Existence of a stationary Navier-Stokes flow past a rigid body, with application to starting problem in higher dimensions

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1 Introduction

We consider the large time behavior of the Navier-Stokes flow past a rigid body $\mathscr{O} \subset \mathbb{R}^n$ ($n \geq 3$). If the body $\mathscr{O}$ translates with a prescribed constant velocity, then we expect from the physical point of view that solutions to the Navier-Stokes equation reflect anisotropic decay structure at spatial infinity. In [5–8], Finn succeeded in constructing a stationary solution, called by physicall reasonable solution, that exhibits a paraboloidal wake region behind the body. For more understanding, he raised a question related to convergence of the nonstationary solution to stationary solution, which is called Finn's starting problem. Finn's starting problem is the following: suppose both a rigid body and fluid filling the outside of the body are initially at rest and the body starts to translate with a velocity which gradually increases and is maintained after a certain finite time, then prove that a nonstationary flow converge to a stationary solution corresponding to a terminal velocity of the body as time goes to infinity. If the problem is proved, the stationary solution is said to be attainable by the terminology of Heywood [14] who first studied Finn's starting problem in L^2 framework. But his result is partial result because stationary solutions do not belong to L^2 in general. We thus need L^q framework and this problem had remained open until Kobayashi and Shibata [15] developed the L^q theory of the linearized problem, which is called the Oseen problem. By using estimates of solution to the Ossen problem established by [15], Finn's starting problem was affirmatively solved by Galdi, Heywood and Shibata [13].

In this paper, we derive new convergence rate that is determined by the summability of stationary solution corresponding to a small terminal velocity. Our convergence rates are the improvement of [13]. Moreover, we extend this result to the case of higher dimensions. This procedure is not obvious because there is less literature for concerning the stationary problem in higher dimensions. We thus first consider the stationary problem and construct a small stationary solution possessing the optimal summability at spatial infinity, which is the same as that of the Oseen fundamental solution.

Let us introduce the mathematical formulation of Finn's starting problem. We suppose that a rigid body $\mathscr{O}$ is translating with a prescribed velocity $-\psi(t)ae_1$, where $a > 0$, $e_1 = (1, 0, \cdots, 0)^\top$ and ψ is a function on $\mathbb{R}$ describing the transition of the translational velocity in such a way that

$$\psi \in C^1(\mathbb{R};\mathbb{R}), \quad |\psi(t)| \le 1 \quad \text{for } t \in \mathbb{R}, \quad \psi(t) = 0 \quad \text{for } t \le 0, \quad \psi(t) = 1 \quad \text{for } t \ge 1. \tag{1.1}$$

Here and hereafter, $(\cdot)^\top$ denotes the transpose. We take the frame attached to the body, then the fluid motion which occupies the exterior domain $D = \mathbb{R}^n \setminus \mathscr{O}$ with C^2 boundary ∂D and is started from rest obeys

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = \Delta u - \psi(t) a \dfrac{\partial u}{\partial x_1} - \nabla p, & x \in D, \ t > 0, \\ \nabla \cdot u = 0, & x \in D, t \ge 0, \\ u|_{\partial D} = -\psi(t) a e_1, & t > 0, \\ u \to 0 & \text{as } |x| \to \infty, \\ u(x, 0) = 0, & x \in D. \end{cases} \tag{1.2}$$

Here, $u = (u_1(x,t), \cdots, u_n(x,t))^\top$ and $p = p(x,t)$ denote unknown velocity and pressure of the fluid, respectively. Since $\psi(t) = 1$ for $t \ge 1$, the large time behavior of solutions is related to the stationary problem

$$\begin{cases} u_s \cdot \nabla u_s = \Delta u_s - a \dfrac{\partial u_s}{\partial x_1} - \nabla p_s, & x \in D, \\ \nabla \cdot u_s = 0, & x \in D, \\ u_s|_{\partial D} = -a e_1, & \\ u_s \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.3}$$

We look for a nonstationary solution to (1.2), which tends to a stationary solution to (1.3) as $t \to \infty$. Moreover, we derive new convergence rate, that is determined by the summability of stationary solution corresponding to a small terminal velocity. We thus first consider the stationary problem. In $n = 3$, the pioneering work is due to Leray [16]. He provided the existence theorem for weak solution to problem (1.3), what is called D-solution, having finite Dirichlet integral without smallness assumption on data. But his solution didn't have the anisotropic decay structure caused by translation. To fill this gap, Finn introduced another class of solutions (physically reasonable solutions) and succeeded in constructing a solution possessing the anisotropic decay structure. We note that D-solutions become physically reasonable solutions, see Babenko [1], Galdi [11] and Farwig and Sohr [4]. But in this paper, we construct a stationary solution having the optimal summability at spatial infinity in $n \ge 3$ and give the shorter proof of the existence theorem for stationary solutions.

Note that the summability of the Oseen fundamental solution

$$\mathbf{E} \in L^q(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad q > \frac{n+1}{n-1}, \qquad \nabla \mathbf{E} \in L^r(\{x \in \mathbb{R}^n \mid |x| > 1\}), \quad r > \frac{n+1}{n}, \tag{1.4}$$

see Galdi [12, Section VII], is optimal summability of stationary solutions at infinity as long as the net force does not vanish. For the proof, we rely on L^q theory of the Oseen problem developed by Galdi, see Proposition 3.1. By making use of his result, we find a certain closed ball N so that a map $\Psi : N \ni v \mapsto u \in N$ which provides the solution to the problem

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + v \cdot \nabla v, & x \in D, \\ \nabla \cdot u = 0, & x \in D, \\ u|_{\partial D} = -ae_1, & \\ u \to 0 & \text{as } |x| \to \infty, \end{cases}$$

is well-defined and contractive. As long as we only use Proposition 3.1, the space in which estimates of Ψ are closed is

$$\{u \in L^{n+1}(D) \mid \nabla u \in L^{\frac{n+1}{2}}(D)\}.$$

But we cannot capture the optimal summability at infinity, thus find a closed ball within the space

$$\{u \in L^{\alpha_1}(D) \cap L^{\alpha_2}(D) \mid \nabla u \in L^{\beta_1}(D) \cap L^{\beta_2}(D)\}. \tag{1.5}$$

It is not straightforward to prove the well-definedness of Ψ in the space (1.5). We in fact take suitable parameters (q_1, q_2, q_3, q_4) satisfying $\alpha_1 \le q_1 \le q_2 \le \alpha_2, \beta_1 \le r_1 \le r_2 \le \beta_2$ and

$$\frac{2}{n} < \frac{1}{q_i} + \frac{1}{r_i} < 1, \quad i = 1, 2$$

required in Proposition 3.1. We then apply Proposition 3.1 to $f = v \cdot \nabla v$ with $v \in L^{q_1}(D), \nabla v \in L^{r_1}(D)$ and $v \in L^{q_2}(D), \nabla v \in L^{r_2}(D)$.

Let us proceed to the starting problem. We prove the attainability of the stationary solution obtained above by applying the L^q-L^r estimate of the Oseen semigroup [3, 15]. Since the fluid is initially at rest and the stationary solution u_s presents in the forcing term of the integral equation for perturbation, we expect that the convergence rate is determined by the summability of u_s, in fact, we derive

$$\|u(t) - u_s\|_q = O(t^{-\frac{1}{2} + \frac{n}{2q} - \frac{\rho_1}{2}}), \quad n \le q \le \infty, \tag{1.6}$$

$$\|\nabla u(t) - \nabla u_s\|_n = O(t^{-\frac{1}{2} - \frac{\rho_1}{2}}) \tag{1.7}$$

as $t \to \infty$, where $\|\cdot\|_q$ denotes the L^q norm and $u_s \in L^{n/(1+\rho_1)}(D)$ with some $\rho_1 > 0$. Our rate is the improvement of the one in Galdi, Heywood and Shibata [13], which is the same as in stability analysis (Shibata [18]). The key step of the proof of (1.6)–(1.7) is the L^n convergence

$$\|u(t)-u_s\|_n = O(t^{-\frac{\rho_1}{2}}) \tag{1.8}$$

as $t \to \infty$. We first derive the slower convergence $\|u(t)-u_s\|_{L^n(D)} = O(t^{-\frac{\rho}{2}})$ with some $\rho \in (0,1)$. This convergence property implies the better convergence properties of the other norm. We then repeat improvements of L^n convergence step by step to obtain (1.8). We note that if $n \geq 4$, L^{q_0} convergence is needed, where $q_0 < n$ is appropriately chosen in Lemma 3.5.

2 Results

First result is the existence and summability of stationary solutions.

Theorem 2.1. *Let $n \geq 3$. For every $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ satisfying*

$$\frac{n+1}{n-1} < \alpha_1 \leq n+1 \leq \alpha_2 < \frac{n(n+1)}{2}, \quad \frac{n+1}{n} < \beta_1 \leq \frac{n+1}{2} \leq \beta_2 < \frac{n(n+1)}{n+2}, \tag{2.1}$$

there exists a constant $\delta = \delta(\alpha_1, \alpha_2, \beta_1, \beta_2, n, D) \in (0,1)$ such that if

$$0 < a^{\frac{n-2}{n+1}} < \delta,$$

problem (1.3) *admits a unique solution u_s along with*

$$\|u_s\|_{\alpha_1} + \|u_s\|_{\alpha_2} \leq Ca^{\frac{n-1}{n+1}}, \quad \|\nabla u_s\|_{\beta_1} + \|\nabla u_s\|_{\beta_2} \leq Ca^{\frac{n}{n+1}}, \tag{2.2}$$

where $C > 0$ is independent of a.

The lower bounds of (2.1) coincide with the optimal summability of the Oseen fundamental solution at spatial infinity, see (1.4), and the upper bounds of (2.1) come from (3.1) with $q < n/2$ in Proposition 3.1.

We next study the starting problem. To prove the attainability of the stationary solution, it is convenient to set

$$\alpha_1 = \frac{n}{1+\rho_1}, \qquad \alpha_2 = \frac{n}{1-\rho_2}, \qquad \beta_1 = \frac{n}{2+\rho_3}, \qquad \beta_2 = \frac{n}{2-\rho_4} \tag{2.3}$$

with $(\rho_1, \rho_2, \rho_3, \rho_4)$ satisfying

$$0 < \rho_1 < \frac{n^2-2n-1}{n+1}, \quad \frac{1}{n+1} \leq \rho_2 < \frac{n-1}{n+1}, \quad 0 < \rho_3 < \frac{n^2-2n-2}{n+1}, \quad \frac{2}{n+1} \leq \rho_4 < \frac{n}{n+1} \tag{2.4}$$

and we impose the additional condition

$$\rho_2 + \rho_4 > 1. \tag{2.5}$$

It is reasonable to look for a nonstationary solution (u,p) by the form

$$u(x,t) = v(x,t) + \psi(t)u_s, \quad p(x,t) = \phi(x,t) + \psi(t)p_s.$$

Then the perturbation (v,ϕ) satisfies the following problem

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v - a\dfrac{\partial v}{\partial x_1} - v\cdot\nabla v - \psi(t)v\cdot\nabla u_s - \psi(t)u_s\cdot\nabla v + (1-\psi(t))a\dfrac{\partial v}{\partial x_1} \\ \qquad\qquad +h_1(x,t) + h_2(x,t) - \nabla\phi, \quad x\in D,\, t>0, \\ \nabla\cdot v = 0, \quad x\in D,\, t\geq 0, \\ v|_{\partial D} = 0, \quad t>0, \\ v\to 0 \quad \text{as } |x|\to\infty, \\ v(x,0) = 0, \quad x\in D, \end{cases}$$

where

$$h_1(x,t) = -\psi'(t)u_s, \tag{2.6}$$

$$h_2(x,t) = \psi(t)\big(1-\psi(t)\big)\Big(u_s\cdot\nabla u_s + a\frac{\partial u_s}{\partial x_1}\Big), \tag{2.7}$$

thus obeys

$$v(t) = \int_0^t e^{-(t-\tau)A_a}P\Big[-v\cdot\nabla v - \psi(\tau)v\cdot\nabla u_s - \psi(\tau)u_s\cdot\nabla v + \big(1-\psi(\tau)\big)a\frac{\partial v}{\partial x_1} + h_1(\tau) + h_2(\tau)\Big]d\tau \tag{2.8}$$

by using the Oseen semigroup e^{-tA_a} (see the next section) and the Fujita-Kato projection P from $L^q(D)$ onto solenoidal L^q space $L^q_\sigma(D)$ associated with the Helmholtz decomposition ([9], [17], [19]). For the attainability of the stationary solution, we have the following.

Theorem 2.2. *Let $n\geq 3$ and let ψ be a function on $\mathbb{R}$ satisfying* (1.1). *We set $M = \max_{t\in\mathbb{R}}|\psi'(t)|$. Suppose that ρ_1, ρ_2, ρ_3 and ρ_4 satisfy* (2.4)–(2.5) *and let δ be the constant in Theorem 2.1 with* (2.3). *Then there exists a constant $\varepsilon = \varepsilon(n,D)\in(0,\delta]$ such that if*

$$0 < (M+1)a^{\frac{n-2}{n+1}} < \varepsilon,$$

(2.8) *admits a unique solution v within the class*

$$Y := \big\{v\in BC([0,\infty);L^n_\sigma(D)) \mid t^{\frac12}v\in BC((0,\infty);L^\infty(D)), t^{\frac12}\nabla v\in BC((0,\infty);L^n(D)), \lim_{t\to 0} t^{\frac12}\big(\|v(t)\|_\infty + \|\nabla v(t)\|_n\big) = 0\big\}. \tag{2.9}$$

Moreover, let $n = 3$. *Then there exists a constant* $\varepsilon_* = \varepsilon_*(D) \in (0, \varepsilon]$ *such that if* $0 < (M+1)a^{1/4} < \varepsilon_*$, *the solution* v *enjoys decay properties*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{3}{2q}-\frac{\rho_1}{2}}), \quad 3 \le \forall q \le \infty, \quad \|\nabla v(t)\|_3 = O(t^{-\frac{1}{2}-\frac{\rho_1}{2}})$$

as $t \to \infty$.

Let $n \ge 4$ *and suppose that* $\rho_3 > 1$ *and* $1 < \rho_1 \le 1 + \rho_3$ *in addition to* (2.4) (*the set of those parameters is nonvoid when* $n \ge 4$). *Then there exists a constant* $\varepsilon_* = \varepsilon_*(n, D) \in (0, \varepsilon]$ *such that if* $0 < (M+1)a^{(n-2)/(n+1)} < \varepsilon_*$, *the solution* v *enjoys*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{n}{2q}-\frac{\rho_1}{2}}), \quad n \le \forall q \le \infty, \quad \|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho_1}{2}})$$

as $t \to \infty$.

3 Proof of results

For the proof of Theorem 2.1, we make use of the result on the Oseen problem due to Galdi [12, Theorem VII.7.1], see also [10] for the first proof.

Proposition 3.1. *Let* $n \ge 3$. *Suppose* $a > 0$ *and* $1 < q < (n+1)/2$. *Given* $f \in L^q(D)$ *and* u_* *belonging to the trace space* $W^{2-1/q,q}(\partial D)$, *problem*

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + f, & x \in D, \\ \nabla \cdot u = 0, & x \in D, \\ u|_{\partial D} = u_*, & \\ u \to 0 & \text{as } |x| \to \infty \end{cases}$$

admits a unique (*up to an additive constant for* p) *solution* (u, p) *within the class*

$$X_q(n) := \Big\{(u,p) \in L^1_{\text{loc}}(D) \,\Big|\, u \in L^{s_2}(D), \nabla u \in L^{s_1}(D), \nabla^2 u \in L^q(D), \frac{\partial u}{\partial x_1} \in L^q(D), \nabla p \in L^q(D)\Big\},$$

where

$$\frac{1}{s_1} = \frac{1}{q} - \frac{1}{n+1}, \quad \frac{1}{s_2} = \frac{1}{q} - \frac{2}{n+1}. \tag{3.1}$$

If, in particular, $a \in (0,1]$ *and* $q < n/2$, *then the solution* (u,p) *obtained above satisfies*

$$a^{\frac{2}{n+1}}\|u\|_{s_2} + a\left\|\frac{\partial u}{\partial x_1}\right\|_q + a^{\frac{1}{n+1}}\|\nabla u\|_{s_1} + \|\nabla^2 u\|_q + \|\nabla p\|_q \le C\big(\|f\|_q + \|u_*\|_{W^{2-\frac{1}{q},q}(\partial D)}\big)$$

with a constant $C > 0$ *dependent on* q, n *and* D, *however, independent of* a.

We define

$$B := \{u \in L^{\alpha_1}(D) \cap L^{\alpha_2}(D) \mid \nabla u \in L^{\beta_1}(D) \cap L^{\beta_2}(D)\},$$

which is a Banach space equipped with the norm

$$\|u\|_B := \sum_{i=1}^{2} (a^{\frac{2}{n+1}} \|u\|_{\alpha_i} + a^{\frac{1}{n+1}} \|\nabla u\|_{\beta_i}).$$

To prove Theorem 2.1, we find a certain closed ball N of B so that a map $\Psi : N \ni v \mapsto u \in N$ which provides the solution to the problem

$$\begin{cases} \Delta u - a\dfrac{\partial u}{\partial x_1} = \nabla p + v \cdot \nabla v, & x \in D, \\ \nabla \cdot u = 0, & x \in D, \\ u|_{\partial D} = -ae_1, & \\ u \to 0 & \text{as } |x| \to \infty, \end{cases}$$

is well-defined and contractive. If α_1 and β_1 are simultaneously close to upper bounds or if α_2 and β_2 are simultaneously close to lower bounds, we cannot apply Proposition 3.1 to $f = v \cdot \nabla v \in L^{\alpha_1}(D) \cap L^{\beta_1}(D)$ and $f = v \cdot \nabla v \in L^{\alpha_2}(D) \cap L^{\beta_2}(D)$ because the relation

$$\frac{2}{n} < \frac{1}{\alpha_2} + \frac{1}{\beta_2} < \frac{1}{\alpha_1} + \frac{1}{\beta_1} < 1$$

required in Proposition 3.1, is not satisfied. To overcome this difficulty, we take (q_1, q_2, r_1, r_2) satisfying

$$\begin{aligned} &\frac{n+1}{n-1} < \alpha_1 \le q_1 \le n+1 \le q_2 \le \alpha_2 < \frac{n(n+1)}{2}, \\ &\frac{n+1}{n} < \beta_1 \le r_1 \le \frac{n+1}{2} \le r_2 \le \beta_2 < \frac{n(n+1)}{n+2}, \\ &\max\left\{\frac{1}{\alpha_1} + \frac{2}{n+1}, \frac{1}{\beta_1} + \frac{1}{n+1}\right\} \le \frac{1}{q_1} + \frac{1}{r_1} < 1, \\ &\frac{2}{n} < \frac{1}{q_2} + \frac{1}{r_2} \le \min\left\{\frac{1}{\alpha_2} + \frac{2}{n+1}, \frac{1}{\beta_2} + \frac{1}{n+1}\right\} \end{aligned}$$

and we apply Proposition 3.1 to $f = v \cdot \nabla v \in L^{q_1}(D) \cap L^{r_1}(D)$ and $f = v \cdot \nabla v \in L^{q_2}(D) \cap L^{r_2}(D)$. We then obtain a solution $u \in B$, thus the map Ψ can be well-defined. Moreover, from the estimate in Proposition 3.1, the map Ψ is contractive, which completes the proof of Theorem 2.1.

We next provide the proof of Theorem 2.2. We define the Oseen operator $A_a : L^q_\sigma(D) \to L^q_\sigma(D)$ $(a > 0, 1 < q < \infty)$ by

$$\mathscr{D}(A_a) = W^{2,q}(D) \cap W^{1,q}_0(D) \cap L^q_\sigma(D), \quad A_a u = -P\left[\Delta u - a\frac{\partial u}{\partial x_1}\right].$$

Here, $W_0^{1,q}(D)$ denotes the completion of $C_0^\infty(D)$ in the Sobolev space $W^{1,q}(D)$. Perturbation argument implies that $-A_a$ generates an analytic C_0-semigroup e^{-tA_a} called the Oseen semigroup in $L^q_\sigma(D)$. The following L^q-L^r estimates of e^{-tA_a} were established by Kobayashi and Shibata [15] in the three-dimensional case and further developed by Enomoto and Shibata [2,3] for $n \geq 3$.

Proposition 3.2 ([2,3,15]). *Let $n \geq 3$, $\sigma_0 > 0$ and assume $|a| \leq \sigma_0$.*

1. *Let $1 < q \leq r \leq \infty$ ($q \neq \infty$). Then we have*
$$\|e^{-tA_a} f\|_r \leq C t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})} \|f\|_q$$
for $t > 0$ and $f \in L^q_\sigma(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.

2. *Let $1 < q \leq r \leq n$. Then we have*
$$\|\nabla e^{-tA_a} f\|_r \leq C t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|f\|_q$$
for $t > 0$ and $f \in L^q_\sigma(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.

3. *Let $n/(n-1) \leq q \leq r \leq \infty$ ($q \neq \infty$). Then we have*
$$\|e^{-tA_a} P\nabla \cdot F\|_r \leq C t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|F\|_q$$
for $t > 0$ and $F \in L^q(D)$, where $C = C(n, \sigma_0, q, r, D) > 0$ is independent of a.

We recall a function space Y defined by (2.9), which is a Banach space endowed with norm $\|\cdot\|_Y = \|\cdot\|_{Y,\infty}$, where

$$\begin{aligned}
&\|v\|_{Y,t} := [v]_{n,t} + [v]_{\infty,t} + [\nabla v]_{n,t},\\
&[v]_{q,t} := \sup_{0<\tau<t} \tau^{\frac{1}{2}-\frac{n}{2q}} \|v(\tau)\|_q, \quad q = n, \infty; \qquad [\nabla v]_{n,t} := \sup_{0<\tau<t} \tau^{\frac{1}{2}} \|\nabla v(\tau)\|_n
\end{aligned}$$

for $t \in (0, \infty]$. By making use of Proposition 3.2, we have the following lemma.

Lemma 3.3. *Suppose that u_s is the stationary solution obtained in Theorem 2.1. For $u, v \in Y$, we set*

$$\begin{aligned}
&G_1(u,v)(t) = \int_0^t e^{-(t-\tau)A_a} P[u \cdot \nabla v](\tau)\, d\tau, \quad G_2(v)(t) = \int_0^t e^{-(t-\tau)A_a} P[\psi(\tau) v \cdot \nabla u_s]\, d\tau,\\
&G_3(v)(t) = \int_0^t e^{-(t-\tau)A_a} P[\psi(\tau) u_s \cdot \nabla v]\, d\tau,\\
&G_4(v)(t) = \int_0^t e^{-(t-\tau)A_a} P\left[(1-\psi(\tau)) a \frac{\partial v}{\partial x_1}(\tau)\right] d\tau,\\
&H_1(t) = \int_0^t e^{-(t-\tau)A_a} P h_1(\tau)\, d\tau, \quad H_2(t) = \int_0^t e^{-(t-\tau)A_a} P h_2(\tau)\, d\tau,
\end{aligned}$$

where h_1 and h_2 are defined by (2.6) *and* (2.7)*, respectively. Then we have $G_1(u,v), G_i(v), H_j \in Y$ $(i = 2,3,4, j = 1,2)$ along with*

$$\begin{aligned}
&\|G_1(u,v)\|_{Y,t} \le C[u]_{n,t}^{\frac{1}{2}}[u]_{\infty,t}^{\frac{1}{2}}[\nabla v]_{n,t},\\
&\|G_2(v)\|_{Y,t} \le C\big(\|\nabla u_s\|_{\frac{n}{2+\rho_3}} + \|\nabla u_s\|_{\frac{n}{2}} + \|\nabla u_s\|_{\frac{n}{2-\rho_4}}\big)[v]_{\infty,t},\\
&\|G_3(v)\|_{Y,t} \le C\big(\|u_s\|_{\frac{n}{1+\rho_1}} + \|u_s\|_n + \|u_s\|_{\frac{n}{1-\rho_2}}\big)[\nabla v]_{n,t}, \quad \|G_4(v)\|_{Y,t} \le Ca[\nabla v]_{n,t},\\
&\|H_1\|_{Y,t} \le CM\|u_s\|_n, \quad \|H_2\|_{Y,t} \le C\big(\|u_s\|_{\frac{n}{1-\rho_2}}\|\nabla u_s\|_{\frac{n}{2-\rho_4}} + a\|\nabla u_s\|_{\frac{n}{2-\rho_4}}\big)
\end{aligned}$$

for all $t \in (0,\infty]$ and

$$\lim_{t\to 0}\|H_j(t)\|_{Y,t} = 0$$

for $j = 1,2$. Here, C is a positive constant independent of u, v, ψ, a and t.

Lemma 3.3 implies that the estimate of terms in the equation (2.8) is closed in Banach space Y. Thus we can obtain a unique global solution v within Y. For the decay properties, we first derive slower decay in the following proposition.

Proposition 3.4. *Given $\rho \in (0,1)$ satisfying $\rho \le \min\{\rho_1, \rho_3\}$, where $u_s \in L^{3/(1+\alpha_1)}(D)$ and $\nabla u_s \in L^{3/(2+\rho_3)}(D)$, the solution $v(t)$ satisfies*

$$\|v(t)\|_q = O(t^{-\frac{1}{2}+\frac{n}{2q}-\frac{\rho}{2}}), \qquad n \le \forall q \le \infty, \tag{3.2}$$

$$\|\nabla v(t)\|_n = O(t^{-\frac{1}{2}-\frac{\rho}{2}}) \tag{3.3}$$

as $t \to \infty$.

When $n = 3$, we can take $\rho := \min\{\rho_1, \rho_3\}$ in Proposition 3.4, thus get better decay properties of the other norms of the solution. With them at hand, we repeat improvement of the estimate of $\|v(t)\|_3$ step by step. Namely, we can prove by induction that

$$\|v(t)\|_3 = O(t^{-\sigma_k}), \qquad \sigma_k := \min\left\{\frac{k}{2}\rho_3,\ \frac{\rho_1}{2}\right\} \tag{3.4}$$

as $t \to \infty$ for all $k \ge 1$. This together with the same argument as in Enomoto-Shibata [3] asserts Theorem 2.2 with $n = 3$.

For $n \ge 4$, we derive the L^{q_0}-decay of the solution with specific $q_0 < n$, see (3.5).

Proposition 3.5. *Let $n \ge 4$. Given γ satisfying*

$$\max\left\{0,\ \frac{\rho_1+3-n}{2}\right\} < \gamma < \frac{1}{2}$$

(*note that* (2.4) *yields $\rho_1 < n-2$), we have $v(t) \in L^{q_0}(D)$ for all $t > 0$ with*

$$\sup_{\tau>0}(1+\tau)^\gamma\|v(\tau)\|_{q_0} < \infty,$$

where

$$q_0 := \frac{n}{1+\rho_1-2\gamma}\ (< n). \tag{3.5}$$

The rate of $L^{n/(1+\rho_1)}$-L^{q_0} estimate is $-\gamma$, thus the relation between γ and q_0 is determined by h_1, see (2.6). To conclude Proposition 3.5, we prove $v_m(t) \in L^{q_0}(D)$ for all $t > 0$ along with $K_m := \sup_{\tau>0}(1+\tau)^\gamma \|v_m(\tau)\|_{q_0} \leq C$ for all $m \geq 1$, where $v_m(t)$ is the approximate solution and C is independent of m. Since $\|v_m(t) - v(t)\|_n \to 0$ as $m \to \infty$ for each $t > 0$, the proof of Proposition 3.5 is complete. Proposition 3.4 and Proposition 3.5 yield Theorem 2.2 with $n \geq 4$.

References

[1] Babenko, K.I.: On stationary solutions of the problem of flow past a body of viscous incompressible fluid. Math. Sb. **91**, 3–27 (1973); English Translation: Math. USSR Sbornik **20**, 1–25 (1973)

[2] Enomoto, Y., Shibata, Y.: Local Energy Decay of Solutions to the Oseen Equation in the Exterior Domains. Indiana Univ. Math. J. **53**, 1291–1330 (2004)

[3] Enomoto, Y., Shibata, Y.: On the Rate of Decay of the Oseen Semigroup in the Exterior Domains and its Application to Navier-Stokes Equation. J. Math. fluid mech. **7**, 339–367 (2005)

[4] Farwig, R., Sohr, H.: Weighted estimates for the Oseen equations and the Navier-Stokes equations in exterior domains. In: Heywood, J.G. et al. (eds.) Theory of the Navier-Stokes Equations, Series on Advances in Mathematics for Applied Sciences, vol. 47, pp. 11–30. World Scientific Publishing, River Edge (1998)

[5] Finn, R.: Estimate at infinity for stationary solutions of the Navier-Stokes equations. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumanie (N.S.) **3**, 387–418 (1959)

[6] Finn, R.: Stationary solutions of the Navier-Stokes equations. In: Proceedings of Symposia in Applied Mathematics, vol. 17, pp. 121–153 (1965)

[7] Finn, R.: On the exterior stationary problem for the Navier-Stokes equations, and associated perturbation problems. Arch. Rational Mech. Anal. **19**, 363–406 (1965)

[8] Finn, R.: Mathematical questions relating to viscous fluid flow in an exterior domain. Rocky Mountain J. Math. **3**, 107–140 (1973)

[9] Fujiwara, D., Morimoto, H.: An L_r-theorem of the Helmholtz decomposition of vector fields. J. Fac. Sci. Univ. Tokyo Sect. IA Math **24**, 685–700 (1977)

[10] Galdi, G.P.: On the Oseen boundary value problem in exterior domains. In: The Navier-Stokes Equations II — Theory and Numerical Methods (Oberwolfach, 1991), vol. 1530 of Lecture Notes in Math., pp. 111–131. Springer, Berlin (1992)

[11] Galdi, G.P.: On the Asymptotic Structure of D-Solutions to Steady Navier-Stokes Equations in Exterior Domains. In: Galdi, G.P. (ed.) Mathematical Problems Related to the Navier-Stokes Equation, Series on Advances in Mathematics for Applied Sciences, vol. 11, pp. 81–104. World Scientific Publishing, River Edge (1992)

[12] Galdi, G.P.: An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems, Second Edition. Springer, New York (2011)

[13] Galdi, G.P., Heywood, J.G., Shibata, Y.: On the global existence and convergence to steady state of Navier-Stokes flow past an obstacle that is started from rest. Arch. Rational Mech. Anal. **138**, 307–318 (1997)

[14] Heywood, J.G.: The exterior nonstationary problem for the Navier-Stokes equations. Acta Math. **129**, 11–34 (1972)

[15] Kobayashi, T., Shibata, Y.: On the Oseen equation in the three dimensional exterior domains. Math. Ann. **310**, 1–45 (1998)

[16] Leray, J.: Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique. J. Math. Pures Appl. **12**, 1–82 (1933)

[17] Miyakawa, T.: On nonstationary solutions of the Navier-Stokes equations in an exterior domain. Hiroshima Math. J. **12**, 115–140 (1982)

[18] Shibata, Y.: On an exterior initial boundary value problem for Navier-Stokes equations. Quart. Appl. Math. **LVII**, 117–155 (1999)

[19] Simader, C.G., Sohr, H.: A new approach to the Helmholtz decomposition and the Neumann problem in L^q-spaces for bounded and exterior domains. In: Galdi, G.P. (ed.) Mathematical Problems Relating to the Navier-Stokes Equations, Series on Advances in Mathematics for Applied Sciences, vol. 11, pp. 1–35. World Scientific Publishing, River Edge (1992)

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