The global well-posedness of the compressible fluid model of Korteweg type in a critical case

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1 Introduction

We consider the global well-posedness of the compressible viscous fluid model of Korteweg type which describes two phase flow with phase transition between liquid and vapor as diffuse interface models. More precisely, we investigate the following system in the N dimensional Euclidean space $\mathbb{R}^N$, $3 \le N \le 7$,

$$\begin{cases} \partial_t \rho + \operatorname{div} \mathbf{m} = 0, \\ \partial_t \mathbf{m} + \operatorname{Div} (\rho^{-1}\mathbf{m} \otimes \mathbf{m}) = \operatorname{Div} \big(\mathbf{S}(\rho^{-1}\mathbf{m}) + \mathbf{K}(\rho) - P(\rho)\mathbf{I}\big), \\ (\rho, \mathbf{m})|_{t=0} = (\rho_* + \rho_0, \mathbf{m}_0). \end{cases} \tag{1.1}$$

Here, $\partial_t = \partial/\partial t$, t is the time variable, $\rho = \rho(x,t)$, $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $\mathbf{m} = (m_1(x,t), \dots, m_N(x,t))^T$ are respective unknown density and momentum, where M^T denotes the transposed M. $P(\rho)$ is the pressure field satisfying a C^∞ function defined on $\rho > 0$, where ρ_* is a positive constant. Moreover, $\mathbf{S}(\mathbf{u})$ is the viscous stress tensor and $\mathbf{K}(\rho)$ is the Korteweg stress tensor read as

$$\begin{aligned} \mathbf{S}(\mathbf{u}) &= 2\mu\mathbf{D}(\mathbf{u}) + (\nu - \mu) \operatorname{div} \mathbf{u}\mathbf{I}, \\ \mathbf{K}(\rho) &= \frac{\kappa}{2}(\Delta\rho^2 - |\nabla\rho|^2)\mathbf{I} - \kappa\nabla\rho \otimes \nabla\rho, \end{aligned}$$

$\mathbf{D}(\mathbf{u})$ denotes the deformation tensor whose (j,k) components are $D_{jk}(\mathbf{u}) = (\partial_j u_k + \partial_k u_j)/2$ with $\partial_j = \partial/\partial x_j$; in addition, $\operatorname{div} \mathbf{u} = \sum_{j=1}^N \partial_j u_j$. For any $N \times N$ matrix field $\mathbf{L}$ with $(j,k)^{\text{th}}$ components L_{jk}, the quantity $\operatorname{Div} \mathbf{L}$ is an N-vector with j^{th} component

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$\sum_{k=1}^N \partial_k L_{jk}$; $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{a} \otimes \mathbf{b}$ denotes an $N \times N$ matrix with $(j,k)^{\text{th}}$ component $a_j b_k$ for any two N-vectors $\mathbf{a} = (a_1, \ldots, a_N)^T$ and $\mathbf{b} = (b_1, \ldots, b_N)^T$. We assume that the viscosity coefficients μ, ν, the capillary coefficient κ, and the mass density ρ_* of the reference body satisfy the conditions:

$$\mu > 0, \quad 2\mu + \nu > 0, \quad \text{and} \quad \kappa > 0. \tag{1.2}$$

Furthermore, we assume that the pressure $P(\rho)$ satisfies

$$P'(\rho_*) = 0. \tag{1.3}$$

Models describing two phase flow with phase transition are classified into two different types: sharp interface models and diffuse interface models. In sharp interface models, two fluids are separated by a phase boundary of zero thickness and physical quantities, such as density or pressure, allow for discontinuities across the interface. On the other hand, in diffuse interface models, the phase boundary is regarded as a narrow layers, which are called transition layer. In this region, physical quantities vary smoothly across the interface. Therefore, it is enough to consider a single system in a single spatial domain. To consider liquid-vapor flows as diffuse interface models, Korteweg [13] proposed the stress tensor including $\nabla\rho \otimes \nabla\rho$ based on Van der Waals's approach [21], later, Dunn and Serrin [7] derived the system (1.1).

An important aspect of diffuse interface models is that the pressure is non-monotone in general because we assume that the Helmholtz free energy is a double-well potential (cf. [6]). Solving the linearized problem around the equilibrium, we can expect the case $P'(\rho_*) < 0$ is unstable; hence we mention mathematical results for $P'(\rho_*) \geq 0$ below.

There are many results on global strong solutions for $P'(\rho_*) > 0$. Bresch, Desjardins, and Lin [2] proved the existence of a global weak solution, later, Haspot improved their result in [8]. Hattori and Li [9, 10] first showed the local and global well-posedness in Sobolev space. They assumed that the initial data $(\rho_0, \mathbf{u}_0)$ belong to $H^{s+1}(\mathbb{R}^N) \times H^s(\mathbb{R}^N)^N$ $(s \geq [N/2] + 3)$. Hou, Peng, and Zhu [11] improved the results [9, 10] for small total energy cases. Wang and Tan [22], Tan and Wang [18], Tan, Wang, and Xu [19], and Tan and Zhang [20] established the optimal decay rates of the global solutions in Sobolev space. Li [14] and Chen and Zhao [3] considered the Navier-Stokes-Korteweg system with external force. Bian, Yao, and Zhu [1] obtained the vanishing capillarity limit of the smooth solution. We also refer to the existence and uniqueness results in critical Besov space proved by Danchin and Desjardins in [5]. Their initial data $(\rho_0, \mathbf{u}_0)$ are assumed to belong to $\dot{B}^{N/2}_{2,1}(\mathbb{R}^N) \cap \dot{B}^{N/2-1}_{2,1}(\mathbb{R}^N) \times \dot{B}^{N/2-1}_{2,1}(\mathbb{R}^N)^N$. Recently, Murata and Shibata [15] proved the global well-posedness in the maximal L_p-L_q regularity class.

In contrast, only a few results are available for $P'(\rho_*) = 0$. Kobayashi and Tsuda [12] proved the existence of global L_2 solutions and the decay estimates. Chikami and Kobayashi [4] improved the result [5]. In particular, for $P'(\rho_*) = 0$, they proved the global estimates under an additional low frequency assumption to control a pressure term. Furthermore, they showed the optimal decay rates of the global solutions in the L_2-framework.

In this paper, we discuss the global existence and uniqueness of strong solutions to (1.1) for small initial data under the assumption (1.3). Consequently, we also prove the decay estimates of the solutions to (1.1). The main tools are the maximal L_p-L_q regularity and L_p-L_q decay properties of the linearized equations.

1.1 Notations

We summarize several symbols and functional spaces used throughout the paper. $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural, real, and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Let q' be the dual exponent of q defined by $q' = q/(q-1)$ for $1 < q < \infty$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$, we write $|\alpha| = \alpha_1+\cdots+\alpha_N$ and $\partial_x^\alpha = \partial_1^{\alpha_1}\cdots\partial_N^{\alpha_N}$ with $x = (x_1, \ldots, x_N)$. For scalar function f and N-vector of functions $\mathbf{g}$, we set

$$\partial_x^k f = (\partial_x^\alpha f \mid |\alpha| = k),\ \ \partial_x^k \mathbf{g} = (\partial_x^\alpha g_j \mid |\alpha| = k,\ \ j = 1, \ldots, N),$$
$$\partial_x^1 f = \partial_x f,\ \ \partial_x^1 \mathbf{g} = \partial_x \mathbf{g}.$$

For any $1 \le p, q \le \infty$, $L_q(\mathbb{R}^N)$, $W_q^m(\mathbb{R}^N)$, and $B^s_{q,p}(\mathbb{R}^N)$ denote the usual Lebesgue space, Sobolev space, and Besov space; respectively, $\|\cdot\|_{L_q(\mathbb{R}^N)}$, $\|\cdot\|_{W_q^m(\mathbb{R}^N)}$, and $\|\cdot\|_{B^s_{q,p}(\mathbb{R}^N)}$ denote their norms. We set $W_q^0(\mathbb{R}^N) = L_q(\mathbb{R}^N)$ and $W_q^s(\mathbb{R}^N) = B^s_{q,q}(\mathbb{R}^N)$ if $s \in \mathbb{R}\setminus\mathbb{N}$. $C^\infty(\mathbb{R}^N)$ denotes the set of all C^∞ functions defined on $\mathbb{R}^N$. For Banach spaces X and Y, $L_p((a,b), X)$ and $W_p^m((a,b), X)$ denote the usual Lebesgue space and Sobolev space of X-valued functions defined on an interval (a, b), respectively. The d-product space of X is defined by $X^d = \{f = (f, \ldots, f_d) \mid f_i \in X\ (i = 1, \ldots, d)\}$; for simplicity, its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$. We set

$$W_q^{m,\ell}(\mathbb{R}^N) = \{(f, \mathbf{g}) \mid f \in W_q^m(\mathbb{R}^N),\ \ \mathbf{g} \in W_q^\ell(\mathbb{R}^N)^N\},$$
$$\|(f, \mathbf{g})\|_{W_q^{m,\ell}(\mathbb{R}^N)} = \|f\|_{W_q^m(\mathbb{R}^N)} + \|\mathbf{g}\|_{W_q^\ell(\mathbb{R}^N)}.$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}_\xi^{-1} = \mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by setting

$$\hat f(\xi) = \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} f(x)\,dx, \quad \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi} g(\xi)\,d\xi.$$

The letter C denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a, b, \ldots$. The values of constants C, and $C_{a,b,\ldots}$ may change from line to line. We use small boldface letters, e.g., $\mathbf{u}$, to denote vector-valued functions and capital boldface letters, e.g., $\mathbf{H}$, to denote matrix-valued functions, respectively. To state our main theorem, we introduce a solution space and several norms:

$$\begin{aligned}
D_{q,p}(\mathbb{R}^N) &= B^{3-2/p}_{q,p}(\mathbb{R}^N) \times B^{2(1-1/p)}_{q,p}(\mathbb{R}^N)^N,\\
X_{p,q,t} &= \{(\rho, \mathbf{m}) \mid \rho \in L_p((0,t), W_q^3(\mathbb{R}^N)) \cap W_p^1((0,t), W_q^1(\mathbb{R}^N)),\\
&\quad \mathbf{m} \in L_p((0,t), W_q^2(\mathbb{R}^N)^N) \cap W_p^1((0,t), L_q(\mathbb{R}^N)^N), \quad \rho_*/4 \le \rho_* + \rho(t,x) \le 4\rho_*\},\\
[\mathbf{u}]_{q,\ell,(a,t)} &= \sup_{a\le s\le t} \langle s\rangle^\ell \|\mathbf{u}(\cdot, s)\|_{L_q(\mathbb{R}^N)}\ \ (a = 0, 2),\ \ [\mathbf{u}]_{q,\ell,t} = [\mathbf{u}]_{q,\ell,(0,t)},\\
\mathcal{N}(\rho, \mathbf{m})(t) &= \sum_{j=0}^{1}\sum_{i=1}^{2}\{[(\partial_x^j\rho, \partial_x^j\mathbf{m})]_{\infty,\frac{N}{q_1}+\frac{j}{2},t}\\
&\quad + [(\partial_x^j\rho, \partial_x^j\mathbf{m})]_{q_1,\frac{N}{2q_1}+\frac{j}{2},t} + [(\partial_x^j\rho, \partial_x^j\mathbf{m})]_{q_2,\frac{N}{2q_2}+1+\frac{j}{2},t}\\
&\quad + \|\langle s\rangle^{\ell_i}(\rho, \mathbf{m})\|_{L_p((0,t),W^{3,2}_{q_i}(\mathbb{R}^N))} + \|\langle s\rangle^{\ell_i}(\partial_s\rho, \partial_s\mathbf{m})\|_{L_p((0,t),W^{1,0}_{q_i}(\mathbb{R}^N))}\},
\end{aligned} \tag{1.4}$$

where $\langle s\rangle = (1 + s)$, $\ell_1 = N/2q_1 - \tau$, $\ell_2 = N/2q_2 + 1 - \tau$;τ is given in Theorem 1.1 below.

1.2 Main theorem

Setting $\rho = \rho_* + \theta$, we can rewrite (1.1) to the following formulation:

$$\begin{cases} \partial_t\theta + \operatorname{div}\mathbf{m} = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \partial_t\mathbf{m} - \dfrac{1}{\rho_*}\operatorname{Div}\mathbf{S}(\mathbf{m}) - \kappa\rho_*\nabla\Delta\theta = \mathbf{g}(\theta,\mathbf{m}) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\theta,\mathbf{m})|_{t=0} = (\rho_0,\mathbf{m}_0) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.5}$$

where

$$\mathbf{g}(\theta,\mathbf{m}) = -\operatorname{Div}\Big[\frac{1}{\rho_*+\theta}\mathbf{m}\otimes\mathbf{m} - \mathbf{S}\Big(\Big(\frac{1}{\rho_*+\theta} - \frac{1}{\rho_*}\Big)\mathbf{m}\Big) - \mathbf{K}(\theta) + \int_0^1 P''(\rho_*+\tau\theta)(1-\tau)\,d\tau\theta^2\mathbf{I}\Big].$$

We now state our main theorem.

Theorem 1.1. *Assume that conditions* (1.2) *and* (1.3) *hold and that* $3 \le N \le 7$. *Let* q_1, q_2, *and* p *be numbers such that*

$$2 < p < \infty, \;\; q_1 < N < q_2, \;\; 2 < q_1 \le 4, \;\; \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{N}, \;\; \frac{2}{p} + \frac{N}{q_2} < 1. \tag{1.6}$$

Let τ *be a number such that*

$$\frac{1}{p} < \tau < \frac{N}{q_2} + \frac{1}{p}. \tag{1.7}$$

Then, there exists a small number $\epsilon > 0$ *such that for any initial data* $(\rho_0,\mathbf{m}_0) \in \cap_{i=1}^2 D_{q_i,p}(\mathbb{R}^N) \cap W^{1,0}_{q_1/2}(\mathbb{R}^N)$ *satisfying*

$$\mathcal{I} := \sum_{i=1}^2 \|(\rho_0,\mathbf{m}_0)\|_{D_{q_i,p}(\mathbb{R}^N)} + \|(\rho_0,\mathbf{m}_0)\|_{W^{1,0}_{\frac{q_1}{2}}(\mathbb{R}^N)} + \|\mathbf{n}_0\|_{L_{\frac{q_1}{2}}(\mathbb{R}^N)} < \epsilon$$

with $\mathbf{m}_0 = \partial_x\mathbf{n}_0$, *problem* (1.5) *admits a solution* $(\theta,\mathbf{m})$ *with*

$$(\theta,\mathbf{m}) \in \cap_{i=1}^2 X_{p,q_i,\infty},$$

satisfying the estimate

$$\mathcal{N}(\theta,\mathbf{m})(\infty) \le L\epsilon$$

with some constant L *independent of* ϵ.

Remark 1.2. (1) In Theorem 1.1, the constant L is defined from several constants appearing in the estimates for the linearized equations and the constant ϵ will be chosen in such a way that $L^2\epsilon < 1$.
(2) We only consider the dimension $3 \le N \le 7$. For $N = 2$, $q_1 < 2$, and so $q_1/2 < 1$. In this case, our argument does not work. Furthermore, we need a restriction $N < 8$ by the condition $q_1 \le 4$.

2 Analysis for the linear problem

In this section, we consider the maximal L_p-L_q regularity and decay properties of solutions, which are the key tools for the proof of Theorem 1.1.

2.1 Maximal L_p-L_q regularity

In this subsection, we state the maximal L_p-L_q regularity for the linear problem:

$$\begin{cases} \partial_t\theta + \operatorname{div}\mathbf{m} = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \partial_t\mathbf{m} - \dfrac{1}{\rho_*}\operatorname{Div}\mathbf{S}(\mathbf{m}) - \kappa\rho_*\nabla\Delta\theta = \mathbf{g} & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\theta,\mathbf{m})|_{t=0} = (\rho_0,\mathbf{m}_0) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

If we extend $\mathbf{g}$ by zero outside of $(0,T)$, by Theorem 2.6 in [16] and the uniquness of solutions, we have the following result.

Theorem 2.1. *Let $T,R>0$ and $1<p,q<\infty$. Then, there exists a constant $\delta_0 \geq 1$ such that the following assertion holds: For any initial data $(\rho_0,\mathbf{m}_0) \in D_{q,p}(\mathbb{R}^N)$ with $\|(\rho_0,\mathbf{m}_0)\|_{D_{q,p}(\mathbb{R}^N)} \leq R$ satisfying the range condition:*

$$\rho_*/2 < \rho_* + \rho_0(x) < 2\rho_* \quad (x \in \mathbb{R}^N), \tag{2.2}$$

and right member $\mathbf{g} \in L_p((0,T),L_q(\mathbb{R}^N)^N)$, problem (2.1) admits a unique solution $(\theta,\mathbf{m}) \in X_{p,q,T}$ possessing the estimate

$$E_{p,q}(\theta,\mathbf{m})(t) \leq C_{p,q,N,\delta_0,R}e^{\delta t}\left(\|(\rho_0,\mathbf{m}_0)\|_{D_{q,p}(\mathbb{R}^N)} + \|\mathbf{g}\|_{L_p((0,t),L_q(\mathbb{R}^N)^N)}\right) \tag{2.3}$$

for any $t \in (0,T]$ and $\delta \geq \delta_0$. Here, we set

$$\begin{aligned} E_{p,q}(\theta,\mathbf{m})(t) &= \|\partial_s\theta\|_{L_p((0,t),W^1_q(\mathbb{R}^N))} + \|\theta\|_{L_p((0,t),W^3_q(\mathbb{R}^N))} \\ &\quad + \|\partial_s\mathbf{m}\|_{L_p((0,t),L_q(\mathbb{R}^N)^N)} + \|\mathbf{m}\|_{L_p((0,t),W^2_q(\mathbb{R}^N)^N)}. \end{aligned}$$

Constant $C_{p,q,N,\delta_0,R}$ is independent of δ and t.

Remark 2.2. Using Theorem 2.1 and employing the same argument as in the proof of Theorem 3.1 in [15], we also have the local well-posedness for (1.1).

2.2 Decay property of solutions

In this subsection, we consider the following linearized problem:

$$\begin{cases} \partial_t\theta + \operatorname{div}\mathbf{m} = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \partial_t\mathbf{m} - \alpha_*\Delta\mathbf{m} - \beta_*\nabla\operatorname{div}\mathbf{m} - \kappa\rho_*\nabla\Delta\theta = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\theta,\mathbf{m})|_{t=0} = (f,\mathbf{g}) & \text{in } \mathbb{R}^N, \end{cases} \tag{2.4}$$

where $\alpha_* = \mu/\rho_*$ and $\beta_* = \nu/\rho_*$. Then, by taking Fourier transform of (2.4) and solving the ordinary differential equation with respect to t,

$$S_1(t)(f,\mathbf{g}) := \theta, \;\; S_2(t)(f,\mathbf{g}) := \mathbf{m} \tag{2.5}$$

satisfy the following formula

(i) If $\delta_* := (\alpha_* + \beta_*)^2/4 - \rho_*\kappa \neq 0$, we have

$$
\begin{aligned}
\theta &= -\mathcal{F}_\xi^{-1}\left[\frac{\lambda_- e^{\lambda_+ t} - \lambda_+ e^{\lambda_- t}}{\lambda_+ - \lambda_-}\hat{f}\right] - \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[\rho_* \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} i\xi_k \hat{g}_k\right], \\
\mathbf{m} &= \mathcal{F}_\xi^{-1}[e^{-\alpha_*|\xi|^2 t}\hat{\mathbf{g}}] - \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[e^{-\alpha_*|\xi|^2 t}\frac{\xi\xi_k}{|\xi|^2}\hat{g}_k\right] - \mathcal{F}_\xi^{-1}\left[\kappa|\xi|^2 \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} i\xi \hat{f}\right] \\
&\quad - \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[\frac{\{(\alpha_* + \beta_*)|\xi|^2 + \lambda_-\}e^{\lambda_+ t} - \{(\alpha_* + \beta_*)|\xi|^2 + \lambda_+\}e^{\lambda_- t}}{|\xi|^2(\lambda_+ - \lambda_-)}\xi\xi_k \hat{g}_k\right],
\end{aligned}
$$

where

$$
\lambda_\pm = \begin{cases} -\dfrac{\alpha_* + \beta_*}{2}|\xi|^2 \pm \sqrt{\delta_*}|\xi|^2 & \delta_* > 0, \\ -\dfrac{\alpha_* + \beta_*}{2}|\xi|^2 \pm i\sqrt{|\delta_*|}|\xi|^2 & \delta_* < 0. \end{cases} \tag{2.6}
$$

(ii) If $\delta_* = 0$, we have

$$
\begin{aligned}
\theta &= \mathcal{F}_\xi^{-1}\left[e^{\lambda_0 t}(1 - \lambda_0 t)\hat{f}\right] - \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[t e^{\lambda_0 t} i\xi_k \hat{g}_k\right], \\
\mathbf{m} &= \mathcal{F}_\xi^{-1}[e^{-\alpha_*|\xi|^2 t}\hat{\mathbf{g}}] - \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[e^{-\alpha_*|\xi|^2 t}\frac{\xi\xi_k}{|\xi|^2}\hat{g}_k\right] - \mathcal{F}_\xi^{-1}\left[e^{\lambda_0 t}\frac{t\lambda_0^2}{|\xi|^2} i\xi \hat{f}\right] \\
&\quad + \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[e^{\lambda_0 t}\frac{1 + t\lambda_0}{|\xi|^2}\xi\xi_k \hat{g}_k\right],
\end{aligned} \tag{2.7}
$$

where

$$
\lambda_0 = -\frac{\alpha_* + \beta_*}{2}|\xi|^2.
$$

To state decay estimates of θ and $\mathbf{m}$, we divide the solution formula into the low and high frequency parts. For this purpose, we introduce a cut off function $\varphi(\xi) \in C^\infty(\mathbb{R}^N)$, which equals 1 for $|\xi| \le \epsilon$ and 0 for $|\xi| \ge 2\epsilon$. Here, ϵ is a suitably small positive constant. Let Φ_0 and Φ_∞ be operators acting on $(f, \mathbf{g}) \in W_q^{1,0}(\mathbb{R}^N)$; they are defined as

$$
\Phi_0(f, \mathbf{g}) = \mathcal{F}_\xi^{-1}[\varphi(\xi)(\hat{f}(\xi), \hat{\mathbf{g}}(\xi))], \quad \Phi_\infty(f, \mathbf{g}) = \mathcal{F}_\xi^{-1}[(1 - \varphi(\xi))(\hat{f}(\xi), \hat{\mathbf{g}}(\xi))].
$$

Theorem 2.3. *Let $S_i(t)$ $(i = 1, 2)$ be the solution operators of* (2.4) *given by* (2.5) *and let $S^0(t)(f, \mathbf{g}) = (S_1^0(t)(f, \mathbf{g}), S_2^0(t)(f, \mathbf{g}))$ and $S^\infty(t)(f, \mathbf{g}) = (S_1^\infty(t)(f, \mathbf{g}), S_2^\infty(t)(f, \mathbf{g}))$ with $S_i^0(t)(f, \mathbf{g}) = S_i(t)\Phi_0(f, \mathbf{g})$ and $S_i^\infty(t)(f, \mathbf{g}) = S_i(t)\Phi_\infty(f, \mathbf{g})$. Then, $S^0(t)$ and $S^\infty(t)$ have the following decay properties*

(i)

$$
\|\partial_x^j S^0(t)(f, \mathbf{g})\|_{L_p(\mathbb{R}^N)} \le C t^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{j}{2}} \|(f, \mathbf{h})\|_{L_q(\mathbb{R}^N)} \tag{2.8}
$$

with $\mathbf{g} = \partial_x \mathbf{h}$, $j \in \mathbb{N}_0$, and some constant C depending on j, p, q, α_* and β_*, where

$$
\begin{cases} 1 < q \le p \le \infty \text{ and } (p, q) \neq (\infty, \infty) & \text{if } 0 < t \le 1, \\ 1 < q \le 2 \le p \le \infty \text{ and } (p, q) \neq (\infty, \infty) & \text{if } t \ge 1. \end{cases} \tag{2.9}
$$

(ii)

$$\|\partial_x^j S^\infty(t)(f,\mathbf{g})\|_{W^{1,0}_p(\mathbb{R}^N)} \le Ct^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{p})-\frac{j}{2}}\|(f,\mathbf{g})\|_{W^{1,0}_q(\mathbb{R}^N)} \tag{2.10}$$

with $j \in \mathbb{N}_0$ and some constant C depending on j, p, q, α_*, and β_*, where

$$1 < q \le p \le \infty \text{ and } (p,q) \ne (\infty,\infty). \tag{2.11}$$

Proof. First, we consider the case $\delta_* \ne 0$. The difference between the cases $P'(\rho_*) = 0$ and $P'(\rho_*) > 0$ is that $\lambda_\pm$ satisfies (2.6) not only for the high frequency part, but also for the low frequency part. Owing to this difference, the second term of $S_1(t)$ given by (2.5) is troublesome because $\lambda_+ - \lambda_- = C_*|\xi|^2$, where $C_* = 2\sqrt{\delta_*}$ for $\delta_* > 0$ and $C_* = 2i\sqrt{|\delta_*|}$ for $\delta_* < 0$. Due to the condition $\mathbf{g} = \partial_x \mathbf{h}$, the second term of $S_1(t)$ satisfies

$$\sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[\rho_* \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-} i\xi_k \hat{g}_k\right] = \sum_{k=1}^N \mathcal{F}_\xi^{-1}\left[\rho_* \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{C_*|\xi|^2}(i\xi_k)(i\xi)^\alpha \hat{h}_k\right],$$

so that we can employ the same calculation as in the proof of Theorem 4.1 in [15].

Next, we consider the case $\delta_* = 0$. By using the condition $\mathbf{g} = \partial_x \mathbf{h}$ and the estimate $(|\xi|t^{1/2})^j e^{-C_0|\xi|^2 t} \le Ce^{-(C_0/2)|\xi|^2 t}$ for $j \in \mathbb{N}_0$ with some constant C_0 depending on α_* and β_*, the solution formula (2.7) can be estimated in the same manner as in the proof of Theorem 4.1 in [15]. This completes the proof of Theorem 2.3. □

3 A proof of Theorem 1.1

We prove Theorem 1.1 by the Banach fixed point argument. Let p, q_1, and q_2 be exponents given in Theorem 1.1. Let ϵ be a small positive number and let $\mathcal{N}(\theta,\mathbf{m})$ be the norm defined in (1.4). We define the underlying space $\mathcal{I}_\epsilon$ as

$$\mathcal{I}_\epsilon = \{(\theta,\mathbf{m}) \in X_{p,q_1,\infty} \cap X_{p,q_2,\infty} \mid (\theta,\mathbf{m})|_{t=0} = (\rho_0,\mathbf{m}_0),\ \mathcal{N}(\theta,\mathbf{m})(\infty) \le L\epsilon\}, \tag{3.1}$$

where L is a constant that will be determined later. Given $(\theta,\mathbf{m}) \in \mathcal{I}_\epsilon$, let $(\omega,\mathbf{w})$ be a solution to the equation:

$$\begin{cases} \partial_t\omega + \rho_* \operatorname{div}\mathbf{w} = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \partial_t\mathbf{w} - \dfrac{1}{\rho_*}\operatorname{Div}\mathbf{S}(\mathbf{w}) - \kappa\rho_*\nabla\Delta\omega = \mathbf{g}(\theta,\mathbf{m}) & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ (\omega,\mathbf{w})|_{t=0} = (\rho_0,\mathbf{m}_0) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.2}$$

We shall prove the following inequality in several steps:

$$\mathcal{N}(\omega,\mathbf{w})(t) \le C(\mathcal{I} + \mathcal{N}(\theta,\mathbf{m})(t)^2), \tag{3.3}$$

where $\mathcal{I}$ is defined in Theorem 1.1. Throughout the following steps, we use the estimate

$$\frac{\rho_*}{4} \le \rho_* + \theta(t,x) \le 4\rho_*, \tag{3.4}$$

which is obtained by $(\theta,\mathbf{m}) \in X_{p,q_1,\infty} \cap X_{p,q_2,\infty}$.

3.1 Estimates of $(\partial_x^j \omega, \partial_x^j \mathbf{w})$ for $j = 0, 1$

3.1.1 Case: $t > 2$

In order to estimate $(\omega, \mathbf{w})$ for $t > 2$, we write $(\omega, \mathbf{w})$ by Duhamel's principle as follows:

$$(\omega, \mathbf{w}) = S(t)(\rho_0, \mathbf{m}_0) + \int_0^t S(t-s)(0, \mathbf{g}(s))\, ds. \tag{3.5}$$

Because $S(t)(\rho_0, \mathbf{m}_0)$ can be estimated directly by Theorem 2.3, we only estimate the second term for low and high frequencies below. For $t > 2$, we divide the second term into three parts as follows:

$$\begin{aligned}\int_0^t \|\partial_x^j S^d(t-s)(0, \mathbf{g}(s))\|_{L_X}\, ds &= \left(\int_0^{t/2} + \int_{t/2}^{t-1} + \int_{t-1}^t\right) \|\partial_x^j S^d(t-s)(0, \mathbf{g}(s))\|_{L_X}\, ds \\ &=: \sum_{k=1}^3 I_X^{k,d},\end{aligned} \tag{3.6}$$

where $d = 0, \infty$ and $X = \infty, q_1, q_2$.

<u>**Estimates for the low frequency part in L_∞**</u>

Using (3.4) and Theorem 2.3 (i) with $(p, q) = (\infty, q_1/2)$ and Hölder's inequality under the condition $q_1/2 \leq 2$, we have

$$I_\infty^{1,0} \leq C \int_0^{t/2} (t-s)^{-\frac{N}{q_1} - \frac{j}{2}} \|\mathbf{h}\|_{L_{q_1/2}(\mathbb{R}^N)}\, ds \leq C \int_0^{t/2} (t-s)^{-\frac{N}{q_1} - \frac{j}{2}} (A_1 + B_1)\, ds, \tag{3.7}$$

where

$$\begin{aligned} A_1 &= \|(\theta, \mathbf{m})\|_{L_{q_1}(\mathbb{R}^N)}^2 + \|(\theta, \mathbf{m})\|_{L_{q_1}(\mathbb{R}^N)} \|(\partial_x \theta, \partial_x \mathbf{m})\|_{L_{q_1}(\mathbb{R}^N)} + \|\partial_x \theta\|_{L_{q_1}(\mathbb{R}^N)}^2, \\ B_1 &= \|\theta\|_{L_{q_1}(\mathbb{R}^N)} \|\partial_x^2 \theta\|_{L_{q_1}(\mathbb{R}^N)}. \end{aligned}$$

A_1 and B_1 satisfy:

$$\begin{aligned} A_1 &\leq \langle s\rangle^{-\frac{N}{q_1}} [(\theta, \mathbf{m})]_{q_1, \frac{N}{2q_1}, t}^2 + \langle s\rangle^{-(\frac{N}{q_1} + \frac{1}{2})} [(\theta, \mathbf{m})]_{q_1, \frac{N}{2q_1}, t} [(\partial_x \theta, \partial_x \mathbf{m})]_{q_1, \frac{N}{2q_1} + \frac{1}{2}, t} \\ &\quad + \langle s\rangle^{-(\frac{N}{q_1} + 1)} [\partial_x \theta]_{q_1, \frac{N}{2q_1} + \frac{1}{2}, t}^2 \\ &\leq \langle s\rangle^{-\frac{N}{q_1}} ([(\theta, \mathbf{m})]_{q_1, \frac{N}{2q_1}, t}^2 + [(\theta, \mathbf{m})]_{q_1, \frac{N}{2q_1}, t} [(\partial_x \theta, \partial_x \mathbf{m})]_{q_1, \frac{N}{2q_1} + \frac{1}{2}, t} + [\partial_x \theta]_{q_1, \frac{N}{2q_1} + \frac{1}{2}, t}^2). \end{aligned} \tag{3.8}$$

$$B_1 \leq \langle s\rangle^{-(\frac{N}{q_1} - \tau)} [\theta]_{q_1, \frac{N}{2q_1}, t} \langle s\rangle^{\frac{N}{2q_1} - \tau} \|\theta\|_{W_{q_1}^2(\mathbb{R}^N)}. \tag{3.9}$$

Because $1 - N/q_1 < 0$ and $1 - (N/q_1 - \tau)p' < 0$, which follow from $q_1 < N$ and $\tau < N/q_2 + 1/p$, using (3.7), (3.8) and (3.9), we obtain

$$\begin{aligned} I_\infty^{1,0} &\leq C t^{-\frac{N}{q_1} - \frac{j}{2}} \int_0^{t/2} \langle s\rangle^{-\frac{N}{q_1}}\, ds ([(\theta, \mathbf{m})]_{q_1, \frac{N}{2q_1}, t}^2 \\ &\quad + [(\theta, \mathbf{m})]_{q_1, \frac{N}{2q_1}, t} [(\partial_x \theta, \partial_x \mathbf{m})]_{q_1, \frac{N}{2q_1} + \frac{1}{2}, t} + [\partial_x \theta]_{q_1, \frac{N}{2q_1} + \frac{1}{2}, t}^2) \\ &\quad + C t^{-\frac{N}{q_1} - \frac{j}{2}} \left(\int_0^{t/2} \langle s\rangle^{-(\frac{N}{q_1} - \tau)p'}\, ds\right)^{1/p'} [\theta]_{q_1, \frac{N}{2q_1}, t} \|\langle s\rangle^{\frac{N}{2q_1} - \tau} \theta\|_{L_p((0,t), W_{q_1}^2(\mathbb{R}^N))} \\ &\leq C t^{-\frac{N}{q_1} - \frac{j}{2}} E_0^0(t), \end{aligned} \tag{3.10}$$

where

$$E_0^0(t) = [(\theta, \mathbf{m})]^2_{q_1, \frac{N}{2q_1}, t} + [(\theta, \mathbf{m})]_{q_1, \frac{N}{2q_1}, t}[(\partial_x\theta, \partial_x\mathbf{m})]_{q_1, \frac{N}{2q_1}+\frac{1}{2}, t} + [\partial_x\theta]^2_{q_1, \frac{N}{2q_1}+\frac{1}{2}, t}$$
$$+ [\theta]_{q_1, \frac{N}{2q_1}, t}\|\langle s\rangle^{\frac{N}{2q_1}-\tau}\theta\|_{L_p((0,t), W^2_{q_1}(\mathbb{R}^N))}.$$

Similarly, we have

$$I^{2,0}_\infty \leq Ct^{-\frac{N}{q_1}-\frac{j}{2}}E_0^0(t). \tag{3.11}$$

We now estimate $I^{3,0}_\infty$. Using (3.4) and Theorem 2.3 (i) with $(p, q) = (\infty, q_2)$, we obtain

$$I^{3,0}_\infty \leq C\int_{t-1}^t (t-s)^{-\frac{N}{2q_2}-\frac{j}{2}}\|\mathbf{h}\|_{L_{q_2}(\mathbb{R}^N)}\,ds \leq C\int_{t-1}^t (t-s)^{-\frac{N}{2q_2}-\frac{j}{2}}(A_2+B_2)\,ds, \tag{3.12}$$

where

$$A_2 = \|(\theta, \mathbf{m})\|_{L_\infty(\mathbb{R}^N)}(\|(\theta, \mathbf{m})\|_{L_{q_2}(\mathbb{R}^N)} + \|(\partial_x\theta, \partial_x\mathbf{m})\|_{L_{q_2}(\mathbb{R}^N)}) + \|\partial_x\theta\|_{L_\infty(\mathbb{R}^N)}\|\partial_x\theta\|_{L_{q_2}(\mathbb{R}^N)},$$
$$B_2 = \|\theta\|_{L_\infty(\mathbb{R}^N)}\|\partial_x^2\theta\|_{L_{q_2}(\mathbb{R}^N)}.$$

A_2 and B_2 satisfy:

$$A_2 \leq \langle s\rangle^{-(\frac{N}{q_1}+\frac{N}{2q_2}+1)}[(\theta, \mathbf{m})]_{\infty, \frac{N}{q_1}, t}[(\theta, \mathbf{m})]_{q_2, \frac{N}{2q_2}+1, t}$$
$$+ \langle s\rangle^{-(\frac{N}{q_1}+\frac{N}{2q_2}+\frac{3}{2})}[(\theta, \mathbf{m})]_{\infty, \frac{N}{q_1}, t}[(\partial_x\theta, \partial_x\mathbf{m})]_{q_2, \frac{N}{2q_2}+\frac{3}{2}, t}$$
$$+ \langle s\rangle^{-(\frac{N}{q_1}+\frac{N}{2q_2}+2)}[\partial_x\theta]_{\infty, \frac{N}{q_1}+\frac{1}{2}, t}[(\partial_x\theta, \partial_x\mathbf{m})]_{q_2, \frac{N}{2q_2}+\frac{3}{2}, t}, \tag{3.13}$$

$$B_2 \leq \langle s\rangle^{-(\frac{N}{q_1}+\frac{N}{2q_2}+1-\tau)}[\theta]_{\infty, \frac{N}{q_1}, t}\langle s\rangle^{\frac{N}{2q_2}+1-\tau}\|\theta\|_{W^2_{q_2}(\mathbb{R}^N)}. \tag{3.14}$$

Because $1-(N/2q_2+j/2) > 0$, $1-(N/2q_2+j/2)p' > 0$, and $N/2q_2+1-\tau > j/2$ as follows from $N < q_2$, $2/p+N/q_2 < 1$ and $\tau < N/q_2+1/p$, using (3.12), (3.13) and (3.14), we obtain

$$I^{3,0}_\infty \leq Ct^{-(\frac{N}{q_1}+\frac{N}{2q_2}+1)}\int_{t-1}^t (t-s)^{-(\frac{N}{2q_2}+\frac{j}{2})}\,ds[(\theta, \mathbf{m})]_{\infty, \frac{N}{q_1}, t}[(\theta, \mathbf{m})]_{q_2, \frac{N}{2q_2}+1, t}$$
$$+ Ct^{-(\frac{N}{q_1}+\frac{N}{2q_2}+\frac{3}{2})}\int_{t-1}^t (t-s)^{-(\frac{N}{2q_2}+\frac{j}{2})}\,ds[(\theta, \mathbf{m})]_{\infty, \frac{N}{q_1}, t}[(\partial_x\theta, \partial_x\mathbf{m})]_{q_2, \frac{N}{2q_2}+\frac{3}{2}, t}$$
$$+ Ct^{-(\frac{N}{q_1}+\frac{N}{2q_2}+2)}\int_{t-1}^t (t-s)^{-(\frac{N}{2q_2}+\frac{j}{2})}\,ds[\partial_x\theta]_{\infty, \frac{N}{q_1}+\frac{1}{2}, t}[(\partial_x\theta, \partial_x\mathbf{m})]_{q_2, \frac{N}{2q_2}+\frac{3}{2}, t}$$
$$+ Ct^{-(\frac{N}{q_1}+\frac{N}{2q_2}+1-\tau)}\left(\int_{t-1}^t (t-s)^{-(\frac{N}{2q_2}+\frac{j}{2})p'}\,ds\right)^{1/p'}[\theta]_{\infty, \frac{N}{q_1}, t}\|\langle s\rangle^{\frac{N}{2q_2}+1-\tau}\theta\|_{L_p((0,t), W^2_{q_2}(\mathbb{R}^N))}$$
$$\leq Ct^{-\frac{N}{q_1}-\frac{j}{2}}E_2^0(t), \tag{3.15}$$

where

$$E_2^0(t) = [(\theta, \mathbf{m})]_{\infty, \frac{N}{q_1}, t}\{[(\theta, \mathbf{m})]_{q_2, \frac{N}{2q_2}+1, t} + [(\partial_x\theta, \partial_x\mathbf{m})]_{q_2, \frac{N}{2q_2}+\frac{3}{2}, t}\}$$
$$+ [\partial_x\theta]_{\infty, \frac{N}{q_1}+\frac{1}{2}, t}[(\partial_x\theta, \partial_x\mathbf{m})]_{q_2, \frac{N}{2q_2}+\frac{3}{2}, t} + [\theta]_{\infty, \frac{N}{q_1}, t}\|\langle s\rangle^{\frac{N}{2q_2}+1-\tau}\theta\|_{L_p((0,t), W^2_{q_2}(\mathbb{R}^N))}.$$

Using (3.10), (3.11) and (3.15), we obtain

$$\int_0^t \|\partial_x^j S^0(t-s)(0,\mathbf{g}(s))\|_{L_\infty}\,ds \le Ct^{-\frac{N}{q_1}-\frac{j}{2}}(E_0^0(t)+E_2^0(t)). \tag{3.16}$$

Estimates for the low frequency part in L_{q_1}

Using (3.4) and Theorem 2.3 (i) with $(p,q) = (q_1, q_1/2)$ and employing the same calculation as in the estimate in L_∞, we obtain

$$I_{q_1}^{1,0} + I_{q_1}^{2,0} \le Ct^{-\frac{N}{2q_1}-\frac{j}{2}}E_0^0(t). \tag{3.17}$$

Using Theorem 2.3 (i) with $(p,q) = (q_1,q_1)$ and employing the same calculation as in the estimate in L_∞ under the conditions $1-(j/2)p' > 0$, and $3N/2q_1 - \tau > N/2q_1 + j/2$, which follow from $p > 2$ and $\tau < N/q_2 + 1/p$, we obtain

$$I_{q_1}^{3,0} \le Ct^{-\frac{N}{2q_1}-\frac{j}{2}}E_1^0(t), \tag{3.18}$$

where

$$\begin{aligned} E_1^0(t) = {} & [(\theta,\mathbf{m})]_{\infty,\frac{N}{q_1},t}\{[(\theta,\mathbf{m})]_{q_1,\frac{N}{2q_1},t} + [(\partial_x\theta,\partial_x\mathbf{m})]_{q_1,\frac{N}{2q_1}+\frac{1}{2},t}\} \\ & + [\partial_x\theta]_{\infty,\frac{N}{q_1}+\frac{1}{2},t}[(\partial_x\theta,\partial_x\mathbf{m})]_{q_1,\frac{N}{2q_1}+\frac{1}{2},t} + [\theta]_{\infty,\frac{N}{q_1},t}\|\langle s\rangle^{\frac{3N}{2q_1}-\tau}\theta\|_{L_p((0,t),W^2_{q_1}(\mathbb{R}^N))}. \end{aligned}$$

Using (3.17) and (3.18), we obtain

$$\int_0^t \|\partial_x^j S^0(t-s)(0,\mathbf{g}(s))\|_{L_{q_1}}\,ds \le Ct^{-\frac{N}{2q_1}-\frac{j}{2}}(E_0^0(t)+E_1^0(t)). \tag{3.19}$$

Estimates for the low frequency part in L_{q_2}

Using (3.4) and Theorem 2.3 (i) with $(p,q) = (q_2, q_1/2)$ and $(p,q) = (q_2,q_2)$, we obtain

$$\int_0^t \|\partial_x^j S^0(t-s)(0,\mathbf{g}(s))\|_{L_{q_2}}\,ds \le Ct^{-\frac{N}{2q_2}-1-\frac{j}{2}}(E_0^0(t)+E_2^0(t)). \tag{3.20}$$

Estimates for the high frequency part

Employing the same calculation for the low frequencies, we have estimates for the high frequencies under the conditions (1.6) and (1.7) as follows:

$$\begin{aligned} &\int_0^t \|\partial_x^j S^\infty(t-s)(0,\mathbf{g}(s))\|_{L_\infty}\,ds \le Ct^{-\frac{N}{q_1}-\frac{j}{2}}(E_0^\infty(t)+E_2^\infty(t)), \\ &\int_0^t \|\partial_x^j S^\infty(t-s)(0,\mathbf{g}(s))\|_{L_{q_1}}\,ds \le Ct^{-\frac{N}{2q_1}-\frac{j}{2}}(E_0^\infty(t)+E_1^\infty(t)), \\ &\int_0^t \|\partial_x^j S^\infty(t-s)(0,\mathbf{g}(s))\|_{L_{q_2}}\,ds \le Ct^{-\frac{N}{2q_2}-1-\frac{j}{2}}(E_0^\infty(t)+E_2^\infty(t)), \end{aligned} \tag{3.21}$$

where

$$
\begin{aligned}
E_0^\infty(t) &= [(\theta,\mathbf{m})]_{q_1,\frac{N}{2q_1},t}[(\partial_x\theta,\partial_x\mathbf{m})]_{q_1,\frac{N}{2q_1}+\frac12,t}\\
&\quad+[\partial_x\theta]_{\infty,\frac{N}{q_1}+\frac12,t}[\mathbf{m}]^2_{q_1,\frac{N}{2q_1},t}+[\partial_x\theta]_{q_1,\frac{N}{2q_1}+\frac12,t}[\partial_x\mathbf{m}]_{q_1,\frac{N}{2q_1}+\frac12,t}\\
&\quad+[\theta]_{q_1,\frac{N}{2q_1},t}\|\langle s\rangle^{\frac{N}{2q_1}-\tau}(\theta,\mathbf{m})\|_{L_p((0,t),W^{3,2}_{q_1}(\mathbb{R}^N))}\\
&\quad+([\mathbf{m}]_{q_1,\frac{N}{2q_1},t}+[\partial_x\theta]_{q_1,\frac{N}{2q_1}+\frac12,t})\|\langle s\rangle^{\frac{N}{2q_1}-\tau}\theta\|_{L_p((0,t),W^2_{q_1}(\mathbb{R}^N))},\\
E_1^\infty(t) &= [(\theta,\mathbf{m})]_{\infty,\frac{N}{q_1},t}[(\partial_x\theta,\partial_x\mathbf{m})]_{q_1,\frac{N}{2q_1}+\frac12,t}\\
&\quad+[\partial_x\theta]_{q_1,\frac{N}{2q_1}+\frac12,t}[\mathbf{m}]^2_{\infty,\frac{N}{q_1},t}+[\partial_x\theta]_{\infty,\frac{N}{q_1}+\frac12,t}[\partial_x\mathbf{m}]_{q_1,\frac{N}{2q_1}+\frac12,t}\\
&\quad+[\theta]_{\infty,\frac{N}{q_1},t}\|\langle s\rangle^{\frac{N}{2q_1}-\tau}(\theta,\mathbf{m})\|_{L_p((0,t),W^{3,2}_{q_1}(\mathbb{R}^N))}\\
&\quad+([\mathbf{m}]_{\infty,\frac{N}{q_1},t}+[\partial_x\theta]_{\infty,\frac{N}{q_1}+\frac12,t})\|\langle s\rangle^{\frac{N}{2q_1}-\tau}\theta\|_{L_p((0,t),W^2_{q_1}(\mathbb{R}^N))},\\
E_2^\infty(t) &= [(\theta,\mathbf{m})]_{\infty,\frac{N}{q_1},t}[(\partial_x\theta,\partial_x\mathbf{m})]_{q_2,\frac{N}{2q_2}+\frac32,t}\\
&\quad+[\partial_x\theta]_{q_2,\frac{N}{2q_2}+\frac32,t}[\mathbf{m}]^2_{\infty,\frac{N}{q_1},t}+[\partial_x\theta]_{\infty,\frac{N}{q_1}+\frac12,t}[\partial_x\mathbf{m}]_{q_2,\frac{N}{2q_2}+\frac32,t}\\
&\quad+[\theta]_{\infty,\frac{N}{q_1},t}\|\langle s\rangle^{\frac{N}{2q_2}+1-\tau}(\theta,\mathbf{m})\|_{L_p((0,t),W^{3,2}_{q_1}(\mathbb{R}^N))}\\
&\quad+([\mathbf{m}]_{\infty,\frac{N}{q_1},t}+[\partial_x\theta]_{\infty,\frac{N}{q_1}+\frac12,t})\|\langle s\rangle^{\frac{N}{2q_2}+1-\tau}\theta\|_{L_p((0,t),W^2_{q_2}(\mathbb{R}^N))}.
\end{aligned}
$$

Using (3.5), (3.16), (3.19), (3.20) and (3.21), we obtain

$$
\begin{aligned}
&\sum_{j=0}^{1}[(\partial_x^j\omega,\partial_x^j\mathbf{w})]_{\infty,\frac{N}{q_1}+\frac{j}{2},(2,t)}\le C(\|(\rho_0,\mathbf{m}_0)\|_{W^{1,0}_{\frac{q_1}{2}}}+\|\mathbf{n}_0\|_{L_{\frac{q_1}{2}}}+E_0(t)+E_2(t)),\\
&\sum_{j=0}^{1}[(\partial_x^j\omega,\partial_x^j\mathbf{w})]_{q_1,\frac{N}{2q_1}+\frac{j}{2},(2,t)}\le C(\|(\rho_0,\mathbf{m}_0)\|_{W^{1,0}_{\frac{q_1}{2}}}+\|\mathbf{n}_0\|_{L_{\frac{q_1}{2}}}+E_0(t)+E_1(t)),\qquad(3.22)\\
&\sum_{j=0}^{1}[(\partial_x^j\omega,\partial_x^j\mathbf{w})]_{q_2,\frac{N}{2q_2}+1+\frac{j}{2},(2,t)}\le C(\|(\rho_0,\mathbf{m}_0)\|_{W^{1,0}_{\frac{q_1}{2}}}+\|\mathbf{n}_0\|_{L_{\frac{q_1}{2}}}+E_0(t)+E_2(t)),
\end{aligned}
$$

where $E_i(t)=E_i^0(t)+E_i^\infty(t)$ with $i=0,1,2$.

3.1.2 Case: $0<t<2$

Estimates in L_{q_i} for $i=1,2$

Using Theorem 2.1 and the following estimate

$$
\|\mathbf{g}\|_{L_p((0,t),L_{q_i}(\mathbb{R}^N))}\le CE_i^\infty(t),
$$

which is calculated in 3.1.1, we obtain

$$
\begin{aligned}
&\|(\omega,\mathbf{w})\|_{L_p((0,2),W^{3,2}_{q_i}(\mathbb{R}^N))}+\|(\partial_s\omega,\partial_s\mathbf{w})\|_{L_p((0,2),W^{1,0}_{q_i}(\mathbb{R}^N))}\\
&\le C\{\|(\rho_0,\mathbf{m}_0)\|_{D_{q_i,p}(\mathbb{R}^N)}+E_i(2)\}\qquad(3.23)
\end{aligned}
$$

for $i=1,2$.

Estimates in L_∞

Using Lemma 1 in [17] and Lemma 3.3 in [15], we obtain

$$\|(\omega,\mathbf{w})\|_{L_\infty((0,2),W^1_\infty(\mathbb{R}^N))}\le C\{\|(\rho_0,\mathbf{m}_0)\|_{D_{q_2,p}(\mathbb{R}^N)}+E_2(2)\}. \tag{3.24}$$

3.1.3 Conclusion

Combining (3.22), (3.23) and (3.24), we obtain

$$\begin{aligned}
&\sum_{j=0}^1[(\partial_x^j\omega,\partial_x^j\mathbf{w})]_{\infty,\frac{N}{q_1}+\frac{j}{2},t}\le C(\mathcal{I}+E_0(t)+E_2(t)),\\
&\sum_{j=0}^1[(\partial_x^j\omega,\partial_x^j\mathbf{w})]_{q_1,\frac{N}{2q_1}+\frac{j}{2},t}\le C(\mathcal{I}+E_0(t)+E_1(t)),\\
&\sum_{j=0}^1[(\partial_x^j\omega,\partial_x^j\mathbf{w})]_{q_2,\frac{N}{2q_2}+1+\frac{j}{2},t}\le C(\mathcal{I}+E_0(t)+E_2(t)).
\end{aligned}\tag{3.25}$$

3.2 Estimates of the weighted norm

In order to estimate the weighted norm in the maximal L_p-L_q regularity class, we consider the following time shifted equations, which is equivalent to the first and the second equations of (3.2):

$$\begin{aligned}
&\partial_s(\langle s\rangle^{\ell_i}\omega)+\delta_0\langle s\rangle^{\ell_i}\omega+\rho_*\operatorname{div}(\langle s\rangle^{\ell_i}\mathbf{w})\\
&=\delta_0\langle s\rangle^{\ell_i}\omega+(\partial_s\langle s\rangle^{\ell_i})\omega\\
&\partial_s(\langle s\rangle^{\ell_i}\mathbf{w})+\delta_0\langle s\rangle^{\ell_i}\mathbf{w}-\alpha_*\Delta(\langle s\rangle^{\ell_i}\mathbf{w})-\beta_*\nabla(\operatorname{div}\langle s\rangle^{\ell_i}\mathbf{w})+\kappa\rho_*\nabla\Delta\langle s\rangle^{\ell_i}\omega\\
&=\langle s\rangle^{\ell_i}\mathbf{g}(\theta,\mathbf{m})+\delta_0\langle s\rangle^{\ell_i}\mathbf{w}+(\partial_s\langle s\rangle^{\ell_i})\mathbf{w},
\end{aligned}$$

where $i=1,2$, $\ell_1=N/2q_1-\tau$ and $\ell_2=N/2q_2+1-\tau$. By Theorem 2.1, we have

$$\begin{aligned}
&\|\langle s\rangle^{\ell_i}(\omega,\mathbf{w})\|_{L_p((0,t),W^{3,2}_{q_i}(\mathbb{R}^N))}+\|\langle s\rangle^{\ell_i}(\partial_s\omega,\partial_s\mathbf{w})\|_{L_p((0,t),W^{1,0}_{q_i}(\mathbb{R}^N))}\\
&\le C(\|(\rho_0,\mathbf{m}_0)\|_{D_{q_i,p}(\mathbb{R}^N)}+\|\langle s\rangle^{\ell_i}\mathbf{g}(\theta,\mathbf{m})\|_{L_p((0,t),L_{q_i}(\mathbb{R}^N))}\\
&+\|\langle s\rangle^{\ell_i}(\omega,\mathbf{w})\|_{L_p((0,t),W^{1,0}_{q_i}(\mathbb{R}^N))}+\|(\partial_s\langle s\rangle^{\ell_i})(\omega,\mathbf{w})\|_{L_p((0,t),W^{1,0}_{q_i}(\mathbb{R}^N))}.
\end{aligned}\tag{3.26}$$

We can estimate the left-hand sides of (3.26) by the same calculation as in [15], we have

$$\begin{aligned}
&\|\langle s\rangle^{\ell_i}(\omega,\mathbf{w})\|_{L_p((0,t),W^{3,2}_{q_i}(\mathbb{R}^N))}+\|\langle s\rangle^{\ell_i}(\partial_s\omega,\partial_s\mathbf{w})\|_{L_p((0,t),W^{1,0}_{q_i}(\mathbb{R}^N))}\\
&\le C(\mathcal{I}+E_0(t)+E_i(t)).
\end{aligned}\tag{3.27}$$

3.3 Conclusion

Combining (3.25) and (3.27), we have (3.3). Recalling that $\mathcal{I}\le\epsilon$, for $(\theta,\mathbf{m})\in\mathcal{I}_\epsilon$, we have

$$\mathcal{N}(\omega,\mathbf{w})(\infty)\le C(\mathcal{I}+\mathcal{N}(\theta,\mathbf{m})(\infty)^2)\le C\epsilon+CL^2\epsilon^2. \tag{3.28}$$

Choosing ϵ so small that $L^2\epsilon \le 1$ and setting $L = 2C$ in (3.28), we have

$$\mathcal{N}(\omega, \mathbf{w})(\infty) \le L\epsilon. \tag{3.29}$$

We define a map Φ acting on $(\theta, \mathbf{m}) \in \mathcal{I}_\epsilon$ by $\Phi(\theta, \mathbf{m}) = (\omega, \mathbf{w})$, and then it follows from (3.29) that Φ is the map from $\mathcal{I}_\epsilon$ into itself. Considering the difference $\Phi(\theta_1, \mathbf{m}_1) - \Phi(\theta_2, \mathbf{m}_2)$ for $(\theta_i, \mathbf{m}_i) \in \mathcal{I}_\epsilon$ $(i = 1, 2)$, employing the same argument as in the proof of (3.28) and choosing $\epsilon > 0$ samller if necessary, we see that Φ is a consraction map on $\mathcal{I}_\epsilon$, and therefore there exists a fixed point $(\omega, \mathbf{w}) \in \mathcal{I}_\epsilon$ which solves the equation (1.5). Since the existence of solutions to (1.5) is proved by the contraction mapping principle, the uniqueness of solutions belonging to $\mathcal{I}_\epsilon$ follows immediately, which completes the proof of Theorem 1.1.

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