# KIM-FORKING AND KIM-DIVIDING IN NATP THEORIES

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## 1. Introduction

This note is based on the author's talk in RIMS Symposia, "Model theoretic aspects of the notion of independence and dimension", held on December 13 − 15, 2021. Kim's lemma has been a key observation in the study of forking and dividing, especially in the proof of the equivalence of forking (Kim-forking) and dividing (Kim-dividing) in simple theories and $\mathrm{NTP}_2$ theories ($\mathrm{NSOP}_1$ theories, resp.). In a collaboration with Bonghun Lee and Hyoyoon Lee, now we are trying to show that Kim-forking and Kim-dividing are equivalent in NATP theories. Following the strategy of [CK12], we obtained a partial result which is expected to play a role in proving the equivalence, namely

**Theorem 1.1.** *In NATP theories, if a formula Kim-divides over a model $M$, then it coheir-divides over $M$.*

## 2. Preliminary

We summarize notations and basic facts of the subjects to be covered in this note. First we recall notations and definitions of tree properties.

**Notation 2.1.** Let $\kappa$ and $\lambda$ be cardinals.

(i) By $\kappa^\lambda$, we mean the set of all functions from $\lambda$ to $\kappa$.
(ii) By $\kappa^{<\lambda}$, we mean $\bigcup_{\alpha<\lambda}\kappa^\alpha$ and call it a tree. If $\kappa = 2$, we call it a binary tree. If $\kappa \geq \omega$, then we call it an infinitary tree.
(iii) By $\emptyset$ or $\langle\rangle$, we mean the empty string in $\kappa^{<\lambda}$, which means the empty set (recall that every function can be regarded as a set of ordered pairs).

Let $\eta, \nu \in \kappa^{<\lambda}$.

(iv) By $\eta \trianglelefteq \nu$, we mean $\eta \subseteq \nu$. If $\eta \trianglelefteq \nu$ or $\nu \trianglelefteq \eta$, then we say $\eta$ and $\nu$ are comparable.
(v) By $\eta \perp \nu$, we mean that $\eta \ntrianglelefteq \nu$ and $\nu \ntrianglelefteq \eta$. We say $\eta$ and $\nu$ are incomparable if $\eta \perp \nu$.
(vi) By $\eta \wedge \nu$, we mean the maximal $\xi \in \kappa^{<\lambda}$ such that $\xi \trianglelefteq \eta$ and $\xi \trianglelefteq \nu$.
(vii) By $l(\eta)$, we mean the domain of $\eta$.
(viii) By $\eta <_{lex} \nu$, we mean that either $\eta \trianglelefteq \nu$, or $\eta \perp \nu$ and $\eta(l(\eta \wedge \nu)) < \nu(l(\eta \wedge \nu))$.
(ix) By $\eta^\frown\nu$, we mean $\eta \cup \{(i + l(\eta), \nu(i)) : i < l(\nu)\}$.

Let $X \subseteq \kappa^{<\lambda}$.

(x) By $\eta^\frown X$ and $X^\frown\eta$, we mean $\{\eta^\frown x : x \in X\}$ and $\{x^\frown\eta : x \in X\}$ respectively.

Let $\eta_0, ..., \eta_n \in \kappa^{<\lambda}$.

(xi) By $\mathrm{cl}(\eta_0, ..., \eta_n)$, we mean a tuple $(\eta_0 \wedge \eta_0, ..., \eta_0 \wedge \eta_n, ..., \eta_n \wedge \eta_0, ..., \eta_n \wedge \eta_n)$.
(xii) We say a subset $X$ of $\kappa^{<\lambda}$ is an *antichain* if the elements of $X$ are pairwise incomparable, *i.e.*, $\eta \perp \nu$ for all $\eta, \nu \in X$).

**Definition 2.2.** Let $\varphi(x, y)$ be an $\mathcal{L}$-formula.

(i) We say $\varphi(x, y)$ has the *tree property* (TP) if there exists a tree-indexed set $(a_\eta)_{\eta\in\omega^{<\omega}}$ of parameters and $k \in \omega$ such that
$\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in \omega^\omega$ (path consistency), and
$\{\varphi(x, a_{\eta^\frown i})\}_{i\in\omega}$ is $k$-inconsistent for all $\eta \in \omega^{<\omega}$, *i.e.*, any subset of $\{\varphi(x, a_{\eta^\frown i})\}_{i\in\omega}$ of size $k$ is inconsistent.
(ii) We say $\varphi(x, y)$ has the *tree property of the first kind* ($\mathrm{TP}_1$) if there is a tree-indexed set $(a_\eta)_{\eta\in\omega^{<\omega}}$ of parameters such that
$\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in \omega^\omega$, and
$\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu \in \omega^{<\omega}$.
(iii) We say $\varphi(x, y)$ has the *tree property of the second kind* ($\mathrm{TP}_2$) if there is an array-indexed set $(a_{i,j})_{i,j\in\omega}$ of parameters such that
$\{\varphi(x, a_{n,\eta(n)})\}_{n\in\omega}$ is consistent for all $\eta \in \omega^\omega$, and
$\{\varphi(x, a_{i,j}), \varphi(x, a_{i,k}))\}$ is inconsistent for all $i, j, k \in \omega$ with $j \neq k$.
(iv) We say $\varphi(x, y)$ has the *1-strong order property* ($\mathrm{SOP}_1$) if there is a binary-tree-indexed set $(a_\eta)_{\eta\in 2^{<\omega}}$ of parameters such that
$\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in 2^\omega$,
$\{\varphi(x, a_{\eta^\frown 1}), \varphi(x, a_{\eta^\frown 0^\frown \nu})\}$ is inconsistent for all $\eta, \nu \in 2^{<\omega}$.
(v) We say $\varphi(x, y)$ has the *2-strong order property* ($\mathrm{SOP}_2$) if there is a binary-tree-indexed set $(a_\eta)_{\eta\in 2^{<\omega}}$ of parameters such that
$\{\varphi(x, a_{\eta\lceil n})\}_{n\in\omega}$ is consistent for all $\eta \in 2^\omega$,
$\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent for all $\eta \perp \nu \in 2^{<\omega}$.
(vi) We say a theory has TP if there is a formula having TP with respect to its monster model of the theory. Sometimes we say that the theory is TP, and we call the theory an TP theory. We define $\mathrm{TP}_1$ theory, $\mathrm{TP}_2$ theory, $\mathrm{SOP}_1$ theory, and $\mathrm{SOP}_2$ theory in the same manner.
(vii) We say a theory is NTP if the theory is not TP, and we call the theory NTP theory. We define $\mathrm{NTP}_1$ theory, $\mathrm{NTP}_2$ theory, $\mathrm{NSOP}_1$ theory, and $\mathrm{NSOP}_2$ theory in the same manner.

The following facts are well known (*cf.* [Con12], [DS04], [KK11], and [She90]).

**Fact 2.3.** (i) *A theory has $TP_1$ if and only if it has $SOP_2$.*
(ii) *A theory has TP if and only if it has $TP_1$ or $TP_2$.*
(iii) *A theory has TP if and only if it has $SOP_1$ or $TP_2$.*
(iv) *If a theory has $SOP_2$, then it has $SOP_1$.*

**Definition 2.4.** [AK20, Definition 4.1] We say a formula $\varphi(x, y)$ has the *antichain tree property* (ATP) if there exists a tree indexed set of paremeters $(a_\eta)_{\eta\in 2^{<\omega}}$ such that

(i) for any antichain $X$ in $2^{<\omega}$, the set $\{\varphi(x, a_\eta) : \eta \in X\}$ is consistent,
(ii) for any $\eta, \nu \in 2^{<\omega}$, if $\eta \trianglelefteq \nu$, then $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent.

We say a theory has ATP if there exists a formula having ATP. If a theory does not have ATP, then we say the theory has NATP.

If a theory has ATP, then the theory always has $\mathrm{TP}_2$ and $\mathrm{SOP}_1$ ([AK20, Proposition 4.4, 4.6]). Thus the class of $\mathrm{NTP}_2$ theories and the class of $\mathrm{NSOP}_1$ theories are subclasses of the class of NATP theories. Therefore we have the following diagram:

$$\begin{array}{ccccccc} \mathrm{NTP}_2 & \longrightarrow & \mathrm{NATP} & & & & \\ \uparrow & & \uparrow & & & & \\ \text{simple} & \longrightarrow & \mathrm{NSOP}_1 & \longrightarrow & \mathrm{NSOP}_2 & \longrightarrow & \cdots \end{array}$$

Now we recall notions of forking, dividing, and pre-independence.

**Definition 2.5.** Let $\mathcal{L}$ be a language, $T$ an $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of $T$, $A \subset \mathbb{M}$ a small set, $\varphi(x,y)$ an $\mathcal{L}$-formula, $p(x)$ a type over $A$.

(i) A global extension $\tilde{p}$ of $p$ is said to be *invariant over* $A$ if $\varphi(x,c') \in \tilde{p}$ whenever $\varphi(x,c) \in \tilde{p}$ and $c \equiv_A c'$.
(ii) A global extension $\tilde{p}$ of $p$ is said to be *Lascar invariant over* $A$ if $\varphi(x,c') \in \tilde{p}$ whenever $\varphi(x,c) \in \tilde{p}$ and $c \equiv^L_A c'$.
(iii) A global extension $\tilde{p}$ of $p$ is called a *coheir of* $p$ *(over* $A$*)* if for all $\varphi(x,c) \in \tilde{p}$, there exists $m \in M$ such that $\models \varphi(m,c)$.
(iv) A sequence $(c_i)_{i\in\omega}$ is called an *invariant sequence over* $A$ *in* $p$ if there exists a global $A$-invariant extension $\tilde{p}$ of $p$ over $A$ such that $c_i \models \tilde{p}|_{Ac_{<i}}$ for all $i \in \omega$.
(v) A sequence $(c_i)_{i\in\omega}$ is called a *coheir sequence over* $A$ *in* $p$ if there exists a global coheir extension $\tilde{p}$ of $p$ over $A$ such that $c_i \models \tilde{p}|_{Ac_{<i}}$ for all $i \in \omega$.
(vi) We say $\varphi(x,c)$ *divides over* $A$ if there exists an $A$-indiscernible sequence $(c_i)_{i\in\omega}$ such that $c_0 = c$ and $\{\varphi(x,c_i)\}_{i\in\omega}$ is inconsistent.
(vii) We say $\varphi(x,c)$ *forks over* $A$ if there exist $\psi_0(x,c_0), ..., \psi_n(x,c_n)$ such that $\varphi(x,c) \vdash \bigvee \psi_i(x,c_i)$ and $\psi_i(x,c_i)$ divides over $A$ for each $i$.
(viii) We say $p(x)$ *divides over* $A$ if there exists $\varphi(x,c)$ such that $p(x) \vdash \varphi(x,c)$ and $\varphi(x,c)$ divides over $A$.
(ix) We say $p(x)$ *forks over* $A$ if there exists $\varphi(x,c)$ such that $p(x) \vdash \varphi(x,c)$ and $\varphi(x,c)$ forks over $A$.
(x) We say $\varphi(x,c)$ *Kim-divides over* $A$ if there exists an $A$-invariant sequence $(c_i)_{i\in\omega}$ in $\mathrm{tp}(c/A)$ such that $\{\varphi(x,c_i)\}_{i\in\omega}$ is inconsistent.
(xi) We say $\varphi(x,c)$ *coheir-divides over* $A$ if there exists a coheir sequence $(c_i)_{i\in\omega}$ over $A$ in $\mathrm{tp}(c/A)$ such that $\{\varphi(x,c_i)\}_{i\in\omega}$ is inconsistent.
(xii) By the same manner above, we define Kim-forking, coheir-forking for formulas, and Kim-dividing, Kim-forking, coheir-dividing, coheir-forking for types.

**Definition 2.6.** [CK12] [Adl08] A ternary relation $\mathop{\smile\hskip-0.9em ^| \ }$ is called a *pre-independence relation* if it is invariant under automorphisms. For a pre-independence relation $\mathop{\smile\hskip-0.9em ^| \ }$, we say

(i) it satisfies *monotonicity* if $aa' \mathop{\smile\hskip-0.9em ^| \ }_A bb'$ implies $a \mathop{\smile\hskip-0.9em ^| \ }_A b$,
(ii) it satisfies *base monotonicity* if $a \mathop{\smile\hskip-0.9em ^| \ }_A bc$ implies $a \mathop{\smile\hskip-0.9em ^| \ }_{Ab} c$,
(iii) it satisfies *transitivity on the left (over* $A$*)* if $a \mathop{\smile\hskip-0.9em ^| \ }_{Ab} c$ and $b \mathop{\smile\hskip-0.9em ^| \ }_A c$ implies $ab \mathop{\smile\hskip-0.9em ^| \ }_A c$,
(iv) it satisfies *right extension (over* $A$*)* if for all $a, b, c, A$ with $a \mathop{\smile\hskip-0.9em ^| \ }_A b$, there exists $c' \equiv_{Ab} c$ such that $a \mathop{\smile\hskip-0.9em ^| \ }_A bc'$,

(v) it satisfies *left extension (over A)* if for all $a, b, c, A$ with $a \mathop{\smile\!\!\!\!\mid}_A b$, there exists $c' \equiv_{Aa} c$ such that $ac' \mathop{\smile\!\!\!\!\mid}_A b$.
(vi) We say a set $A$ is an *extension base for* $\mathop{\smile\!\!\!\!\mid}$ if $a \mathop{\smile\!\!\!\!\mid}_A A$ for all $a$.
(vii) We say $\mathop{\smile\!\!\!\!\mid}$ *preserves indiscernibility over a set A* if $I$ is $aA$-indiscernible whenever $a \mathop{\smile\!\!\!\!\mid}_A I$ and $I$ is $A$-indiscernible.

**Definition 2.7.** [CK12] [Adl08] Let $\mathcal{L}$ be a language, $T$ an $\mathcal{L}$-theory, $\mathbb{M}$ a monster model of $T$, $A, B \subset \mathbb{M}$ small sets, and $a \in \mathbb{M}$. Suppose $A \subseteq B$.

(i) $a \mathop{\smile\!\!\!\!\mid}^d_A B$ if $\text{tp}(a/B)$ does not divide over $A$.
(ii) $a \mathop{\smile\!\!\!\!\mid}^f_A B$ if $\text{tp}(a/B)$ does not fork over $A$.
(iii) $a \mathop{\smile\!\!\!\!\mid}^K_A B$ if $\text{tp}(a/B)$ does not Kim-fork over $A$.
(iv) $a \mathop{\smile\!\!\!\!\mid}^i_A B$ if there exists a global Lascar invariant type over $A$ containing $\text{tp}(a/B)$.
(v) $a \mathop{\smile\!\!\!\!\mid}^u_A B$ if there exists a global coheir over $A$ containing $\text{tp}(a/B)$.

**Fact 2.8.** [CK12] [Adl08] *For all $a, b$ and a model $M$ $a \mathop{\smile\!\!\!\!\mid}^u_M b \Rightarrow a \mathop{\smile\!\!\!\!\mid}^i_M b \Rightarrow a \mathop{\smile\!\!\!\!\mid}^f_M b \Rightarrow a \mathop{\smile\!\!\!\!\mid}^d_M b$ and $a \mathop{\smile\!\!\!\!\mid}^f_M b \Rightarrow a \mathop{\smile\!\!\!\!\mid}^K_M b$. If $M$ is a model, then $\mathop{\smile\!\!\!\!\mid}^u$ satisfies monotonicity, base monotonicity, transitivity on the left over $M$, right extension over $M$, and left extension over $M$.*

**Definition 2.9.** Let $p(x) = \text{tp}(a/M)$.

(i) A global extension $\tilde{p}$ of $p$ is called a strong coheir of $p$ (over $M$) if $\tilde{p}$ is a coheir over $M$ and for any $A \supseteq M$, if $a' \models \tilde{p}|_A$ then $A \mathop{\smile\!\!\!\!\mid}^K_M a'$.
(ii) A sequence $(a_i)_{i\in\omega}$ is called a strong coheir sequence over $M$ in $p$ if there exists a global strong coheir extension $\tilde{p}$ of $p$ over $M$ such that $a_i \models \tilde{p}|_{Ma<i}$ for all $i \in \omega$.
(iii) We say a formula $\varphi(x, a)$ strong coheir-divides over $M$ if there exists a strong coheir sequence $(a_i)_{i\in\omega}$ over $M$ in $p$ such that $\{\varphi(x, a_i)\}_{i\in\omega}$ is inconsistent.
(iv) We say that $\text{tp}(a/Bb)$ is strictly invariant over B (denoted by $a \mathop{\smile\!\!\!\!\mid}^{ist}_B b$) if there is a global extension $p$, which is Lascar invariant over $B$ (so $a \mathop{\smile\!\!\!\!\mid}^i_B b$) and for any $C \supseteq Bb$, if $c \models p|_C$ then $C \mathop{\smile\!\!\!\!\mid}^f_B c$.
(v) We write $a \mathop{\smile\!\!\!\!\mid}^{iKst}_B b$ if there is a global extension $p$ of $\text{tp}(a/Bb)$, which is Lascar invariant over $B$ (so $a \mathop{\smile\!\!\!\!\mid}^i_B b$) and for any $C \supseteq Bb$, if $c \models p|_C$ then $C \mathop{\smile\!\!\!\!\mid}^K_B c$.
(vi) We write $a \mathop{\smile\!\!\!\!\mid}^{uKst}_B b$ if there is a global strong coheir of $\text{tp}(a/Bb)$.

## 3. Kim-dividing and coheir-dividing on NATP theories

In [CK12], Chernikov and Kaplan proved that forking and dividing are equivalent over models, in $\text{NTP}_2$ theories. In other words, they proved that in $\text{NTP}_2$ theories, for a model $M$, if a formula forks over $M$, then it divides over $M$.

The strategy they have taken can be divides into two main part. First they show that in $\text{NTP}_2$ and over a set $B$, if a pre-independence relation $\mathop{\smile\!\!\!\!\mid}$ satisfies every property in Definition 2.6, then every formula dividing over $B$ coheir-divides over $B$. In particular, in $\text{NTP}_2$ theories, for any model $M$, if a formula divides over $M$, then it coheir-divides over $M$. As a corollary they show that in $\text{NTP}_2$ theories, $\mathop{\smile\!\!\!\!\mid}^{ist}$ is an extension base over models.

As the second step, they prove Kim's lemma for $NTP_2$ theories over models. That is, if a formula $\varphi(x,a)$ divides over a model $M$, then for any sequence $(a_i)_{i\in\omega}$ such that $a_0 = a$ and $a_i \mathop{\smile\!\!\!\!\!\!\mid}^{ist}_M a_{<i}$ for all $i \in \omega$, the set $\{\varphi(x,a_i)\}_{i\in\omega}$ is inconsistent.

Using these observation, they prove that forking and dividing are equivalent over models in $NTP_2$ theories. If a formula $\varphi(x,a)$ forks over a model $M$, then there exist $\psi_0(x,a_0)$, ... , $\psi_n(x,a_n)$ such that $\psi_i$ divides over $M$ for each $i \le n$ and $\varphi(x,a) \vdash \bigvee \psi_i(x,a_i)$. Since they prove that $\mathop{\smile\!\!\!\!\!\!\mid}^{ist}$ is an extension base over models, there exists a global extension $p(x,x_0,...,x_n)$ of $\mathrm{tp}(a,a_0,...,a_n/M)$ satisfying (iv) in Definition 2.9. If we choose any sequence $(a^j,a^j_0,...,a^j_n)_{j\in\omega}$ such that $(a^j,a^j_0,...,a^j_n) \models p|_M a^{<j}, a^{<j}_0, ..., a^{<j}_n$ for each $j \in \omega$, then by Kim's lemma, $\{\varphi(x,a^j)\}_{j\in\omega}$ is inconsistent. Thus $\varphi(x,a)$ divides over $M$.

Following their strategy, we are trying to show that Kim-forking and Kim-dividing are equivalent over models in NATP theories, and obtained a small result which corresponds to their first step. Note that we can not expect to prove that forking and dividing are equivalent over models since they are not equivalent in $NSOP_1$ theories.

To explain our result, we need a notion of $\mathcal{T}^\delta_\alpha$, which describes trees that grow downward. The notion is originally from [KR20].

**Definition 3.1.** Suppose $\alpha$ and $\delta$ are ordinals. We define $\mathcal{T}^\delta_\alpha$ to be the set of functions $\eta$ so that

(i) $\mathrm{dom}(\eta)$ is an end-segment of $\alpha$ of the form $[\beta,\alpha)$ for $\beta$ equal to 0 or a successor ordinal. If $\alpha$ is a successor or 0, we allow $\beta = \alpha$, i.e. $\mathrm{dom}(\eta)=\emptyset$.
(ii) $\mathrm{ran}(\eta) \subseteq \delta$
(iii) finite support: the set $\{\gamma \in \mathrm{dom}(\eta) : \eta(\gamma) \ne 0\}$ is finite.

We interpret $\mathcal{T}^\delta_\alpha$ as an $\mathcal{L}_{s,\alpha}$-structure by defining

(iv) $\eta \trianglelefteq \nu$ if and only if $\eta \subseteq \nu$. Write $\eta \perp \nu$ if $\neg(\eta \trianglelefteq \nu)$ and $\neg(\nu \trianglelefteq \eta)$.
(v) $\eta \wedge \nu = \eta|_{[\beta,\alpha)} = \nu|_{[\beta,\alpha)}$ where $\beta = \min\{\gamma : \eta|_{[\gamma,\alpha)} = \nu|_{[\gamma,\alpha)}\}$, if non-empty (note that $\beta$ will not be a limit, by finite support). Define $\eta \wedge \nu$ to be the empty function if this set is empty (note that this cannot occur if $\alpha$ is a limit).
(vi) $\eta <_{lex} \nu$ if and only if $\eta \triangleleft \nu$ or, $\eta \perp \nu$ with $\mathrm{dom}(\eta \wedge \nu)=[\gamma+1,\alpha)$ and $\eta(\gamma) < \nu(\gamma)$.
(vii) for all $\beta \in \alpha \setminus \lim(\alpha)$, $P_\beta = \{\eta \in \mathcal{T}^\delta_\alpha : \mathrm{dom}(\eta) = [\beta,\alpha)\}$ (the $\beta$-th floor).

We will also need the following notation.

(viii) (canonical inclusion) For $\alpha' > \alpha$, $\mathcal{T}^2_\alpha$ can be embedded in $\mathcal{T}^2_{\alpha'}$ with respect to $\mathcal{L}_{s,\alpha'}$ by a map $f_{\alpha,\alpha'} : \mathcal{T}^2_\alpha \to \mathcal{T}^2_{\alpha'} : \eta \mapsto \eta \cup \{(\beta,0) : \beta \in \alpha' \setminus \alpha\}$. Unless otherwise stated, we regard $\mathcal{T}^2_\alpha$ as $f_{\alpha,\alpha'}(\mathcal{T}^2_\alpha)$ in $\mathcal{T}^2_{\alpha'}$
(ix) $\eta \perp_{lex} \nu$ if and only if $\eta <_{lex} \nu$ and $\eta \not\trianglelefteq \nu$. For an indexed set $\{a_\eta\}_{\eta\in\mathcal{T}^2_\alpha}$ and $\eta \in \mathcal{T}^2_\alpha$, by $a_{\perp_{lex}\eta}$ we mean the set $\{a_\nu : \nu \perp_{lex} \eta\}$.
(x) for each $\eta \in \mathcal{T}^\delta_\alpha$, let $h(\eta)$ be an ordinal such that $\mathrm{dom}(\eta) = [h(\eta),\alpha)$.
(xi) for each $\eta \in \mathcal{T}^\delta_\alpha$, let $C_\eta$ (the cone on $\eta$) be the set of all $\nu \in \mathcal{T}^\delta_\alpha$ such that $\eta \trianglelefteq \nu$.

**Remark 3.2.** Let $\alpha, \beta$ be ordinals and suppose $\beta < \alpha$. Let us consider $\mathcal{T}^2_\alpha$.

(i) By finite support, $<_{lex}$ is a well-ordering on $P_\beta$.
(ii) If $\alpha$ is a cardinal, then $|P_\beta| = \alpha$.
(iii) If $\alpha$ is a cardinal, then $|C_\eta| < \alpha$ for all $\eta \in \mathcal{T}^2_\alpha$.

**Lemma 3.3.** *Let $\theta$ and $\kappa$ be cardinals such that $\kappa$ is uncountable, $\theta^+ < \kappa$ and $\mathrm{cf}(\kappa) = \kappa$, where $\theta^+$ is the successor cardinal of $\theta$. Let $c$ be a $\theta$-coloring on $\mathcal{T}^2_\kappa$ (a function from $\mathcal{T}^2_\kappa$ to $\theta$) which satisfies:*

(*) *for each $\beta < \alpha$ and $\eta, \nu \in P_\beta$, if $\eta <_{lex} \nu$ then $c(\eta) \leq c(\nu)$.*

*Then for any infinite cardinal $\lambda < \kappa$, there exists an $\mathcal{L}$-embedding $f : \mathcal{T}^2_\lambda \to \mathcal{T}^2_\kappa$ and $\iota < \theta$ such that $c(\eta) = \iota$ for all $\eta \in f(\mathcal{T}^2_\lambda)$.*

*Proof.* By Remark 3.2 (ii), $|P_\beta| = \kappa$ for each $\beta < \kappa$. Thus for each $\beta < \kappa$, there exists $\iota_\beta < \theta$ such that $|\{\eta \in P_\beta : c(\eta) = \iota_\beta\}| = \kappa$. By Remark 3.2 (i), $\{\eta \in P_\beta : c(\eta) = \iota_\beta\}$ has the least element with respect to $<_{lex}$, say $\eta_\beta$. We show that $c(\eta) = \iota_\beta$ for all $\eta \in P_\beta$ with $\eta_\beta < \eta$. Suppose not. Then there exists $\eta \in P_\beta$ such that $\eta_\beta < \eta$ and $\iota_\beta < c(\eta)$. Let $\nu = \eta_\beta \wedge \eta$. Then $\{\eta \in P_\beta : c(\eta) = \iota_\beta\} \subseteq C_\nu$ by (*). But by Remark 3.2 (iii), $|C_\nu| < \kappa$. It is a contradiction.

Let $\lambda$ be an infinite cardinal such that $\theta^+ \leq \lambda < \kappa$. For each $\beta < \lambda$, let $\gamma_\beta$ be the largest ordinal such that $\eta_\beta(\gamma_\beta) = 1$. We can find such an ordinal by finite support. Since $\mathrm{cf}(\kappa) = \kappa$, we can choose an ordinal $\gamma < \kappa$ such that $\lambda < \gamma$, and $\gamma_\beta < \gamma$ for all $\beta < \lambda$. Let $\xi$ be an element of $\mathcal{T}^2_\kappa$ such that $\mathrm{dom}(\xi) = [\gamma, \kappa)$, $\xi(\gamma) = 1$ and $\xi(\gamma') = 0$ for all $\gamma' > \gamma$. Then each floor in $C_\xi$ is monochromatic with respect to $c$. Since the cone has at least $\lambda$-many floors and we have only $\theta$-many colors, we can find $\lambda$-many floors in the cone which have the same color. This completes the proof. □

*Proof of the main theorem.* Suppose $\varphi(x, a)$ Kim-divieds over $M$. Then there exists a global $M$-invariant type $p(y)$ containing $\mathrm{tp}(a/M)$ such that for all $(a_i)_{i\in\omega} \models p^{\otimes\omega}|_M$, the set $\{\varphi(x, a_i)\}_{i\in\omega}$ is $k$-inconsistent for some $k \in \omega$. To get a contradiction, we assume there is no global coheir extension $q(y)$ of $\mathrm{tp}(a/M)$ such that for all $(a_i)_{i\in\omega} \models q^{\otimes\omega}|_M$, the set $\{\varphi(x, a_i)\}_{i\in\omega}$ is inconsistent. Note that the number of all global coheir extensions of $\mathrm{tp}(a/M)$ is bounded. Let $\{q_i\}_{i<\theta}$ be an enumeration of all global coheir extension of $\mathrm{tp}(a/M)$, where $\theta$ is a cardinal number. Let $\kappa$ be a cardinal such that $\theta^+ < \kappa$ and $\mathrm{cf}(\kappa) = \kappa$.

*Claim.* For any ordinal $\alpha$ and any set $A \supseteq M$, there exists $\{b_\eta\}_{\eta\in\mathcal{T}^2_\alpha}$ which satisfies:

(i) $b_\eta \models p|_{Mb_{\rhd\eta}}$ for each $\eta \in \mathcal{T}^2_\alpha$, and
(ii) For each $\eta \in \mathcal{T}^2_\alpha$, there exists $i < \theta$ such that $b_\eta \models q_i|_{Ab_{\perp_{lex}\eta}}$ and $b_\eta \not\models q_{i'}|_{Ab_{\perp_{lex}\eta}}$ for all $i' < i$.
(iii) $A' \mathop{\smile\!\!\!\!\!\!\;\mid}^u_M A$ where $A' = \{b_\eta\}_{\eta\in\mathcal{T}_\alpha}$.

Furthermore, for any set $A \supseteq M$, we can construct a sequence $\{\{b_\eta\}_{\eta\in\mathcal{T}^2_\alpha}\}_{\alpha<\kappa}$ whose elements satisfy (i) and (ii) above, and $\{b_\eta\}_{\eta\in\mathcal{T}^2_\alpha} \subseteq \{b_\eta\}_{\eta\in\mathcal{T}^2_{\alpha'}}$ for $\alpha < \alpha' < \kappa$ by canonical inclusion.

*Proof of Claim.* First we construct $\{b_\eta\}_{\eta\in\mathcal{T}^2_0}$. Choose any $A \supseteq M$. Then there exist and $b \models p|_M$ and $b' \models q_0|_A$ . Since $b' \mathop{\smile\!\!\!\!\!\!\;\mid}^u_M A$, by left extension, there exists $b^* \equiv_{Mb'} b$ such that $b^* b' \mathop{\smile\!\!\!\!\!\!\;\mid}^u_M A$. In particular, $b^* \models p|_M$ and $b^* \mathop{\smile\!\!\!\!\!\!\;\mid}^u_M A$. Thus there exists $i < \theta$ such that $b^* \models q_i|_A$. We may assume $i$ is the smallest one of those indices. Let $b_\emptyset := b^*$. Then $\{b_\eta\}_{\eta\in\mathcal{T}^2_\alpha}$ satisfies (i), (ii), and (iii).

Now we consider the successor case. Assume that the statement is true for some ordinal $\beta$. Let $\alpha := \beta + 1$. Choose any $A \supseteq M$ and suppose we have constructed $\{\{b_\eta\}_{\eta\in\mathcal{T}^2_{\beta'}}\}_{\beta'<\beta}$ whose members satisfies (i), (ii), (iii), and $\{b_\eta\}_{\eta\in\mathcal{T}^2_{\beta'}} \subseteq \{b_\eta\}_{\eta\in\mathcal{T}^2_{\beta''}}$ for each $\beta' < \beta'' \leq \beta$ by canonical inclusion. For each $\eta \in \mathcal{T}^2_\beta$, let $b^0_\eta := b_\eta$. By the induction hypothesis, we have $\{b^1_\eta\}_{\eta\in\mathcal{T}^2_\beta}$ satisfing (i), (ii), and (iii) over $A\cup A^0$ where $A^0=\{b^0_\eta\}_{\eta\in\mathcal{T}^2_\beta}$. Let $A^1=\{b^1_\eta\}_{\eta\in\mathcal{T}^2_\beta}$.

For each $\eta\in\mathcal{T}^2_\alpha\setminus\{\emptyset\}$, let $b_\eta:=b^{\eta(\beta)}_{\eta|_\beta}$. Choose any $b\models p|_{AA^0A^1}$. Note that $A^0A^1\mathop{\smile\!\!\!\!\!\!\mid}^u_M A$ by left transitivity. Hence we can apply left extension to get $b^*\equiv_{A^0A^1}b$ such that $b^*A^0A^1\mathop{\smile\!\!\!\!\!\!\mid}^u_M A$. In partinular, $b^*\mathop{\smile\!\!\!\!\!\!\mid}^u_M A$. Thus there exists $i<\theta$ such that $b^*\models q_i|_A$. We may assume $i$ is the smallest one of those indices. Let $b_\emptyset:=b^*$. Then $\{b_\eta\}_{\eta\in\mathcal{T}^2_\alpha}$ satisfies (i), (ii), and (iii). Clearly $\{b_\eta\}_{\eta\in\mathcal{T}^2_\beta}\subseteq\{b_\eta\}_{\eta\in\mathcal{T}^2_\alpha}$ by canonical inclusion.

Finally, let $\alpha$ be a limit ordinal and assume that the statement is true for all $\beta<\alpha$. Suppose we have constructed a sequence $\{\{b_\eta\}_{\eta\in\mathcal{T}^2_\beta}\}_{\beta<\alpha}$ such that $\{b_\eta\}_{\eta\in\mathcal{T}^2_\beta}\subseteq\{b_\eta\}_{\eta\in\mathcal{T}^2_{\beta'}}$ for all $\beta<\beta'<\alpha$, and $\{b_\eta\}_{\eta\in\mathcal{T}^2_\beta}$ satisfies (i),(ii), and (iii) for each $\beta<\alpha$. Then the union $\{b_\eta\}_{\eta\in\mathcal{T}^2_\alpha}:=\bigcup_{\beta<\alpha}\{b_\eta\}_{\eta\in\mathcal{T}^2_\beta}$ satisfies (i), (ii), and (iii). This completes the proof of the claim. □

Now we have $\{b_\eta\}_{\eta\in\mathcal{T}^2_\kappa}$ satisfies (i) and (ii) over $M$, by the claim above. Let $c$ be a $\theta$-coloring on $\mathcal{T}^2_\kappa$ such that $c(\eta)=i$ if and only if $b_\eta\models q_i|_{Mb_{\perp_{lex}\eta}}$ and $b_\eta\not\models q_{i'}|_{Mb_{\perp_{lex}\eta}}$ for all $i'<i$, as in (ii). By the construction, if $\eta$ and $\nu$ are in the same floor and $\eta<_{lex}\nu$, then $\{b_\xi\}_{\xi\in C_\eta}\equiv_{Mb_{\perp_{lex}\eta}}\{b_\xi\}_{\xi\in C_\nu}$, and hence $c(\eta)\le c(\nu)$ by the minimality of the index $i$ in (ii). Thus this coloring satisfies the hypothesis in Lemma 3.3, and we can find an $\mathcal{L}$-embedding $f:\mathcal{T}^2_\omega\to\mathcal{T}^2_\kappa$ such that $c(f(\mathcal{T}^2_\omega))=\iota$ for some $\iota<\theta$. For each $\eta\in\mathcal{T}^2_\omega$, let $c_\eta:=b_{f(\eta)}$. Then $\{c_\eta\}_{\eta\in\mathcal{T}^2_\omega}$ satisfies the following conditions:

(iv) $c_\eta\models p|_{Mc_{\triangleright\eta}}$ for each $\eta\in\mathcal{T}^2_\omega$.
(v) $c_\eta\models q_\iota|_{Mc_{\perp_{lex}\eta}}$ for each $\eta\in\mathcal{T}^2_\omega$.

In particula, every finite antichain $X$ in $\mathcal{T}^2_\omega$ is an initial segment of some coheir sequence in $q_\iota$ over $M$. Thus the set $\{\varphi(x,c_\eta)\}_{\eta\in X}$ is consistent. On the other hand, every finite path $X$ in $\mathcal{T}^2_\omega$ with $|X|\ge k$ is an initial segment of some Morley sequence in $p$ over $M$. Thus the set $\{\varphi(x,c_\eta)\}_{\eta\in X}$ is inconsistent. Therefore by compactness, we can construct an antichain tree, which yields a contradiction with the assumption that $T$ is NATP. □

The following two corollaries can be shown by using the same argument in [CK12, Lemma 3.1, Proposition 3.7].

**Corollary 3.4.** *Suppose that $T$ is NATP. If a formula Kim-forks over a model $M$, then it quasi divides over $M$.*

**Corollary 3.5.** *In NATP theories, every model is an extension base for $\mathop{\smile\!\!\!\!\!\!\mid}^{uKst}$.*

Thus if we can prove Kim's lemma for Kim-dividing in NATP, we can prove that Kim-forking and Kim-dividing are equivalent in NATP, over models.

**Conjecture 3.6.** In NATP thoeries, if a formula $\varphi(x,a)$ Kim-divides over a model $M$, then $\{\varphi(x,a_i)\}_{i\in\omega}$ is inconsistent for any strong coheir sequence $(a_i)_{i\in\omega}$ over $M$.

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