_1 & \text{on} \quad \Gamma(t), \\ \partial_z\Phi_1 = 0 & \text{on} \quad \Sigma_1, \end{cases} \quad \text{and} \quad \begin{cases} \Delta\Phi_2 + \delta^{-2}\partial_z^2\Phi_2 = 0 & \text{in} \quad \Omega_2(t), \\ \Phi_2 = \phi_2 & \text{on} \quad \Gamma(t), \\ \nabla b\cdot\nabla\Phi_2 - \delta^{-2}\partial_z\Phi_2 = 0 & \text{on} \quad \Sigma_2. \end{cases}$$

It follows from the kinematical boundary conditions on the interface that ϕ_1 and ϕ_2 are related by $\Lambda_1(\zeta)\phi_1 + \Lambda_2(\zeta)\phi_2 = 0$. Introducing the canonical variable $\phi := \underline{\rho}_2\phi_2 - \underline{\rho}_1\phi_1$, ϕ_1 and ϕ_2 can be written in terms of the canonical variables (ζ,ϕ) as

$$\begin{cases} \phi_1 = -(\underline{\rho}_1\Lambda_2(\zeta) + \underline{\rho}_2\Lambda_1(\zeta))^{-1}\Lambda_2(\zeta)\phi, \\ \phi_2 = (\underline{\rho}_1\Lambda_2(\zeta) + \underline{\rho}_2\Lambda_1(\zeta))^{-1}\Lambda_1(\zeta)\phi. \end{cases}$$

Therefore, the total energy $\mathscr{E}$ can be written in terms of the canonical variables (ζ,ϕ) as

$$\mathscr{E} = \frac{1}{2}((\underline{\rho}_1\Lambda_2(\zeta) + \underline{\rho}_2\Lambda_1(\zeta))^{-1}\Lambda_1(\zeta)\phi, \Lambda_2(\zeta)\phi)_{L^2} + \frac{1}{2}\|\zeta\|_{L^2}^2,$$

which will be denoted by $\mathscr{H}^{\mathrm{IW}}(\zeta,\phi)$. This is the Hamiltonian of the full model for the interfacial gravity waves found by T. B. Benjamin and T. J. Bridges [1]. See also W. Craig and M. D. Groves [3].

As in the case of the full model, the Kakinuma model (25) has a conserved quantity

$$\mathscr{E}^{\mathrm{K}} = \sum_{\ell=1,2} \iint_{\Omega_\ell(t)} \frac{1}{2}\underline{\rho}_\ell \bigl(|\nabla \Phi_\ell^{\mathrm{app}}(\boldsymbol{x},z,t)|^2 + \delta^{-2}(\partial_z \Phi_\ell^{\mathrm{app}}(\boldsymbol{x},z,t))^2 \bigr) \mathrm{d}\boldsymbol{x}\mathrm{d}z + \int_{\mathbf{R}^n} \frac{1}{2}\zeta(\boldsymbol{x},t)^2 \mathrm{d}\boldsymbol{x},$$

where Φ_ℓ^{app} for $\ell = 1,2$ are given by (28). This is nothing but the total energy, which can be written explicitly in terms of $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$. Now, we define the canonical variable ϕ by (31). Then, by Proposition 2 we can determine $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ from the canonical variables (ζ, ϕ), so that the total energy $\mathscr{E}^{\mathrm{K}}$ can be expressed in terms of (ζ, ϕ), which will be denoted by $\mathscr{H}^{\mathrm{K}}(\zeta, \phi)$. This is a Hamiltonian of the Kakinuma model. In fact, we have the following theorem.

Theorem 3 ([4]) *Let m be an integer such that $m > \frac{n}{2} + 1$ and $b \in W^{m,\infty}$. Then, the Kakinuma model* (25) *is equivalent to Hamilton's canonical equations*

$$\partial_t \zeta = \frac{\delta \mathscr{H}^{\mathrm{K}}}{\delta \phi}, \quad \partial_t \phi = -\frac{\delta \mathscr{H}^{\mathrm{K}}}{\delta \zeta}, \tag{33}$$

as long as $\zeta(\cdot,t) \in H^m$ satisfies

$$\inf_{\boldsymbol{x}\in\mathbf{R}^n} \bigl(1 - \underline{h}_1^{-1}\zeta(\boldsymbol{x},t)\bigr) > 0, \qquad \inf_{\boldsymbol{x}\in\mathbf{R}^n} \bigl(1 + \underline{h}_2^{-1}\zeta(\boldsymbol{x},t) - \underline{h}_2^{-1}b(\boldsymbol{x})\bigr) > 0$$

and $\nabla\phi(\cdot,t) \in L^2$. More precisely, for any regular solution $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ to the Kakinuma model (25)*, if we define ϕ by* (31)*, then (ζ, ϕ) satisfies Hamilton's canonical equations* (33)*. Conversely, for any regular solution (ζ, ϕ) to Hamilton's canonical equations* (33)*, if we define $(\boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ as a solution to the compatibility conditions* (29) *and the relation* (31)*, then $(\zeta, \boldsymbol{\phi}_1, \boldsymbol{\phi}_2)$ satisfies the Kakinuma model* (25)*.*

The following theorem states that the Hamiltonian $\mathscr{H}^{\mathrm{K}}(\zeta, \phi)$ of the Kakinuma model is a higher order shallow water approximation of the Hamiltonian $\mathscr{H}^{\mathrm{IW}}(\zeta, \phi)$ of the full model. This is one version of the consistency of the Kakinuma model with the full model. We put $\mathring{H}^m = \{\phi\,;\, \nabla\phi \in H^{m-1}\}$.

Theorem 4 ([5]) *Let $c, M, \underline{h}_{\min}$ be positive constants and m an integer such that $m > \frac{n}{2} + 1$ and $m \geq 4(N+1)$. There exists a positive constant C such that for any positive parameters $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ satisfying the natural restrictions* (22)*, $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, and the condition $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$, and for any canonical variables $(\zeta, \phi) \in H^m \times \mathring{H}^m$ and the bottom topography $b \in W^{m+1,\infty}$ satisfying*

$$\begin{cases} \underline{h}_1^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|\zeta\|_{H^m} + \underline{h}_2^{-1}\|b\|_{W^{m+1,\infty}} \leq M, \\ H_1(\boldsymbol{x}) \geq c, \quad H_2(\boldsymbol{x}) \geq c \quad \textit{for} \quad \boldsymbol{x} \in \mathbf{R}^n, \end{cases}$$

we have

$$|\mathscr{H}^{\mathrm{K}}(\zeta,\phi) - \mathscr{H}^{\mathrm{IW}}(\zeta,\phi)| \leq C\|\nabla\phi\|_{H^{4N+3}}^2 ((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2}).$$

8 Conditionally rigorous justification of the Kakinuma model

In order to give a rigorous justification of the Kakinuma model as a higher order shallow water approximation, we need to give an error estimate between the solution to the Kakinuma model and the solution to the full model. However, we cannot expect to construct a solution to the initial value problem for the full model in Sobolev spaces with uniform bounds with respect to the shallowness parameters $\delta_1 = \underline{h}_1\delta$ and $\delta_2 = \underline{h}_2\delta$ because the initial value problem for the full model is ill-posed. Nevertheless, if we assume the existence of the solution to the full model with a uniform bound, then we can give an error estimate between the solutions by making use of the well-posedness of the initial value problem to the Kakinuma model.

To state the result, we remark that the full model (24) for interfacial gravity waves can be written in a more compact and closed form as

$$\begin{cases} \partial_t\zeta + \Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 = 0, \\ \partial_t\zeta - \Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 = 0, \\ \underline{\rho}_1\left(\partial_t\phi_1 + \dfrac{1}{2}|\nabla\phi_1|^2 - \dfrac{1}{2}\delta^2\dfrac{(\Lambda_1(\zeta,\delta,\underline{h}_1)\phi_1 - \nabla\zeta\cdot\nabla\phi_1)^2}{1+\delta^2|\nabla\zeta|^2}\right) \\ \quad -\underline{\rho}_2\left(\partial_t\phi_2 + \dfrac{1}{2}|\nabla\phi_2|^2 - \dfrac{1}{2}\delta^2\dfrac{(\Lambda_2(\zeta,b,\delta,\underline{h}_2)\phi_2 + \nabla\zeta\cdot\nabla\phi_2)^2}{1+\delta^2|\nabla\zeta|^2}\right) - \zeta = 0, \end{cases} \tag{34}$$

where $\phi_\ell = \Phi_\ell|_{z=\zeta}$ for $\ell = 1,2$. Here, we remind that $\Lambda_1(\zeta,\delta,\underline{h}_1)$ and $\Lambda_2(\zeta,b,\delta,\underline{h}_2)$ are the Dirichlet-to-Neumann maps for Laplace's equation defined by (32).

Theorem 5 ([5]) *Let $c_0, c, M, \underline{h}_{\min}$ be positive constants and m an integer such that $m > \frac{n}{2} + 4N + 5$. Then, there exist a time $T > 0$ and a constant $C > 0$ such that the following holds true: Let $\underline{\rho}_1, \underline{\rho}_2, \underline{h}_1, \underline{h}_2, \delta$ be positive parameters satisfying the natural restrictions (22), $\underline{h}_1\delta, \underline{h}_2\delta \leq 1$, and the condition $\underline{h}_{\min} \leq \underline{h}_1, \underline{h}_2$ and let $b \in W^{m+1,\infty}$ satisfy $\underline{h}_2^{-1}\|b\|_{W^{m+1,\infty}} \leq M$. Suppose that the full model for interfacial gravity waves (34) possesses a solution $(\zeta^{\mathrm{IW}}, \phi_1^{\mathrm{IW}}, \phi_2^{\mathrm{IW}}) \in C([0,T_1]; H^{m+1}\times \mathring{H}^{m+1}\times\mathring{H}^{m+1})$ satisfying a uniform bound*

$$\begin{cases} \|\zeta^{\mathrm{IW}}(t)\|_{H^{m+1}}^2 + \displaystyle\sum_{\ell=1,2}\underline{\rho}_\ell\underline{h}_\ell\|\nabla\phi_\ell^{\mathrm{IW}}(t)\|_{H^m}^2 \leq M, \\ 1 - \underline{h}_1^{-1}\zeta^{\mathrm{IW}}(\boldsymbol{x},t) \geq c, \quad 1 + \underline{h}_2^{-1}\zeta^{\mathrm{IW}}(\boldsymbol{x},t) - \underline{h}_2^{-1}b(\boldsymbol{x}) \geq c \quad \text{for} \quad \boldsymbol{x}\in\mathbf{R}^n,\ t\in[0,T_1]. \end{cases}$$

Let $\zeta_{(0)} := \zeta^{\mathrm{IW}}|_{t=0}$ and $\phi_{(0)} := (\underline{\rho}_2\phi_2^{\mathrm{IW}} - \underline{\rho}_1\phi_1^{\mathrm{IW}})|_{t=0}$ be the initial data for the canonical variables, and let $(\boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ be the initial data to the Kakinuma model constructed from $(\zeta_{(0)}, \phi_{(0)})$ by Proposition 2. Assume moreover that the initial data $(\zeta_{(0)}, \boldsymbol{\phi}_{1(0)}, \boldsymbol{\phi}_{2(0)})$ satisfy the stability condition (27), let $(\zeta^{\mathrm{K}}, \boldsymbol{\phi}_1^{\mathrm{K}}, \boldsymbol{\phi}_2^{\mathrm{K}})$ be the solution to the initial value problem *for the Kakinuma model* (25) *and* (19) *on the time interval* $[0,T]$ *whose unique existence is guaranteed by Theorem* 2*, and put*

$$\phi_1^{\mathrm{K}} = \sum_{j=0}^{N} H_1^{2j}\phi_{1,j}^{\mathrm{K}}, \qquad \phi_2^{\mathrm{K}} = \sum_{j=0}^{N^*} H_2^{p_j}\phi_{2,j}^{\mathrm{K}}.$$

Then, we have the error bound

$$\|\zeta^{\mathrm{K}}(t) - \zeta^{\mathrm{IW}}(t)\|_{H^{m-(4N+5)}} + \sum_{\ell=1,2}\sqrt{\underline{\rho}_\ell \underline{h}_\ell}\|\nabla\phi_\ell^{\mathrm{K}}(t) - \nabla\phi_\ell^{\mathrm{IW}}(t)\|_{H^{m-(4N+5)}} \le C((\underline{h}_1\delta)^{4N+2} + (\underline{h}_2\delta)^{4N+2})$$

for $0 \le t \le \min\{T, T_1\}$*.*

Acknowledgements This work was partially supported by JSPS KAKENHI Grant Number JP17K18742 and JP17H02856.

References

[1] T. B. Benjamin and T. J. Bridges, Reappraisal of the Kelvin–Helmholtz problem. I. Hamiltonian structure, J. Fluid Mech., **333** (1997), 301–325.

[2] D. Bresch and M. Renardy, Well-posedness of two-layer shallow-water flow between two horizontal rigid plates, Nonlinearity **24** (2011), 1081–1088.

[3] W. Craig and M. D. Groves, Normal forms for wave motion in fluid interfaces, Wave Motion, **31** (2000), no. 1, 21–41.

[4] V. Duchêne and T. Iguchi, A mathematical analysis of the Kakinuma model for interfacial gravity waves. Part I: Structures and well-posedness, arXiv:2103.12392.

[5] V. Duchêne and T. Iguchi, A mathematical analysis of the Kakinuma model for interfacial gravity waves. Part II: Justification as a shallow water approximation, in preparation.

[6] T. Iguchi, Isobe–Kakinuma model for water waves as a higher order shallow water approximation, J. Differential Equations, **265** (2018), 935–962.

[7] T. Iguchi, A mathematical justification of the Isobe–Kakinuma model for water waves with and without bottom topography, J. Math. Fluid Mech., **20** (2018), 1985–2018.

[8] T. Iguchi, N. Tanaka, and A. Tani, On the two-phase free boundary problem for two-dimensional water waves, Math. Ann., **309** (1997), 199–223.

[9] 柿沼 太郎, 非線形緩勾配方程式の内部波への拡張, 海岸工学論文集, 第 47 巻 (2000), 土木学会 1–5.

[10] T. Kakinuma, A set of fully nonlinear equations for surface and internal gravity waves, Coastal Engineering V: Computer Modelling of Seas and Coastal Regions, 225–234, WIT Press, 2001.

[11] T. Kakinuma, A nonlinear numerical model for surface and internal waves shoaling on a permeable beach, Coastal engineering VI: Computer Modelling and Experimental Measurements of Seas and Coastal Regions, 227–236, WIT Press, 2003.

[12] V. Kamotski and G. Lebeau, On 2D Rayleigh–Taylor instabilities, Asymptotic Analysis, **42** (2005), 1–27.

[13] D. Lannes, A stability criterion for two-fluid interfaces and applications, Arch. Ration. Mech. Anal., **208** (2013), 481–567.

[14] Y. Murakami and T. Iguchi, Solvability of the initial value problem to a model system for water waves, Kodai Math. J., **38** (2015), 470–491.

[15] R. Nemoto and T. Iguchi, Solvability of the initial value problem to the Isobe–Kakinuma model for water waves, J. Math. Fluid Mech., **20** (2018), 631–653.

Department of Mathematics
Faculty of Science and Technology
Keio University
3-14-1 Hiyoshi, Kohoku-ku
Yokohama 223-8522
JAPAN
E-mail address: `iguchi@math.keio.ac.jp`

慶應義塾大学・理工学部　井口　達雄