

A CONJECTURE TO FIND FOUR-TERM ARITHMETIC PROGRESSIONS OF PIATETSKI-SHAPIRO SEQUENCES

KOTA SAITO
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY

ABSTRACT. For every non-integral $\alpha > 1$, the sequence of the integer parts of n^α ($n = 1, 2, \dots$) is called the Piatetski-Shapiro sequence with exponent α . Let $\text{PS}(\alpha)$ be the set of all terms of this sequence. The aim of this article is to propose a conjecture to find infinitely many four-term arithmetic progressions of $\text{PS}(\alpha)$.

1. INTRODUCTION

We let $[x]$ denote the integer part of $x \in \mathbb{R}$. For every non-integral $\alpha > 1$, the sequence $([n^\alpha])_{n=1}^\infty$ is called the Piatetski-Shapiro sequence with exponent α , and we let $\text{PS}(\alpha)$ be the set of all terms of this sequence. A real sequence $(a_j)_{j=0}^{k-1}$ is called a k -term arithmetic progression (k -AP) if there exists $\ell > 0$ such that

$$a_j = a_0 + j\ell$$

for all $j = 0, 1, \dots, k-1$. In this article, we discuss APs of $\text{PS}(\alpha)$. By the result of Frantzikinakis and Wierdl [FW09], $\text{PS}(\alpha)$ contains arbitrarily long APs for all $1 < \alpha < 2$. Further, Matsusaka and the author recently showed that for all $2 < \beta < \gamma$, there are uncountably many $\alpha \in [\beta, \gamma]$ such that $\text{PS}(\alpha)$ contains infinitely many 3-APs [MS21]. More precisely, for any fixed $a, b, c \in \mathbb{N}$, they showed that the Hausdorff dimension of

$$\{\alpha \in [\beta, \gamma]: ax + by = cz \text{ has infinitely many solutions} \\ (x, y, z) \in \text{PS}(\alpha)^3 \text{ with } \#\{x, y, z\} = 3\}$$

is greater than or equal to $1/s^3$. By substituting $a = b = 1$ and $c = 2$, they obtained the result on 3-APs of $\text{PS}(\alpha)$. However, there is no research to find 4-APs of $\text{PS}(\alpha)$ when $\alpha > 2$ is non-integral. The aim of this article is to propose a sufficient condition to find infinitely many 4-APs of $\text{PS}(\alpha)$. Let $\alpha > 1$, and we define the following condition which depends on α .

Condition 1.1. *There exists $\xi > \alpha^2$ such that for infinitely many tuples $(p, q, r, s) \in \mathbb{N}^4$ with $p < q < r < s$, one has*

$$(1.1) \quad |p^\alpha + r^\alpha - 2q^\alpha| \leq q^{-\xi}, \quad |q^\alpha + s^\alpha - 2r^\alpha| \leq q^{-\xi}.$$

Theorem 1.2. *Assume that there exists $\alpha > 1$ satisfying Condition 1.1. Then $\text{PS}(\alpha)$ contains infinitely many four-term arithmetic progressions.*

In view of this theorem, we would expect to find infinitely many 4-APs of $\text{PS}(\alpha)$ by using simultaneous Diophantine approximations. However, we do not find any α which satisfies Condition 1.1.

Question 1.3. Let $1 < \beta < \gamma$. What is a lower bound for the Hausdorff dimension of

$$\{\alpha \in (\beta, \gamma) : \text{for infinitely many } (p, q, r, s) \in \mathbb{N}^2 \text{ with } p < q < r < s, \\ |p^\alpha + r^\alpha - 2q^\alpha| \leq q^{-\gamma^2}, \quad |q^\alpha + s^\alpha - 2r^\alpha| \leq q^{-\gamma^2}\}?$$

If the set had positive Hausdorff dimension, then by Theorem 1.2, we would find uncountably many $\alpha \in (\beta, \gamma)$ such that $\text{PS}(\alpha)$ contains infinitely many four-term arithmetic progressions.

Notation 1.4. Let \mathbb{N} be the set of all positive integers. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x . For all $\ell \in \mathbb{N}$, we define $[\ell] = \mathbb{N} \cap [1, \ell]$. Let $\sqrt{-1}$ denote the imaginary unit, and define $e(x)$ by $e^{2\pi\sqrt{-1}x}$ for all $x \in \mathbb{R}$.

2. PREPARATION

Let $d \in \mathbb{N}$. We mainly discuss the case when $d = 1$ or 2 . For all $\mathbf{x} = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^d$, we define $\{\mathbf{x}\} = (\{x^{(1)}\}, \dots, \{x^{(d)}\})$. Let $(\mathbf{x}_n)_{n=1}^N$ be a sequence composed of $\mathbf{x}_n \in \mathbb{R}^d$ for all $1 \leq n \leq N$. We define the *discrepancy* of $(\mathbf{x}_n)_{n=1}^N$ by

$$\mathcal{D}_{1 \leq n \leq N}(\mathbf{x}_n) = \sup_{\substack{0 \leq a_i < b_i \leq 1 \\ \forall i \in [d]}} \left| \frac{\#\left\{n \in \mathbb{N} \cap [1, N] : \{\mathbf{x}_n\} \in \prod_{i=1}^d [a_i, b_i]\right\}}{N} - \prod_{i=1}^d (b_i - a_i) \right|.$$

We can find upper bounds of the discrepancy from evaluating exponential sums by the following inequality. This is shown by Koksma [Kok50] and Szűsz [Szű52] independently: there exists $C_d > 0$ which depends only on d such that for all $H \in \mathbb{N}$, we have

$$(2.1) \quad \mathcal{D}_{1 \leq n \leq N}(\mathbf{x}_n) \leq C_d \left(\frac{1}{H} + \sum_{\substack{0 < \|\mathbf{h}\|_\infty \leq H \\ \mathbf{h} \in \mathbb{Z}^d}} \frac{1}{\nu(\mathbf{h})} \left| \frac{1}{N} \sum_{n=1}^N e(\langle \mathbf{h}, \mathbf{x}_n \rangle) \right| \right),$$

where we let $\langle \cdot, \cdot \rangle$ denote the standard inner product and define

$$\|\mathbf{h}\|_\infty = \max\{|h^{(1)}|, \dots, |h^{(d)}|\}, \quad \nu(\mathbf{h}) = \prod_{i=1}^d \max\{1, |h^{(i)}|\}$$

for all $\mathbf{h} = (h^{(1)}, \dots, h^{(d)}) \in \mathbb{R}^d$. This inequality is sometimes referred as the Erdős-Turán-Koksma inequality. We refer [DT97, Theorem 1.21] to the readers for more details. In order to evaluate upper bounds for the right-hand side on (2.1), we will use the following lemma which is called van der Corput's k -th derivative test.

Lemma 2.1. Let V_1, V_2 be real numbers with $V_2 - V_1 \geq 1$. Let $f : [V_1, V_2] \rightarrow \mathbb{R}$ be a function which has continuous derivatives up to the k -th order, where $k \geq 4$. Let λ_k and T be positive real numbers. Suppose that

$$\lambda_k \leq |f^{(k)}(x)| \leq T\lambda_k$$

for all $x \in [V_1, V_2]$. Then there exists $C(T, k) > 0$ such that

$$\left| \sum_{V_1 < n \leq V_2} e(f(n)) \right| \leq C(T, k) \left((V_2 - V_1)\lambda_k^{1/(2^k-2)} + (V_2 - V_1)^{1-2^{2-k}} \lambda_k^{-1/(2^k-2)} \right).$$

Proof. See the book written by Titchmarsh [Tit86, Theorem 5.13]. □

3. LEMMA

We write $O(1)$ for a bounded quantity. If this bound depends only on some parameters a_1, \dots, a_n , then for instance we write $O_{a_1, a_2, \dots, a_n}(1)$. As is customary, we often abbreviate $O(1)X$ and $O_{a_1, \dots, a_n}(1)X$ to $O(X)$ and $O_{a_1, \dots, a_n}(X)$ respectively for a non-negative quantity X . We also state $f(X) \ll g(X)$ and $f(X) \ll_{a_1, \dots, a_n} g(X)$ as $f(X) = O(g(X))$ and $f(X) = O_{a_1, \dots, a_n}(g(X))$ respectively, where $g(X)$ is non-negative.

Lemma 3.1. *Let $\alpha > 1$, $\xi > \alpha^2$, $0 < \epsilon < (\xi - \alpha^2)/\alpha$, and $\delta > 0$. Then there exists $\theta = \theta(\alpha, \xi, \epsilon) > 0$ such that for all $p, q \in \mathbb{N}$ with $p < q$, by setting $V = (\delta q^\xi)^{1/\alpha}$ and $U = q^{(\xi - \alpha^2)/\alpha^2 - \epsilon/\alpha}$, either one has*

$$\mathcal{D}_{V < v \leq 2V}((vp)^\alpha, (vq)^\alpha) \ll_{\alpha, \xi, \epsilon, \delta} q^{-\theta},$$

or there exist $h_1, h_2 \in \mathbb{N}$ with $h_2 < h_1 \leq q^\theta$ such that $|h_1(p/q)^\alpha - h_2| \leq q^{-\xi/\alpha + \epsilon}$ and

$$\mathcal{D}_{U < u \leq 2U}((uq)^\alpha/h_1) \ll_{\alpha, \xi, \epsilon, \delta} q^{-2\theta}.$$

Proof. Take any $p, q \in \mathbb{N}$ with $p < q$. Take a small parameter $\theta = \theta(\alpha, \xi, \epsilon) > 0$, and large parameter $q_0 = q_0(\alpha, \xi, \epsilon)$ which satisfies $q_0^\theta \geq 2$. Let $\eta = \xi/\alpha - \epsilon + 2\theta$. We may assume that $q \geq q_0$. Let $H = \lfloor q^\theta \rfloor$, and let $L(h_1, h_2) = h_1(p/q)^\alpha + h_2$ for all $h_1, h_2 \in \mathbb{Z}$. By (2.1), we have

$$\mathcal{D}_{V < v \leq 2V}((vp)^\alpha, (vq)^\alpha) \ll \frac{1}{H} + \sum_{0 < \|(h_1, h_2)\|_\infty \leq H} \frac{1}{\nu(h_1, h_2)} |S(h_1, h_2)|,$$

where $S(h_1, h_2) = \frac{1}{V} \sum_{V < v \leq 2V} e(L(h_1, h_2)q^\alpha v^\alpha)$. Firstly, we discuss the case when

$$(3.1) \quad |L(h_1, h_2)| \geq q^{-\eta}$$

for all $h_1, h_2 \in \mathbb{Z}$ with $0 < \|(h_1, h_2)\|_\infty \leq H$. Let $k = \lfloor \alpha(\alpha + \xi)/\xi \rfloor + 1$. Here $\alpha(\alpha + \xi)/\xi > \alpha > 1$, which implies that $k \geq 2$. In addition, $k = \lfloor \alpha(\alpha + \xi)/\xi \rfloor + 1 > \alpha(\alpha + \xi)/\xi > \alpha$. Fix any $h_1, h_2 \in \mathbb{Z}$ with $0 < \|(h_1, h_2)\|_\infty \leq H$. Define $f(x) = L(h_1, h_2)q^\alpha x^\alpha$. Then

$$|L(h_1, h_2)|q^\alpha V^{\alpha-k} \ll_{\alpha, \xi} |f^{(k)}(x)| \ll_{\alpha, \xi} |L(h_1, h_2)|q^\alpha V^{\alpha-k}$$

for all real numbers $x \in (V, 2V]$. Therefore by Lemma 2.1, we obtain

$$(3.2) \quad |S(h_1, h_2)| \ll_{\alpha, \xi} (|L(h_1, h_2)|q^\alpha V^{\alpha-k})^{1/(2^k-2)} + V^{-2^{2-k}} (|L(h_1, h_2)|q^\alpha V^{\alpha-k})^{-1/(2^k-2)}.$$

Let S_1 and S_2 be the first and second term on the right-hand side of (3.2), respectively. Then we have

$$S_1^{2^k-2} \leq |L(h_1, h_2)|q^\alpha V^{\alpha-k} \ll_{\alpha, \xi, \delta} q^\theta q^\alpha q^{(\alpha-k)\xi/\alpha}.$$

Further, we observe that

$$\alpha + \frac{(\alpha - k)\xi}{\alpha} < \alpha + \frac{(\alpha - \alpha(\alpha + \xi)/\xi)\xi}{\alpha} = 0.$$

Therefore, by taking small $\theta > 0$, one has $S_1 \ll_{\alpha, \xi, \delta} q^{-2\theta}$.

Let us next evaluate S_2 . By (3.1), it follows that

$$S_2^{\alpha(2^k-2)} \ll_{\alpha, \xi, \delta} q^{-2^{2-k}(2^k-2)\xi} q^{\eta\alpha} q^{-\alpha^2} q^{\xi(k-\alpha)}.$$

By $2 \leq k \leq \alpha(\alpha + \xi)/\xi + 1$, the exponent of q is

$$\begin{aligned} & -2^{2-k}(2^k-2)\xi + \eta\alpha - \alpha^2 + \xi(k-\alpha) \\ & \leq (-4 + 2^{3-k})\xi + \xi - \epsilon\alpha + 2\theta\alpha - \alpha^2 + \xi(\alpha(\alpha + \xi)/\xi + 1 - \alpha) \\ & \leq -\epsilon\alpha + 2\theta\alpha < 0 \end{aligned}$$

if θ is sufficiently small. Therefore, by taking small $\theta > 0$, one has $S_2 \ll_{\alpha, \xi} q^{-2\theta}$. Hence

$$\mathcal{D}_{V < v \leq 2V}((vp)^\alpha, (vq)^\alpha) \ll_{\alpha, \xi, \delta} q^{-\theta} + q^{-2\theta} (\log H)^2.$$

This implies that

$$\mathcal{D}_{V < v \leq 2V}((vp)^\alpha, (vq)^\alpha) \ll_{\alpha, \xi, \epsilon, \delta} q^{-\theta}.$$

Let us next discuss the case when there exist $h_1, h_2 \in \mathbb{Z}$ with $0 < \|(h_1, h_2)\|_\infty \leq H$ such that $|L(h_1, h_2)| \leq q^{-\eta}$. In this case, it follows that either $h_1 < 0 < h_2$ or $h_2 < 0 < h_1$. Indeed, if $h_1, h_2 \geq 0$ or $h_1, h_2 \leq 0$ holds, then by $0 < \|(h_1, h_2)\|_\infty \leq H$, one has

$$q^{-\eta} \geq |L(h_1, h_2)| = |h_1|(p/q)^\alpha + |h_2| \geq (1/q)^\alpha = q^{-\alpha}.$$

Therefore, $\xi/\alpha - \epsilon \leq \eta \leq \alpha$ which contradicts $\epsilon < (\xi - \alpha^2)/\alpha$. Hence

$$|L(h_1, h_2)| = \left| |h_1|(p/q)^\alpha - |h_2| \right| \leq q^{-\eta}.$$

In addition, this implies that $|h_1| \geq (q/p)^\alpha |h_2| - q^{\alpha-\eta}/p^\alpha > |h_2|$ since $\eta > \alpha$ and $q > p$. We replace $|h_1|$ and $|h_2|$ with h_1 and h_2 , respectively. Let $\psi = \eta - 2\theta = \xi/\alpha - \epsilon$. Let $U = q^{(\psi-\alpha)/\alpha}$. Let $K = \lfloor q^{2\theta} \rfloor$. By (2.1),

$$\mathcal{D}_{U < u \leq 2U}((uq)^\alpha/h_1) \ll \frac{1}{K} + \sum_{1 \leq h \leq K} \frac{|T(h)|}{h},$$

where $T(h) = \frac{1}{U} \sum_{U < u \leq 2U} e(h(uq)^\alpha/h_1)$. Let $\ell = \lfloor \alpha\psi/(\psi - \alpha) \rfloor + 1$. We define $g(x) = (xq)^\alpha/h_1$. Then for all real numbers $x \in (U, 2U]$

$$q^\alpha U^{\alpha-\ell}/h_1 \ll_{\ell, \alpha} |g^{(\ell)}(x)| \ll_{\ell, \alpha} q^\alpha U^{\alpha-\ell}/h_1.$$

Hence, by Lemma 2.1, one has

$$T(h) \ll_{\ell, \alpha} (q^\alpha U^{\alpha-\ell}/h_1)^{1/(2^\ell-2)} + U^{-2^{2-\ell}} (q^\alpha U^{\alpha-\ell}/h_1)^{-1/(2^\ell-2)}.$$

Let T_1 and T_2 be the first and second term on the right-hand side of this equation. Then

$$T_1^{2^\ell-2} = q^\alpha U^{\alpha-\ell}/h_1 \leq q^\alpha q^{(\alpha-\ell)(\psi-\alpha)/\alpha}.$$

The exponent of q is

$$\alpha + (\alpha - \ell)(\psi - \alpha)/\alpha < \alpha + (\alpha - \psi\alpha/(\psi - \alpha))(\psi - \alpha)/\alpha = 0.$$

Therefore, by taking small $\theta > 0$, we have $T_1 \ll_{\alpha, \ell} q^{-3\theta}$. Let us evaluate T_2 . We have

$$T_2^{\alpha(2^\ell-2)} \leq U^{-2^{2-\ell}(2^\ell-2)\alpha} H^\alpha q^{-\alpha^2} U^{(\ell-\alpha)\alpha} \ll_{\alpha, \ell} q^{-2^{2-\ell}(2^\ell-2)(\psi-\alpha)} q^{-\alpha^2} q^{(\ell-\alpha)(\psi-\alpha)} H^\alpha.$$

Let $\xi' = \psi - \alpha$. Note that $\xi' > 0$ holds since $\xi > \alpha^2 + \alpha\epsilon$ and $\psi = \xi/\alpha - \epsilon$. From $2 \leq \ell \leq \alpha(\alpha + \xi')/\xi' + 1$, the exponent of q is

$$\begin{aligned} & -2^{2-\ell}(2^\ell-2)\xi' - \alpha^2 + (\ell-\alpha)\xi' \\ & \leq -4\xi' + 2^{3-2}\xi' - \alpha^2 + (\alpha(\alpha + \xi')/\xi' + 1 - \alpha)\xi' \\ & \leq -2\xi' - \alpha^2 + (\alpha^2/\xi' + 1)\xi' = -\xi'. \end{aligned}$$

Therefore $T_2^{\alpha(2^\ell-2)} \ll_{\alpha, \ell} q^{-\xi'+\alpha\theta}$. By taking sufficiently small $\theta = \theta(\alpha, \xi, \epsilon)$, we have $T_2 \ll_{\alpha, \ell} q^{-3\theta}$. Hence we obtain

$$\mathcal{D}_{U < u \leq 2U}((uq)^\alpha/h_1') \ll_{\alpha, \ell} q^{-2\theta} + q^{-3\theta} \log q \ll_{\alpha, \xi, \epsilon} q^{-2\theta}.$$

□

4. PROOF OF THEOREM 1.2

Let $\alpha > 1$. Suppose that α satisfies Condition 1.1. Then there exists $\xi > \alpha^2$ such that infinitely many (p, q, r, s) 's with $p < q < r < s$ satisfy (1.1). Let $((p_n, q_n, r_n, s_n))_{n=1}^\infty$ be a sequence of which each term satisfies (1.1). We may assume that $q_1 < q_2 < \dots \rightarrow \infty$. If not, then there are at most finitely many (p, q, r, s) 's satisfy (1.1). This is a contradiction. Let $0 < \epsilon < (\xi - \alpha^2)/\alpha$, and $\delta = 2^{-6}$. Take any large $n \in \mathbb{N}$. By Lemma 3.1, by setting $V_n = (\delta q_n^\xi)^{1/\alpha}$ and $U_n = q_n^{(\xi - \alpha^2)/\alpha^2 - \epsilon/\alpha}$, either we have

$$\mathcal{D}_{V_n < v \leq 2V_n}((vp)^\alpha, (vq)^\alpha) \ll_{\alpha, \epsilon, \xi, \delta} q_n^{-\theta},$$

or there exist $h_1, h_2 \in \mathbb{N}$ with $h_2 < h_1 \leq q_n^\theta$ such that $|h_1(p_n/q_n)^\alpha - h_2| \leq q_n^{-\xi/\alpha + \epsilon}$ and

$$\mathcal{D}_{U_n < u \leq 2U_n}((uq_n)^\alpha/h_1) \ll_{\alpha, \epsilon, \xi, \delta} q_n^{-2\theta}.$$

In the former case, let

$$A_n = \{v \in (V_n, 2V_n] \cap \mathbb{N} : (\{vp_n\}^\alpha, \{vq_n\}^\alpha) \in [1/8, 1/4) \times [0, 1/8)\}.$$

By the definition of the discrepancy, it follows that

$$\#A_n = V_n/64 + O(V_n q_n^{-\theta}).$$

Therefore $A_n \neq \emptyset$ if n is sufficiently large. Fix any $v_n \in A$. Then we have

$$\begin{aligned} (v_n r_n)^\alpha &= 2(v_n q_n)^\alpha - (v_n p_n)^\alpha + E_n^{(1)} \\ &= 2\lfloor (v_n q_n)^\alpha \rfloor - \lfloor (v_n p_n)^\alpha \rfloor + 2\{(v_n q_n)^\alpha\} - \{(v_n p_n)^\alpha\} + E_n^{(1)} \end{aligned}$$

where we let $E_n^{(1)} = (v_n r_n)^\alpha - 2(v_n q_n)^\alpha + (v_n p_n)^\alpha$. By $v_n \in A_n$, it follows that

$$1/8 = 1/4 - 1/8 \leq 2\{(v_n q_n)^\alpha\} - \{(v_n p_n)^\alpha\} \leq 1/2.$$

Since $|E_n^{(1)}| \leq \delta = 2^{-6}$, we find that $2\{(v_n q_n)^\alpha\} - \{(v_n p_n)^\alpha\} + E_n^{(1)} \in [0, 1)$. Thus

$$\{(v_n r_n)^\alpha\} = 2\{(v_n q_n)^\alpha\} - \{(v_n p_n)^\alpha\} + E_n^{(1)}.$$

Let $E_n^{(2)} = (v_n q_n)^\alpha + (v_n s_n)^\alpha - 2(v_n r_n)^\alpha$. Then we have

$$\begin{aligned} (v_n s_n)^\alpha &= 2(v_n r_n)^\alpha - (v_n q_n)^\alpha + E_n^{(2)} \\ &= 2\lfloor (v_n r_n)^\alpha \rfloor - \lfloor (v_n q_n)^\alpha \rfloor + 2\{(v_n r_n)^\alpha\} - \{(v_n q_n)^\alpha\} + E_n^{(2)}. \end{aligned}$$

By $v_n \in A_n$, we obtain that

$$\begin{aligned} 2\{(v_n r_n)^\alpha\} - \{(v_n q_n)^\alpha\} + E_n^{(2)} &= 3\{(v_n q_n)^\alpha\} - 2\{(v_n p_n)^\alpha\} + 2E_n^{(1)} + E_n^{(2)}, \\ 1/8 = 3/8 - 1/4 &\leq 3\{(v_n q_n)^\alpha\} - 2\{(v_n p_n)^\alpha\} \leq 3/4, \\ |2E_n^{(1)} + E_n^{(2)}| &\leq 3\delta \leq 2^{-4}. \end{aligned}$$

These imply that $\{(v_n s_n)^\alpha\} = 2\{(v_n r_n)^\alpha\} - \{(v_n q_n)^\alpha\} + E_n^{(2)}$. Therefore

$$\begin{aligned} &| \lfloor (v_n p_n)^\alpha \rfloor + \lfloor (v_n r_n)^\alpha \rfloor - 2\lfloor (v_n q_n)^\alpha \rfloor | \\ &\leq |E_n^{(1)}| + | \{ \{(v_n p_n)^\alpha\} + \{(v_n r_n)^\alpha\} - 2\{(v_n q_n)^\alpha\} \} | \\ &\leq \delta + \delta = 2\delta = 2^{-5}, \end{aligned}$$

and

$$\begin{aligned}
& |[(v_n q_n)^\alpha] + [(v_n s_n)^\alpha] - 2[(v_n r_n)^\alpha]| \\
& \leq |E_n^{(2)}| + |\{(v_n q_n)^\alpha\} + \{(v_n s_n)^\alpha\} - 2\{(v_n r_n)^\alpha\}| \\
& \leq \delta + \delta = 2\delta = 2^{-5}.
\end{aligned}$$

Hence we conclude that

$$[(v_n p_n)^\alpha] + [(v_n r_n)^\alpha] = 2[(v_n q_n)^\alpha], \quad [(v_n q_n)^\alpha] + [(v_n s_n)^\alpha] = 2[(v_n r_n)^\alpha].$$

In the latter case when there exist $h_1, h_2 \in \mathbb{N}$ with $h_2 < h_1 \leq q_n^\theta$ such that

$$|h_1(p_n/q_n)^\alpha - h_2| \leq q_n^{-\xi/\alpha + \epsilon - 2\theta}, \quad \mathcal{D}_{U_n < u \leq 2U_n}((uq_n)^\alpha/h_1) \ll_{\alpha, \epsilon, \xi} q_n^{-2\theta}$$

where $U_n = q_n^{(\xi - \alpha^2)/\alpha^2 - \epsilon/\alpha}$. Let

$$B_n = \{u \in (U_n, 2U_n] : h_1^{-1}2^{-6} \leq \{(u_n q_n)^\alpha/h_1\} < h_1^{-1}2^{-5}\}$$

By the definition of the discrepancy, it follows that

$$\#B_n = U_n h_1^{-1}2^{-6} + O(U_n q_n^{-2\theta}) \geq U_n q_n^{-\theta}2^{-6} + O(U_n q_n^{-2\theta}) > 0.$$

Therefore $B_n \neq \emptyset$ if n is sufficiently large. Fix any $u_n \in B_n$. Then we observe that

$$(u_n q_n)^\alpha = h_1 [(u_n q_n)^\alpha/h_1] + h_1 \{(u_n q_n)^\alpha/h_1\}.$$

Since $u_n \in B_n$, we also observe that $2^{-6} \leq h_1 \{(u_n q_n)^\alpha/h_1\} \leq 2^{-5}$. Thus

$$(4.1) \quad \{(u_n q_n)^\alpha\} = h_1 \{(u_n q_n)^\alpha/h_1\}.$$

Let $E_n^{(3)} = (u_n p_n)^\alpha - (h_2/h_1)(u_n q_n)^\alpha$. It follows that

$$\begin{aligned}
(u_n p_n)^\alpha &= (h_2/h_1)(u_n q_n)^\alpha + E_n^{(3)} \\
&= h_2 [(u_n q_n)^\alpha/h_1] + h_2 \{(u_n q_n)^\alpha/h_1\} + E_n^{(3)}.
\end{aligned}$$

By $u_n \in B_n$, $1 \leq h_2 < h_1 \leq q_n^\theta$, and $u_n \leq U_n = q_n^{(\xi - \alpha^2)/\alpha^2 - \epsilon/\alpha}$, we have

$$q_n^{-\theta}2^{-6} \leq h_2 \cdot h_1^{-1}2^{-6} \leq h_2 \{(u_n q_n)^\alpha/h_1\} \leq h_2 \cdot h_1^{-1}2^{-5} \leq 2^{-5} - q_n^{-\theta}2^{-5},$$

$$|E_n^{(3)}| = (u_n q_n)^\alpha |h_1(p_n/q_n)^\alpha - h_2|/h_1 \leq U_n q_n^\alpha q_n^{-\xi/\alpha + \epsilon - 2\theta} = q_n^{(\xi - \alpha^2)/\alpha - \epsilon} q_n^\alpha q_n^{-\xi/\alpha + \epsilon - 2\theta} = q_n^{-2\theta}.$$

Therefore, we obtain $\{(u_n p_n)^\alpha\} = h_2 \{(u_n q_n)^\alpha/h_1\} + E_n^{(3)}$ by taking large n . Further, let $E_n^{(4)} = (u_n r_n)^\alpha - 2(u_n q_n)^\alpha + (u_n p_n)^\alpha$. We observe that

$$\begin{aligned}
(u_n r_n)^\alpha &= 2(u_n q_n)^\alpha - (u_n p_n)^\alpha + E_n^{(4)} \\
&= 2h_1 [(u_n q_n)^\alpha] - [(u_n p_n)^\alpha] + 2\{(v_n q_n)^\alpha\} - \{(v_n p_n)^\alpha\} + E_n^{(4)}.
\end{aligned}$$

In addition, we have $|E_n^{(4)}| \leq U_n^\alpha q_n^{-\xi} = q_n^{(\xi - \alpha^2)/\alpha - \epsilon} q_n^{-\xi} = q_n^{-(1-1/\alpha)\xi - \alpha - \epsilon}$. Further,

$$\begin{aligned}
& 2\{(u_n q_n)^\alpha\} - \{(u_n p_n)^\alpha\} + E_n^{(4)} \\
&= 2h_1 \{(u_n q_n)^\alpha/h_1\} - \{(u_n p_n)^\alpha\} + E_n^{(4)} \\
&= 2h_1 \{(u_n q_n)^\alpha/h_1\} - h_2 \{(u_n q_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)} \\
&= (2h_1 - h_2) \{(u_n q_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)}
\end{aligned}$$

Note that $h_1 > h_2$ and $u_n \in B_n$ imply

$$2^{-6} = h_1 h_1^{-1}2^{-6} \leq (2h_1 - h_2) \{(u_n q_n)^\alpha/h_1\} \leq 2h_1 \cdot h_1^{-1}2^{-5} = 2^{-4}.$$

Therefore we have $\{(u_n r_n)^\alpha\} = 2\{(u_n q_n)^\alpha\} - \{(u_n p_n)^\alpha\} + E_n^{(4)}$ for sufficiently large n . We next let $E_n^{(5)} = (u_n q_n)^\alpha + (u_n s_n)^\alpha - 2(u_n r_n)^\alpha$. Similarly to the evaluation of $E^{(4)}$, it follows that $|E_n^{(5)}| \leq q_n^{-(1-1/\alpha)\xi - \alpha - \epsilon}$. In addition, we observe that

$$(u_n s_n)^\alpha = 2[(u_n r_n)^\alpha] - [(u_n q_n)^\alpha] + 2\{(u_n r_n)^\alpha\} - \{(u_n q_n)^\alpha\} + E_n^{(5)},$$

and

$$\begin{aligned} & 2\{(u_n r_n)^\alpha\} - \{(u_n q_n)^\alpha\} \\ &= 2(2h_1 - h_2)\{(u_n q_n)^\alpha/h_1\} - h_1\{(u_n q_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)} \\ &= (3h_1 - 2h_2)\{(u_n q_n)^\alpha/h_1\} - E_n^{(3)} + E_n^{(4)}. \end{aligned}$$

Note that $h_1 > h_2$ and $u_n \in B_n$ imply

$$2^{-6} = h_1 \cdot h_1^{-1} 2^{-6} \leq (3h_1 - 2h_2)\{(u_n q_n)^\alpha/h_1\} \leq 3h_1 \cdot h_1^{-1} 2^{-5} \leq 2^{-3}.$$

Therefore $\{(u_n s_n)^\alpha\} = 2\{(u_n r_n)^\alpha\} - \{(u_n q_n)^\alpha\} + E_n^{(5)}$. Similarly to the former case, by taking sufficiently large $n \in \mathbb{N}$, we conclude that

$$[(u_n p_n)^\alpha] + [(u_n r_n)^\alpha] = 2[(u_n q_n)^\alpha], \quad [(u_n q_n)^\alpha] + [(u_n s_n)^\alpha] = 2[(u_n r_n)^\alpha].$$

Hence $\text{PS}(\alpha)$ contains infinitely many 4-APs assuming that α satisfies Condition 1.1

ACKNOWLEDGEMENT

The author would like to thank Professor Yuzuru Sato, Professor Kouji Yano, and Professor Hiroki Sumi for the opportunity to give a talk in RIMS Workshop 2021 ‘‘Integrated Research on Random Dynamical Systems and Multi-Valued Dynamical Systems’’. This work was supported by JSPS KAKENHI Grant Number JP19J20878.

REFERENCES

- [DT97] Michael Drmota and Robert F. Tichy, *Sequences, discrepancies and applications*, Lecture Notes in Mathematics, vol. 1651, Springer-Verlag, Berlin, 1997. MR 1470456
- [FW09] N. Frantzikinakis and M. Wierdl, *A Hardy field extension of Szemerédi’s theorem*, Adv. Math. **222** (2009), no. 1, 1–43. MR 2531366
- [Kok50] Jurjen Ferdinand Koksma, *Some theorems on Diophantine inequalities*, Scriptum no. 5, Math. Centrum Amsterdam, 1950. MR 0038379
- [MS21] T. Matsusaka and K. Saito, *Linear Diophantine equations in Piatetski-Shapiro sequences*, Acta Arith. **200** (2021), no. 1, 91–110. MR 4319608
- [Szi52] Péter Szüsz, *Über ein Problem der Gleichverteilung*, Comptes Rendus du Premier Congrès des Mathématiciens Hongrois, 27 Août–2 Septembre 1950, Akadémiai Kiadó, Budapest, 1952, pp. 461–472. MR 0056036
- [Tit86] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, second ed., The Clarendon Press, Oxford University Press, New York, 1986, Edited and with a preface by D. R. Heath-Brown. MR 882550

KOTA SAITO, GRADUATE SCHOOL OF MATHEMATICS, NAGOYA UNIVERSITY, FURO-CHO, CHIKUSA-KU, NAGOYA, 464-8602, JAPAN

Email address: m17013b@math.nagoya-u.ac.jp