A free-boundary problem for the spherically symmetric motion of a viscous heat-conducting and self-gravitating gas

宮崎大学・工学部　　梅原　守道
Morimichi Umehara*
University of Miyazaki

1 Introduction

We consider a system of equations describing a spherically symmetric n-dimensional ($n \geq 3$) movement around a rigid core (sphere), of a viscous and heat-conducting gas. The gas is bounded by a free-surface, and its motion is driven by both an external body force and a surface force on the free-boundary from outside of the media. Equations describing the motion mentioned above are, in the Lagrangian-mass coordinate system, for $(x,t) \in \Omega \times (0,\infty)$ with $\Omega := (0,1)$

$$\begin{cases} v_t = w_x, \\ u_t = r^{n-1}\left(-p + \dfrac{w_x}{v}\right)_x + f, \\ e_t = \left(-p + \dfrac{w_x}{v}\right) w_x - 2(n-1)\mu\left(\dfrac{w^2}{r^n}\right)_x + \left(\kappa r^{2n-2}\dfrac{\theta_x}{v}\right)_x, \end{cases} \tag{1.1}$$

where $w = r^{n-1}u$; $p = p(v,\theta)$ is the pressure and $e = e(v,\theta)$ is the internal energy par unit mass; $f = f[v] = \hat{f}(r[v],x)$ is the external body force per unit mass; $r = r(x,t) = r[v](x,t)$ is given by

$$r[v](x,t) = \left(1 + n\int_0^x v(s,t)\,\mathrm{d}s\right)^{1/n}.$$

Imposed boundary conditions are

$$\begin{cases} (u,\theta_x)\big|_{x=0} = (0,0), \\ \left(-p + \dfrac{w_x}{v} - 2(n-1)\mu\dfrac{w}{r^n},\, \theta_x\right)\Big|_{x=1} = (\bar{\sigma},0) \end{cases} \tag{1.2}$$

with given $\bar{\sigma} = \bar{\sigma}(t)$. Unknown quantities are the specific volume $v = v(x,t)$, the velocity $u = u(x,t)$ and the absolute temperature $\theta = \theta(x,t)$. In this paper we focus

*Department of Applied Physics, Faculty of Engineering, University of Miyazaki, 1-1 Gakuen Kibanadai Nishi, Miyazaki 889-2192, JAPAN; Fax: +81 985587378; E-mail: umehara@cc.miyazaki-u.ac.jp

our discussion only on a typical case:

$$p(v,\theta) = \frac{\theta}{v}, \quad e(v,\theta) = \frac{\theta}{\gamma - 1}, \tag{1.3}$$

$$\hat{f}(r,x) = -\frac{G(x)}{r^{n-1}}, \quad G(x) = G_n(\beta + x), \tag{1.4}$$

$$\bar{\sigma}(t) = -p_e, \quad p_e = \text{const.} > 0, \tag{1.5}$$

where γ is the specific heat ratio assumed to be a constant with $\gamma > 1$; $G_n = (n-2)G_0$ and G_0 is the gravitational constant ($G_0 > 0$); β is a constant meaning the mass of the central rigid sphere ($\beta > 0$). Equations (1.3)-(1.5) say, respectively, that the gas is ideal; the external body force is given by the gravitation due to both central rigid core and the gas itself; the surface force is the external pressure. Here κ and μ are assumed to be constants with $\kappa > 0$ and $0 < \mu \leq \frac{n}{2(n-1)}$.

We seek to find unknown functions (v, u, θ) for given initial data

$$(v,u,\theta)|_{t=0} = (v_0, u_0, \theta_0) \tag{1.6}$$

satisfying the compatibility conditions

$$\begin{cases} (u_0, {\theta_0}')(0) = (0,0), \\ \left(-p_0 + \dfrac{{w_0}'}{v_0} - 2(n-1)\mu\dfrac{w_0}{{r_0}^n},\ {\theta_0}'\right)(1) = (\bar{\sigma}(0), 0) \end{cases} \tag{1.7}$$

with $p_0 = \theta_0/v_0$ and $w_0 = {r_0}^{n-1}u_0$.

Our system of equations (1.1) and (1.2) arises as some stellar models in some astrophysical arguments (see, for example, [1]). In the paper [2] a large-time behaviour of the flow of a stellar model closely-similar to ours was discussed. However, it seems as through some statements and proofs in [2] are with ambiguities: for example, it looks rather hard to accept that the estimate

$$\int_0^t \int_0^M \theta {v_x}^2 \, \mathrm{d}x \, \mathrm{d}\tau \leq C \quad \text{for any} \ \ t > 0$$

holds with some constant $C > 0$ independent of time (Lemma 9 in [2]), by which the proof of the main theorem of the concerned paper was anchored (In our case, we have Lemma 3.4). Although the system of equations (1.1)-(1.6) was investigated in [3] by Ducomet and Nečasová under the condition $\mu = 0$, the asymptotic behaviour of the flow was not discussed in that paper. Large-time behaviours of viscous gases have been investigated, for example, in [4,6] with no external force fields; in [9] with an attractive force due to a central core in a fixed annulus domain; in [8] with the self-gravitation of the gas in the framework that the gaseous motion is one-dimensional.

From physical point of view, it is expected that the solution of our problem (1.1)-(1.6) converges to a steady state as $t \to \infty$ in some sense, and the steady state be a certain stationary solution. For some barotropic viscous fluids, Ducomet and Zlotnik show this story in [10] under a certain restricted condition. In the present paper we obtain the condition similar to the one in [10] (see (2.5) or (2.7)) guaranteeing unique existence of the stationary solution of our problem, and by using the stationary solution one sees that the solution of our problem converges to the stationary solution as $t \to \infty$.

2 Statement of theorems

Let $Q_T := \Omega \times (0, T)$. First we note that we already have the following result concerning temporally global and unique solvability of our problem up to any fixed time T (Some notations are found in, for example, [5]).

Theorem 1 *Let* $\alpha \in (0, 1)$*. Assume that the initial data in* (1.6)

$$(v_0,\, u_0,\, \theta_0) \in C^{1+\alpha}(\overline{\Omega}) \times (C^{2+\alpha}(\overline{\Omega}))^2$$

satisfy (1.7) *and* $v_0,\, \theta_0 > 0$ *on* $\overline{\Omega}$*. Then there exists a unique solution* (v, u, θ) *of the initial-boundary value problem* (1.1)-(1.6) *such that*

$$(v,\, v_x,\, v_t,\, u,\, \theta) \in (C^{\alpha,\, \frac{\alpha}{2}}_{x,t}(\overline{Q_T}))^3 \times (C^{2+\alpha,\, 1+\frac{\alpha}{2}}_{x,t}(\overline{Q_T}))^2 \tag{2.1}$$

for any positive number T *and the following inequalities hold*:

$$|v,\, v_x,\, v_t|^{(\alpha)}_{Q_T} + |u,\, \theta|^{(2+\alpha)}_{Q_T} \leq C, \qquad v,\, \theta \geq C^{-1} \ \ in \ \overline{Q_T}, \tag{2.2}$$

where C *is a positive constant dependent on the initial data and* T*.*

For the proof of Theorem 1, see [7], in which the case of not only ideal gas, but also radiative and reactive gas was discussed.

In order to investigate large-time behaviour of the solution, we need to derive some uniform-in-time estimates of the solution. To this end, consider the function $V = V(x)$ and the constant Θ satisfying

$$\begin{cases} p(V, \Theta) = -\bar{\sigma} + \displaystyle\int_x^1 \left(-\frac{f[V]}{r^{n-1}} \right) \mathrm{d}s \quad (x \in \overline{\Omega}), \\ e(V, \Theta) = K - E_1[V] \end{cases} \tag{2.3}$$

for some constant K. Here

$$E_1[V] := \int_0^1 (-\bar{\sigma} V - F[V])\, \mathrm{d}x, \quad F[V] = \hat{F}(r[V], x),$$

where the function $\hat{F}(r, x)$ is arbitrarily chosen in such a way as to satisfy

$$\frac{\partial \hat{F}}{\partial r} = \hat{f}, \quad \lim_{r \to \infty} \hat{F}(r, x) = 0 \quad (x \in \overline{\Omega}).$$

From $\hat{f} = -G(x)/r^{n-1}$ we may take

$$\hat{F}(r, x) = \frac{1}{n-2} \frac{G(x)}{r^{n-2}}.$$

It is easy to see that $(v, u, \theta) = (V, 0, \Theta)$ becomes a certain stationary solution of the problem (1.1) and (1.2). The second equation of (2.3) says that it is the sationary solution "having the total energy K" that we especially seek. Namely, by putting $K = E_0$ in (2.3) for

$$E_0 := \int_0^1 \left(\frac{1}{2} {u_0}^2 + e(v_0, \theta_0) \right) \mathrm{d}x + E_1[v_0]$$

and noting the equality (see later, Lemma 3.3)

$$\int_0^1 \left(\frac{1}{2} u^2 + e(v, \theta) \right) \mathrm{d}x + E_1[v] = E_0, \tag{2.4}$$

we may say that our stationary solution (if it exists) also have the energy E_0.

Let $a := G_n(\beta + 1/2)$. The following theorem says that the stationary solution uniquely exists under some restricted situation:

Theorem 2 *Assume that $K > E_1[0]$ and that*

$$\frac{1}{\gamma - 1} > 1 + \frac{a}{p_e} + 2(n-1)\frac{a}{p_e}\frac{K - E_1[0]}{p_e}. \tag{2.5}$$

Then there exists a unique solution (V, Θ) with $V = V(x)$ and $\Theta = \mathrm{const.}$ of the problem (2.3) such that $V \in C^2(\overline{\Omega})$, $V > 0$ on $\overline{\Omega}$ and $\Theta > 0$. Moreover, the inequality $K > E_1[\bar{v}]$ holds with

$$\bar{v} := \frac{(\gamma - 1)(K - E_1[0])}{p_e},$$

and the following inequalities hold:

$$\frac{(\gamma - 1)(K - E_1[\bar{v}])}{p_e + a} \leqslant V(x) \leqslant \bar{v} \quad (x \in \overline{\Omega}), \quad K - E_1[\bar{v}] \leqslant \Theta \leqslant p_e \bar{v}. \tag{2.6}$$

By using the steady state $(V, 0, \Theta)$ obtained as above, we reach the following main theorem concerning the asymptotic behaviour of the solution (v, u, θ).

Theorem 3 *Let T be an arbitrary positive number, and α and the initial data satisfy the hypotheses of Theorem 1. Assume that*

$$\frac{1}{\gamma - 1} > 1 + \frac{a}{p_e} + 2(n-1)\frac{a}{p_e}\frac{E_0 - E_1[0]}{p_e}. \tag{2.7}$$

Then there exists a positive constant C independent of T such that the inequalities (2.2) holds for the solution (v, u, θ) of the problem (1.1)-(1.6) in the class (2.1). Moreover, the solution (v, u, θ) converges to a steady state $(V, 0, \Theta)$ with $V = V(x)$ and $\Theta = \mathrm{const.}$ as $t \to \infty$ in the sense of $H^1(\Omega) \cap C(\overline{\Omega})$. Here V and Θ satisfy the inequalities (2.6) with $K = E_0$.

Remark 2.1 For any (admissible) initial data $E_0 - E_1[0] > 0$ because of

$$E_1[0] = \int_0^1 (-F[0])\,\mathrm{d}x = \int_0^1 \left(-\frac{G(x)}{n-2}\right)\mathrm{d}x = -\frac{a}{n-2}.$$

3 Outline of proofs of theorems

3.1 Sketch of the proof of Theorem 2

Let

$$\hat{v}(p,\theta) := \frac{\theta}{p}, \quad H[v] := p_e + \int_x^1 \left(-\frac{f}{r^{n-1}}\right)\mathrm{d}s, \quad I[v] := (\gamma-1)(K - E_1[v]).$$

Construct the sequence $\{v_i\}$ $(v_i = v_i(x))$ such that

$$\begin{cases} v_0 = M, \\ v_{i+1} = \hat{v}(p_i,\theta_i), \quad p_i = H[v_i], \quad \theta_i = I[v_i] \quad (i = 0,1,\ldots), \end{cases} \tag{3.1}$$

where the constant M taken arbitrarily as to satisfy

$$\underline{v} := \hat{v}(\bar{p},\underline{\theta}) \leqslant M \leqslant \hat{v}(p_e,\theta^*) = \bar{v}$$

with

$$\bar{p} := p_e + \int_0^1 (-\hat{f}(1,x))\,\mathrm{d}x = p_e + a,$$

$$\underline{\theta} := I[\bar{v}], \quad \theta^* := I[0] = (\gamma-1)\left(K + \frac{a}{n-2}\right)$$

Noting that, for the function

$$E_1(y) := p_e y - \int_0^1 \hat{F}(x, r(y))\,\mathrm{d}x, \quad r(y) := (1+nyx)^{1/n}, \quad x \in \overline{\Omega},$$

the positivities of p_e and $\partial\hat{f}/\partial r$ and the negativity of f lead $E_1{}'(y) > 0$ and $E_1{}''(y) < 0$ for any $y \in \overline{\Omega}$, we have bounds of v_i and θ_i from both above and below uniformly in i.

Lemma 3.1 *If the conditions $K > E_1[0]$ and*

$$\frac{1}{\gamma-1} \geqslant 1 + \frac{G_n}{p_e}\left(\frac{\beta}{2} + \frac{1}{3}\right)$$

are satisfied, then, for any $K > E_1[0]$, it holds that $\underline{\theta} > 0$ and

$$\underline{v} \leqslant v_i \leqslant \bar{v}, \quad \underline{\theta} \leqslant \theta_i < \theta^* \quad (i = 0,1,\ldots)$$

Next, let $(\Delta g)_i := g_{i+1} - g_i$. From $(3.1)^2$ we have

$$(\Delta v)_{i+1} = \frac{1}{p_{i+1}}\big((\Delta\theta)_i - v_i(\Delta p)_i\big). \tag{3.2}$$

We note that, for any $k \in \mathbb{N}$ and $r_j = r[v_j]$ $(j = 1, 2)$,

$$A_k(r_1, r_2) := -\left(\frac{1}{{r_2}^k} - \frac{1}{{r_1}^k}\right)\left[\int_0^x (v_2 - v_1)\,\mathrm{d}s\right]^{-1}$$

is a function of r_1 and r_2, and has the estimate

$$0 < A_k(r_1, r_2) \leqslant k.$$

Then we have from (3.2)

$$\begin{aligned} p_{i+1}(\Delta v)_{i+1} \;&=\; -(\gamma-1)\int_0^1\left(p_e + \int_x^1 \frac{A_{n-2}(r_{i+1}, r_i)(s)}{n-2} G(s)\,\mathrm{d}s\right)(\Delta v)_i(s)\,\mathrm{d}x \\ &\quad - v_i \int_x^1 A_{2n-2}(r_{r+1}, r_i)(s) G(s)\left(\int_0^s (\Delta v)_i(s')\,\mathrm{d}s'\right)\mathrm{d}s. \end{aligned}$$

Estimating the left-hand side of this equality from below and the right-hand side, from above, with using upper bound of v_i obtained in Lemma 3.1, one can get the convergence of the numerical sequence $\{J_i\}$ for $J_i = |(\Delta v)_i|^{(0)}$ $(i = 0, 1, \ldots)$ under the condition (2.5). Thus we obtain

Lemma 3.2 *If $K > E_1[0]$ and the condition (2.5) is satisfied, then there exists a function $v^* = v^*(x)$ such that $v^* \in C(\overline{\Omega})$ and $v_i \to v^*$ as $i \to \infty$ in the sense of $C(\overline{\Omega})$.*

Put $\theta^* := I[v^*]$. And see that $\theta_i \to \theta^*$ in $C(\overline{\Omega})$ $(i \to \infty)$ by virtue of Lemma 3.2. The conbination of the function v^* and the constant θ^* becomes the soluion (V, Θ) in Theorem 2.

3.2 Outline of the proof of Theorem 3

To prove Theorem 3, first we get the following estimates in suitable Sobolev spaces. Namely, for some generalized derivatives of the solution

$$(v_{xt},\, v_{tt},\, u_{xxx},\, u_{xt},\, \theta_{xxx},\, \theta_{xt}) \in (L^2(0,T; L^2(\Omega)))^6$$

we find a constant C independent of T such that

$$\begin{cases} \sup\limits_{t\in[0,T]} \|(v,\, v_x,\, v_t,\, v_{xt})(t)\| \le C, \\ \|u,\; \theta - \bar{\theta}\|_{\mathcal{E}(Q_T)} + \displaystyle\int_0^T \|(v - \tilde{v},\; v_x - \tilde{v}_x,\, v_{tt})(t)\|^2\,\mathrm{d}t \le C, \\ C^{-1} \le v,\; \theta \le C, \quad |u| \le C \quad \text{in } \overline{Q_T} \end{cases}$$

with the norm

$$\|u\|_{\mathcal{E}(Q_T)} := \|u\|_{\mathcal{E}_1(Q_T)} + \|u_x\|_{\mathcal{E}_1(Q_T)} + \|u_{xx}\|_{\mathcal{E}_1(Q_T)} + \|u_t\|_{\mathcal{E}_1(Q_T)},$$

$$\|u\|_{\mathcal{E}_1(Q_T)} := \left(\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T \|u_x(t)\|^2 \,\mathrm{d}t \right)^{\frac{1}{2}}.$$

In addition to these uniform bounds, we will also obtain

$$\|v - V\|_{H^1(\Omega)\cap C(\overline{\Omega})} + \|u,\, \theta - \Theta\|_{H^2(\Omega)\cap C(\overline{\Omega})} + \|u_t, \theta_t\| + \sup_{x\in\overline{\Omega}} |u_x, \theta_x| \to 0 \quad (t \to \infty).$$

Here let us get

Lemma 3.3 *The equality* (2.4) *holds for any* $t \in [0, T]$.

Proof. One can rewrite $(1.1)^2$ as

$$\left(\frac{w}{r^{2n-2}}\right)_t = \left(\frac{w_x}{v} + q - p\right)_x \tag{3.3}$$

with

$$q := \hat{q} - 2(n-2)\mu\frac{w}{r^n}(1,t) + \int_x^1 (n-2)\frac{w^2}{r^{3n-2}}\,\mathrm{d}s, \quad \hat{q} := -\bar{\sigma} + \int_x^1 \frac{-f}{r^{n-1}}\,\mathrm{d}s.$$

Note that from (1.2) it holds

$$\left(\frac{w_x}{v} + q - p\right)\Big|_{x=1} = 0.$$

From (3.3) we also have

$$u_t = r^{n-1}\left(\frac{w_x}{v} + \hat{q} - p\right)_x.$$

Multiplying this equation by u, adding it to $(1.1)^3$ and integrating that over $[0, 1]$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{1}{2}u^2 + \frac{\theta}{\gamma - 1}\right)\mathrm{d}x = -\int_0^1 \hat{q} w_x \,\mathrm{d}x,$$

whose right-hand side becomes

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(-\bar{\sigma}v - \hat{F}\right)\mathrm{d}x$$

by using $(1.1)^1$, the equation $r_t = u$ and integration by part. This gives (2.4) by the integration with respect to t. □

The next lemma plays an essential roll in deriving estimates in Sobolev spaces.

Lemma 3.4 *If the condition* (2.7) *is satisfied, then it holds*

$$\int_0^t \|(v-V)(\tau)\|^2 \, \mathrm{d}\tau \le C \tag{3.4}$$

for any $t \in [0,T]$, where C is a positive constant independent of T.

In order to prove this lemma, focus the following equality got from (3.3) for any differentiable and integrable function $g(v)$

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int^v g(v)\,\mathrm{d}v + \frac{w}{r^{2n-2}}\int_0^x vg(v)\,\mathrm{d}s\right) + v(q-p)g(v) \\ &\quad = \left[\left(\frac{w_x}{v}+q-p\right)\int_0^x vg(v)\,\mathrm{d}s\right]_x + \frac{w}{r^{2n-2}}\int_0^x (vg(v))_v w_x\,\mathrm{d}s. \end{aligned} \tag{3.5}$$

Substituting $g(v) = v - V$ into (3.5), noting

$$v(q-p) = q(v-V) - \big[\theta - \Theta - V(q-\tilde{p})\big]$$

with $\tilde{p} := H[V]$ and integrating it with respect to both x and t, one can get (3.4).

Detailed proofs of Theorems 2 and 3 will be found in forthcoming publications by the present authors.

References

[1] Chandrasekhar, S., *An Introduction to the Study of Stellar Structure*, Dover, New York, 1957.

[2] Ducomet, B., Some asymptotics for a reactive Navier-Stokes Poisson system, *Math. Mod. Meth. Appl. Sci.*, **9** (1999), 1039-1076.

[3] Ducomet, B. and Š. Nečasová, Free boundary problem for the equations of spherically symmetric motion of compressible gas with density-dependent viscosity, *J. Evol. Equ.*, **9** (2009), 469-490.

[4] Kazhikhov, A. V., To the theory of boundary value problems for equations of a one-dimensional non-stationary motion of a viscous heat-conductive gas, *Din. Sploshn. Sredy*, **50** (1981), 37-62. (Russian)

[5] Ladyžhenskaja, O. A., Solonnikov, V. A. and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type*, Transl. Math. Monogr. **23**, American Mathematical Society, Providence, R. I., 1968.

[6] Nagasawa, T., On the outer pressure problem of the one-dimensional polytropic ideal gas, *Japan J. Appl. Math.*, **5** (1988), 53-85.

[7] Umehara, M. and A. Tani, Temporally global solution to the equations for a spherically symmetric viscous radiative and reactive gas over the rigid core, *Anal. Appl.* (*Singap.*), **6** (2008), 183-211.

[8] Umehara, M. and A. Tani, Free-boundary problem of the one-dimensional equations for a viscous and heat-conductive gaseous flow under the self-gravitation, *Math. Mod. Meth. Appl.*, **23** (2013), 1377-1419.

[9] Yanagi, S., Asymptotic stability of the spherically symmetric solutions for a viscous polytropic gas in a field of external forces, Transport. *Theor. Statist. Phys.*, **29** (2000), 333-353.

[10] Zlotnik, A. and B. Ducomet, Stabilization rate and stability for viscous compressible barotropic symmetric flows with free boundary for a general mass force, *Mat. Sb.*, **196** (2005), 33-84. (Russian); translation in *Sb. Math.*, **196** (2005), 1745–1799.