

Random Dynamical Systems of Regular Polynomial Maps on \mathbb{C}^2

Hiroki Sumi

Graduate School of
Human and Environmental Studies
Kyoto University
Japan

E-mail: sumi@math.h.kyoto-u.ac.jp

<http://www.math.h.kyoto-u.ac.jp/~sumi/index.html>

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Abstract

We introduce the notion of mean stability in i.i.d. random (holomorphic) 2-dimensional dynamical systems. We can see that a **generic** random dynamical system of regular polynomial maps on \mathbb{P}^2 (the complex 2-dimensional projective space) having an attractor in the line at infinity, is **mean stable**. If a random holomorphic dynamical system on \mathbb{P}^2 is mean stable then for each $z \in \mathbb{P}^2$, for a.e. orbit starting with z , **the Lyapunov exponent is negative**. If a random holomorphic dynamical system on \mathbb{P}^2 is mean stable, then for any $z \in \mathbb{P}^2$, the orbit of the Dirac measure at z under the iterations of the dual map of the transition operator **converges to a periodic cycle** of probability measures. Note that the above statements **cannot hold for deterministic dynamics of a single regular polynomial map f with $\deg(f) \geq 2$** .

We see many **randomness-induced phenomena** (phenomena in random dynamical systems which cannot hold for iteration dynamics of single maps). In this talk, we have seen **randomness-induced order**.

Motivation.

- Nature has a lot of random (noise) terms. Thus it is natural and important to consider **random dynamical systems**.
- **Holomorphic dynamical systems** have been deeply investigated. The study of them helps us to investigate real dynamical systems.
- Combining the above two ideas, we consider **random holomorphic dynamical systems**.
- We want to find new phenomena (so called **randomness-induced phenomena**) in random dynamical systems which **cannot hold in deterministic iteration dynamical systems of single maps**.
- Other motivations: **Random relaxed Newton's method** (in which we can find roots of polynomials **more easily than the deterministic methods**, see S, [S21]). **The action of holomorphic automorphisms** on complex manifolds. **The action of mapping class groups** of the Riemann surfaces on the character varieties, etc.

Definition 1.

- (1) Let \mathbb{C}^2 be the 2-dimensional complex Euclidean space. Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a polynomial map, i.e., if we write $f(x, y) = (g(x, y), h(x, y))$, then $g(x, y)$ and $h(x, y)$ are polynomials of (x, y) .

We say that f is a regular polynomial map on \mathbb{C}^2 if f extends to a holomorphic map on \mathbb{P}^2 (the complex 2-dimensional projective space), i.e.,

$$\mathbb{P}^2 = \{[u : v : w] \mid (u, v, w) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}\}.$$

Note that we regard \mathbb{C}^2 as a subset of \mathbb{P}^2 via the following canonical identification and inclusion:

$$\mathbb{C}^2 \cong \{[u : v : 1] \in \mathbb{P}^2 \mid (u, v) \in \mathbb{C}^2\} \subset \mathbb{P}^2.$$

Remark: Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ be a polynomial map.

Then f is a regular polynomial map if and only if the following (*) holds.

- (*) Let $f(x, y) = (g(x, y), h(x, y))$. Let $g_1(x, y)$ be the highest degree term of $g(x, y)$ and let $h_1(x, y)$ be the highest degree term of $h(x, y)$. Then $\deg(g_1) = \deg(h_1)$ and

$$g_1(x, y) = h_1(x, y) = 0 \Leftrightarrow (x, y) = (0, 0).$$

Example: Let $f(x, y) =$

$$(a_1x^2 + a_2xy + a_3y^2 + b_1x + b_2y + b_3, c_1y^2 + c_2x + c_3y + c_4),$$

where $a_1, c_1 \in \mathbb{C} \setminus \{0\}$, $a_2, a_3, b_1, b_2, b_3, c_2, c_3, c_4 \in \mathbb{C}$.

Then f is a regular polynomial map on \mathbb{C}^2 .

- (3) If f is a regular polynomial map on \mathbb{C}^2 ,
then we regard f as a holomorphic map on \mathbb{P}^2 .
We call such a holomorphic map f on \mathbb{P}^2
“ a regular polynomial map on \mathbb{P}^2 ”.
- (4) Let X be the space of all regular polynomial maps on \mathbb{P}^2 of degree two or more, endowed with the distance η which is defined as $\eta(f, g) = \sup_{z \in \mathbb{P}^2} d(f(z), g(z))$, where d denotes the distance on \mathbb{P}^2 induced by the Fubini-Study metric on \mathbb{P}^2 .
- (5) We denote by $\mathfrak{M}_1(X)$ the space of all Borel probability measures on X .
Also, we set $\mathfrak{M}_{1,c}(X) := \{\tau \in \mathfrak{M}_1(X) \mid \text{supp } \tau \text{ is a compact subset of } X\}$.
We endow $\mathfrak{M}_{1,c}(X)$ with a topology \mathcal{O} which satisfies that
 $\tau_n \rightarrow \tau$ as $n \rightarrow \infty$ if and only if
- (a) for each bounded continuous function $\varphi : X \rightarrow \mathbb{C}$, we have $\int \varphi d\tau_n \rightarrow \int \varphi d\tau$ as $n \rightarrow \infty$, and
- (b) $\text{supp } \tau_n \rightarrow \text{supp } \tau$ as $n \rightarrow \infty$ with respect to the Hausdorff metric in the space of all non-empty compact subsets of X .

For each $\tau \in \mathfrak{M}_{1,c}(X)$, we consider i.i.d. random dynamical system on \mathbb{P}^2 such that at every step we choose a map $f \in X$ according to τ . This defines a Markov process whose state space is \mathbb{P}^2 and whose transition probability $p(z, A)$ from a point $z \in \mathbb{P}^2$ to a Borel subset A of \mathbb{P}^2 satisfies $p(z, A) = \tau(\{h \in X \mid h(z) \in A\})$.

- (6) For $\forall \tau \in \mathfrak{M}_{1,c}(X)$, let $G_\tau := \{\gamma_n \circ \dots \circ \gamma_1 \mid n \in \mathbb{N}, \gamma_j \in \text{supp } \tau(\forall j)\}$. This is a **semigroup** with the semigroup operation being the functional composition. (It is important to study the dynamics of G_τ .)
- (7) We say that an element $\tau \in \mathfrak{M}_{1,c}(X)$ is mean stable if there exist an $n \in \mathbb{N}$, an $m \in \mathbb{N}$, non-empty open subsets U_1, \dots, U_m of \mathbb{P}^2 , a non-empty compact subset K of $\cup_{j=1}^m U_j$, and a constant c with $0 < c < 1$ such that the following (a) and (b) hold.

- (a) For each $(\gamma_1, \dots, \gamma_n) \in (\text{supp } \tau)^n$, we have

$$\gamma_n \circ \dots \circ \gamma_1(\cup_{j=1}^m U_j) \subset K.$$

Moreover, for each $j = 1, \dots, m$, for all $x, y \in U_j$ and for each $(\gamma_1, \dots, \gamma_n) \in (\text{supp } \tau)^n$, we have

$$d(\gamma_n \circ \dots \circ \gamma_1(x), \gamma_n \circ \dots \circ \gamma_1(y)) \leq cd(x, y).$$

- (b) For each $z \in \mathbb{P}^2$, there exists an element $h_z \in G_\tau$ such that $h_z(z) \in U$.

Remark 2. Let $\mathcal{MS} := \{\tau \in \mathfrak{M}_{1,c}(X) \mid \tau \text{ is mean stable}\}$. Then \mathcal{MS} is **non-empty and open** in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$.

Example. Let $f_1, f_2 \in X$ be elements defined by

$$f_1(x, y) = (x^2, y^2), \quad f_2(x, y) = \left(\frac{1}{4}x^2, \frac{1}{2}y^2\right), \quad (x, y) \in \mathbb{C}^2.$$

Let $\tau = \frac{1}{2}\delta_{f_1} + \frac{1}{2}\delta_{f_2} \in \mathfrak{M}_{1,c}(X)$, where δ_{f_i} denotes the Dirac measure concentrated at $f_i \in X$ for each $i = 1, 2$. Then $\tau \in \mathcal{MS}$.

Problem 3. (Open Problem.) Is \mathcal{MS} dense in $(\mathfrak{M}_{1,c}(X), \mathcal{O})$?

(Remark: This kind of statement is **true for random dynamical systems of 1-dimensional complex polynomial maps on $\mathbb{P}^1 \cong \hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ (the Riemann sphere) of degree two or more.** S., 2013 ([S13], Adv. Math.), [SW21] (Takayuki Watanabe's presentation).)

Definition 4. Let

$$\mathbb{P}_\infty^1 := \{[u : v : 0] \in \mathbb{P}^2 \mid (u, v) \in \mathbb{C}^2 \setminus \{(0, 0)\}\}.$$

This is called the line at infinity.

Remark: Let $f \in X$. Then $f(\mathbb{P}_\infty^1) = \mathbb{P}_\infty^1$, $f^{-1}(\mathbb{P}_\infty^1) = \mathbb{P}_\infty^1$, and for each neighborhood B of \mathbb{P}_∞^1 , there exists an open neighborhood C of \mathbb{P}_∞^1 with $C \subset B$ such that $\overline{f(C)} \subset C$.

Definition 5. Let Ψ be the set of all $\tau \in \mathfrak{M}_{1,c}(X)$ satisfying the following condition. There exist two non-empty open subsets U, V of \mathbb{P}_∞^1 and an $n \in \mathbb{N}$ such that all of the following (i)(ii)(iii) hold.

(i) $\#(\mathbb{P}_\infty^1 \setminus U) \geq 3$.

(ii) $\overline{V} \subset U$, where \overline{V} denotes the closure of V in \mathbb{P}_∞^1 .

(iii) For each $(\gamma_1, \dots, \gamma_n) \in (\text{supp } \tau)^n$, we have $\gamma_n \circ \dots \circ \gamma_1(U) \subset V$.

Remark 6. Ψ is a **non-empty open** subset of $(\mathfrak{M}_{1,c}(X), \mathcal{O})$.

Example: Let $Y (\subset X)$ be the set of all regular polynomial maps $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ of the form

$$f(x, y) = (a_1x^2 + a_2xy + a_3y^2 + b_1x + b_2y + b_3, c_1y^2 + c_2x + c_3y + c_4), (x, y) \in \mathbb{C}^2,$$

where $a_1, c_1 \in \mathbb{C} \setminus \{0\}$, $a_2, a_3, b_1, b_2, b_3, c_2, c_3, c_4 \in \mathbb{C}$.

Note that $Y \cong (\mathbb{C} \setminus \{0\})^2 \times \mathbb{C}^8$.

Let τ be a Borel probability measure on Y with compact support.

Then

$$\tau \in \Psi.$$

In fact, for any $f \in Y$ of the above form, via the identification

$$\mathbb{P}_\infty^1 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}, \quad [z : 1 : 0] \leftrightarrow z \ (z \in \mathbb{C}), \quad [1 : 0 : 0] \leftrightarrow \infty,$$

$f|_{\mathbb{P}_\infty^1} : \mathbb{P}_\infty^1 \rightarrow \mathbb{P}_\infty^1$ is equal to the map $z \mapsto \frac{1}{c_1}(a_1z^2 + a_2z + a_3)$ ($z \in \mathbb{C}$) on $\hat{\mathbb{C}}$, and so $[1 : 0 : 0] \in \mathbb{P}_\infty^1$ is a common attracting fixed point of any $f \in Y$.

Theorem 7. Let $\mathcal{A} := \Psi \cap \mathcal{MS} = \{\tau \in \Psi \mid \tau \text{ is mean stable}\}$.

Then \mathcal{A} is **open and dense** in Ψ .

Moreover, for each mean stable $\tau \in \mathfrak{M}_{1,c}(X)$ (in particular, for each $\tau \in \mathcal{A}$), we have all of the following (1)–(7).

(1) There exists a constant c_τ with $c_\tau < 0$ such that the following holds.

– For each $z \in \mathbb{P}^2$, there exists a Borel subset $B_{\tau,z}$ of $X^{\mathbb{N}}$ with $(\otimes_{n=1}^{\infty} \tau)(B_{\tau,z}) = 1$ such that for each $(\gamma_1, \gamma_2, \dots) \in B_{\tau,z}$, we have

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|D(\gamma_n \circ \dots \circ \gamma_1)_z\| \leq c_\tau < 0.$$

Here, for each $f \in X$ and each $z \in \mathbb{P}^2$, we denote by $\|Df_z\|$ the norm of the differential of f at z with respect to the Fubini-Study metric in \mathbb{P}^2 .

(2) For each $z \in \mathbb{P}^2$, there exists a Borel subset $C_{\tau,z}$ of $X^{\mathbb{N}}$ with $(\otimes_{n=1}^{\infty} \tau)(C_{\tau,z}) = 1$ such that for each $\gamma = (\gamma_1, \gamma_2, \dots) \in C_{\tau,z}$, there exists a number $r = r(\tau, z, \gamma) > 0$ satisfying that

$$\text{diam}(\gamma_n \circ \dots \circ \gamma_1(B(z, r))) \rightarrow 0 \text{ as } n \rightarrow \infty$$

exponentially fast, where $B(z, r)$ denotes the ball with center z and radius r with respect to the distance d induced by the Fubini-Study metric on \mathbb{P}^2 , and for each subset A of \mathbb{P}^2 , we set $\text{diam}A := \sup_{x,y \in A} d(x, y)$.

(3) Let $\text{Min}(\tau)$ be the set of all minimal sets of τ . Then,

$$1 \leq \#\text{Min}(\tau) < \infty.$$

Here, we say that a non-empty compact subset L of \mathbb{P}^2 is a minimal set of τ if for each $z \in L$, we have $L = \overline{\bigcup_{h \in G_\tau} \{h(z)\}}$.

(4) For each $z \in \mathbb{P}^2$, there exists a Borel subset $D_{\tau,z}$ of $X^{\mathbb{N}}$ with $(\otimes_{n=1}^{\infty} \tau)(D_{\tau,z}) = 1$ such that for each $(\gamma_1, \gamma_2, \dots) \in D_{\tau,z}$,

$$d(\gamma_n \circ \dots \circ \gamma_1(z), \cup_{L \in \text{Min}(\tau)} L) \rightarrow 0 \text{ as } n \rightarrow \infty$$

exponentially fast.

- (5) Let $C(\mathbb{P}^2)$ be the Banach space of all continuous complex-valued functions on \mathbb{P}^2 endowed with the supremum norm.

Let $M_\tau : C(\mathbb{P}^2) \rightarrow C(\mathbb{P}^2)$ be the linear operator defined by

$$M_\tau(\varphi)(z) = \int_X \varphi(h(z)) d\tau(h), \text{ for } \varphi \in C(\mathbb{P}^2), z \in \mathbb{P}^2.$$

Then there exists a finite dimensional subspace $W_\tau \neq \{0\}$ of $C(\mathbb{P}^2)$ with $M_\tau(W_\tau) = W_\tau$ such that for each $\varphi \in C(\mathbb{P}^2)$,

$\{M_\tau^n(\varphi)\}_{n=1}^\infty$ tends to W_τ as $n \rightarrow \infty$.

Also, the map $\nu \mapsto W_\nu$ is **continuous** on \mathcal{MS} w.r.t. the topology \mathcal{O} .

- (6) There exists a number $0 < \alpha < 1$ such that the following (a)(b)(c) hold.

(a) The space W_τ in (3) is included in the Banach space $C^\alpha(\mathbb{P}^2)$ of all α -Hölder continuous functions on \mathbb{P}^2 endowed with α -Hölder norm.

(b) For each $\varphi \in C^\alpha(\mathbb{P}^2)$, $\{M_\tau^n(\varphi)\}_{n=1}^\infty$ tends to W_τ **exponentially fast**. (Thus $M_\tau : C^\alpha(\mathbb{P}^2) \rightarrow C^\alpha(\mathbb{P}^2)$ has the “**spectral gap property**”.)

(c) For each $L \in \text{Min}(\tau)$, let $T_{L,\tau} : \mathbb{P}^2 \rightarrow [0,1]$ be the function of probability of tending to L . That is,

$$\begin{aligned} & T_{L,\tau}(z) \\ &= (\otimes_{n=1}^\infty \tau) \{(\gamma_1, \gamma_2, \dots) \in X^\mathbb{N} \mid d(\gamma_n \circ \dots \circ \gamma_1(z), L) \rightarrow 0 \text{ as } n \rightarrow \infty\} \end{aligned}$$

for each $z \in \mathbb{P}^2$. Then,

$$T_{L,\tau} \in W_\tau \subset C^\alpha(\mathbb{P}^2).$$

Moreover, for each $z \in \mathbb{P}^2$, we have

$$\sum_{L \in \text{Min}(\tau)} T_{L,\tau}(z) = 1.$$

- (7) Let $F(G_\tau) := \{z \in \mathbb{P}^2 \mid \exists U : \text{ nbd of } z \text{ s.t. } G_\tau \text{ is equicontinuous on } U\}$.
 (This is called the Fatou set of semigroup G_τ .)
 Then, for each $L \in \text{Min}(\tau)$ and for each connected component U of $F(G_\tau)$, there exists a **constant** $c_U \in [0, 1]$ such that

$$T_{L,\tau}|_U = c_U \text{ on } U.$$

Thus $T_{L,\tau}$ is a continuous function on \mathbb{P}^2 which varies only on $J(G_\tau) := \mathbb{P}^2 \setminus F(G_\tau)$ (this $J(G_\tau)$ is called the Julia set of G_τ).

Remark 8.

- (1) **None of statements (1)–(6) in Theorem 7 can hold for deterministic dynamics of a single $f \in X$.** In fact, in the Julia set $J(f|_{\mathbb{P}_\infty^1})$ of $f|_{\mathbb{P}_\infty^1}$, we have a chaotic phenomenon. See Mañé's paper (1988)[Ma88] etc. Therefore, the statements (1)–(6) describe **randomness-induced phenomena** (phenomena in random dynamical systems which cannot hold for iteration dynamics of single maps). In this presentation, we have seen **randomness-induced order**.
- (2) Even if a system induced by an element $\tau \in \Psi$ is mean stable and we have randomness-induced order in the system, **the system still may have a kind of complexity**. In fact, if τ has multiple minimal sets, and if it satisfies some conditions, then there exists a minimal set L of τ and a number $\beta \in (0, 1)$ such that $T_{L,\tau}$ does not belong to $C^\beta(\mathbb{P}^2)$. It implies that $M_\tau : C^\beta(\mathbb{P}^2) \rightarrow C^\beta(\mathbb{P}^2)$ does not have spectral gap property anymore, and there exists an element $\varphi \in C^\beta(\mathbb{P}^2)$ such that $\|M_\tau^n(\varphi)\|_\beta \rightarrow \infty$ as $n \rightarrow \infty$ (see [S11, S13]).

Thus, in this case, there exists a number $\alpha_\tau \in (0, 1)$ such that (i) for each $\alpha \in (0, \alpha_\tau)$, the operator $M_\tau : C^\alpha(\mathbb{P}^2) \rightarrow C^\alpha(\mathbb{P}^2)$ has the spectral gap property and for each $\varphi \in C^\alpha(\mathbb{P}^2)$, $\{M_\tau^n(\varphi)\}_{n=1}^\infty$ tends to the finite dimensional subspace W_τ of $C^\alpha(\mathbb{P}^2)$, but, (ii) for each $\alpha \in (\alpha_\tau, 1)$, the operator $M_\tau : C^\alpha(\mathbb{P}^2) \rightarrow C^\alpha(\mathbb{P}^2)$ does not have the spectral gap property anymore (and M_τ might not behave well on $C^\alpha(\mathbb{P}^2)$, e.g., there exists a $\varphi \in C^\alpha(\mathbb{P}^2)$ such that $\|M_\tau^n(\varphi)\|_\alpha \rightarrow \infty$ as $n \rightarrow \infty$). As we have seen above, even if we have randomness-induced order, we have to check the **gradation between chaos and order**. The above quantity α_τ seems to be a kind of quantity which describes the gradation between chaos and order.

Rough idea of the proofs of Theorem 7

To show the density of \mathcal{A} in Ψ , let $\tau \in \Psi$. Let L be a minimal set of τ , i.e., L is a non-empty compact subset of \mathbb{P}^2 such that for each $z \in L$, we have $L = \overline{\cup_{h \in G_\tau} \{h(z)\}}$.

- If L is “attracting” for τ , then even if we enlarge the support of τ a little bit and we obtain a new $\nu \in \Psi$, there exists a minimal set L_ν of ν which is close to L and is **still attracting for ν** .
- If L is not attracting for τ , then if we enlarge the support of τ a little bit and we obtain a new $\nu \in \Psi$, then L is **broken** (there is no minimal set of ν around L).

Thus, if we enlarge the support of τ a little bit and obtain a new $\nu \in \Psi$, then **every minimal set L of ν is attracting for ν** . Then it is easy to see that ν is **mean stable**.

Hence the set of all mean stable $\nu \in \Psi$ is open and **dense** in Ψ .

Summary

- (1) We introduce the notion of mean stability in i.i.d. random (holomorphic) 2-dimensional dynamical systems.
- (2) We can see that a **generic** random dynamical system of regular polynomial maps on \mathbb{P}^2 having an attractor in the line at infinity, is **mean stable**.
- (3) If a random holomorphic dynamical system on \mathbb{P}^2 is mean stable then for each $z \in \mathbb{P}^2$, for a.e. orbit starting with z , **the Lyapunov exponent is negative**.
- (4) If a random holomorphic dynamical system on \mathbb{P}^2 is mean stable, then for any $z \in \mathbb{P}^2$, the orbit of the Dirac measure at z under the iterations of the dual map of the transition operator **converges to a periodic cycle** of probability measures.
- (5) Note that the statements of (3) and (4) **cannot hold for deterministic dynamics of a single regular polynomial map f with $\deg(f) \geq 2$** .

We see many **randomness-induced phenomena** (phenomena in random dynamical systems which cannot hold for iteration dynamics of single maps). In this presentation, we have seen **randomness-induced order**.

Many kinds of maps in one random dynamical system automatically cooperate together to make the chaoticity weaker. We call such phenomena

Cooperation Principle.

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