SHELAH-STRONG TYPE AND ALGEBRAIC CLOSURE OVER A HYPERIMAGINARY

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Abstract. We characterize Shelah-strong type over a hyperimagianary with the algebraic closure of a hyperimaginary. Also, we present and take a careful look at an example that witnesses $\operatorname{acl}^{\mathrm{eq}}(\boldsymbol{e})$ is not interdefinable with $\operatorname{acl}(\boldsymbol{e})$ where $\boldsymbol{e}$ is a hyperimaginary.

Fix a first order language $\mathcal{L}$, complete theory T and monster model $\mathcal{M}$. **Throughout, fix a hyperimaginary $\boldsymbol{e} = a_E$ where a is a (possibly infinite) real tuple and E is an $\emptyset$-type-definable equivalence relation on $\mathcal{M}^{|a|}$.**

Most of the facts and remarks whose proofs are omitted can be found in the author's dissertation [6].

Fact 1.

(1) *A real tuple b is simply $b/(\bigwedge_{i<\alpha} x_i = y_i)$ where $b = (b_i)_{i<\alpha}$, hence can be seen as (that is, interdefinable with) a hyperimaginary; an imaginary tuple $(b_i/F_i)_{i<\alpha}$ is $(b_i)_{i<\alpha}/(\bigwedge_{i<\alpha} F_i(x_i, y_i))$ where all x_i, y_i's are disjoint, hence is a hyperimaginary as well.* ***In this regard, considering over a set of real elements or a set of imaginaries can be safely replaced by considering over a single hyperimaginary.***

(2) *In the same manner as above, a sequence of hyperimaginaries can be regarded as a single hyperimaginary: A tuple of hyperimaginaries $(b_i/F_i)_{i<\alpha}$ is interdefinable with $(b_i)_{i<\alpha}/(\bigwedge_{i<\alpha} F_i(x_i, y_i))$ where all x_i, y_i's are disjoint.*

Definition 2.

(1) For any hyperimaginary $\boldsymbol{e}'$, we denote $\boldsymbol{e}' \in \operatorname{dcl}(\boldsymbol{e})$ and say $\boldsymbol{e}'$ is *definable over* $\boldsymbol{e}$ if $f(\boldsymbol{e}') = \boldsymbol{e}'$ for all $f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M})$.

(2) For any hyperimaginary $\boldsymbol{e}'$, we denote $\boldsymbol{e}' \in \operatorname{bdd}(\boldsymbol{e})$ and say $\boldsymbol{e}'$ is *bounded over* $\boldsymbol{e}$ if $\{f(\boldsymbol{e}') : f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M})\}$ is bounded.

Remark 3. In Definition 2, $\boldsymbol{e}' \in \operatorname{dcl}(\boldsymbol{e})$ and $\boldsymbol{e}' \in \operatorname{bdd}(\boldsymbol{e})$ are independent of the choice of a monster model $\mathcal{M}$.

Proof. It is easy, but anyway we prove it. Let $\mathcal{M} \prec \mathcal{M}'$ be monster models of T. Suppose that there are only κ-many automorphic images of $\boldsymbol{e}'$ in $\mathcal{M}$, whereas there are at least κ^+ images in $\mathcal{M}'$. Say $\boldsymbol{e}' = b_F$ where b is a real tuple and F is an $\emptyset$-type-definable equivalence relation. Let $(b_i/F)_{i<\kappa^+}$ be an enumeration of automorphic images of b_F in $\mathcal{M}'$. Since there is $(b'_i)_{i<\kappa^+} \equiv_a (b_i)_{i<\kappa^+}$ where each $b'_i \in \mathcal{M}$, there are at least κ^+-many conjugates of b_F in $\mathcal{M}$ (recall $\boldsymbol{e} = a/E$), a contradiction. □

Fact 4.

(1) *A hyperimaginary b_F is called* countable *if $|b|$ is countable. It's not so difficult to prove that any hyperimaginary is interdefinable with a sequence of countable hyperimaginaries(see, for example* [5, Lemma 4.1.3]*).*

(2) *From now on,* definable closure of $\boldsymbol{e}$, $\operatorname{dcl}(\boldsymbol{e})$ *will be seen as an actual (small) set, the set of all countable hyperimaginaries which are definable over* $\boldsymbol{e}$*: In this way,* $\boldsymbol{e}' \in \operatorname{dcl}(\boldsymbol{e})$ *now means that there is a sequence of countable hyperimaginaries that is interdefinable with* $\boldsymbol{e}'$ *and fixed by any* $f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M})$*. Also note that* $f \in \operatorname{Aut}_{\operatorname{dcl}(\boldsymbol{e})}(\mathcal{M})$ *if and only if* f *fixes all hyperimaginaries that are definable over* $\boldsymbol{e}$*. As pointed out in Fact 1(2),* $\operatorname{dcl}(\boldsymbol{e})$ *also can be seen as a single hyperimaginary.*

(3) *Likewise, the* bounded closure of $\boldsymbol{e}$, $\operatorname{bdd}(\boldsymbol{e})$ *is the set of all countable hyperimaginaries which are bounded over* $\boldsymbol{e}$*. In the same way as above,* $\boldsymbol{e}' \in \operatorname{bdd}(\boldsymbol{e})$ *means that there is a sequence of countable hyperimaginaries that is interdefinable with* $\boldsymbol{e}'$*, and the number of* $\boldsymbol{e}$*-automorphic images of it is bounded. Again,* $f \in \operatorname{Aut}_{\operatorname{bdd}(\boldsymbol{e})}(\mathcal{M})$ *is equivalent to saying that* f *fixes all hyperimaginaries that are bounded over* $\boldsymbol{e}$*.*

Remark & Definition 5.

(1) For a hyperimaginary $\boldsymbol{e}'$, denote $\boldsymbol{e}' \in \operatorname{acl}(\boldsymbol{e})$ and say $\boldsymbol{e}'$ is *algebraic over* $\boldsymbol{e}$ if $\{f(\boldsymbol{e}') : f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M})\}$ is finite. As in Remark 3, this definition is independent of the choice of a monster model.

(2) As in Fact 4, the *algebraic closure* of $\boldsymbol{e}$, $\operatorname{acl}(\boldsymbol{e})$ can be regarded as a bounded set of countable hyperimaginaries, which is interdefinable with a single hyperimaginary $b_F \in \operatorname{bdd}(\boldsymbol{e})$ (but possibly $b_F \notin \operatorname{acl}(\boldsymbol{e})$).

(3) Note that given $d_i/L_i \in \operatorname{acl}(\boldsymbol{e})$ $(i \leq n)$, as pointed out in Fact 1, $(d_0/L_0, \cdots, d_n/L_n)$ is interdefinable with a single $d_L \in \operatorname{acl}(\boldsymbol{e})$. Hence by compactness, for any hyperimaginaries b_F and c_F,

$$b_F \equiv_{\operatorname{acl}(\boldsymbol{e})} c_F \text{ if and only if } b_F \equiv_{d_L} c_F \text{ for any } d_L \in \operatorname{acl}(\boldsymbol{e}).$$

Definition 6.

(1) $\operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M}) = \{f \in \operatorname{Aut}(\mathcal{M}) : f(\boldsymbol{e}) = \boldsymbol{e}\}$ (f may permute the elements of $\boldsymbol{e}$).

(2) $\operatorname{Autf}_{\boldsymbol{e}}(\mathcal{M})$ is a subgroup of $\operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M})$ generated by

$$\{f \in \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M}) : f \in \operatorname{Aut}_M(\mathcal{M}) \text{ for some } M \models T \text{ such that } \boldsymbol{e} \in \operatorname{dcl}(M)\}.$$

It can be easily seen that $\operatorname{Autf}_{\boldsymbol{e}}(\mathcal{M})$ is a normal subgroup of $\operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M})$.

(3) The *Lascar group over of* T $\boldsymbol{e}$ is the quotient group

$$\operatorname{Gal_L}(T, \boldsymbol{e}) = \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M}) / \operatorname{Autf}_{\boldsymbol{e}}(\mathcal{M}).$$

Remark 7.

(1) Up to isomorphism, $\operatorname{Gal_L}(T, \boldsymbol{e})$ is independent of the choice of a monster model $\mathcal{M}$.

(2) There are well-defined maps μ and ν such that:

$$\begin{gathered} \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M}) \overset{\mu}{\to} S_M(M) \overset{\nu}{\to} \operatorname{Gal_L}(T, \boldsymbol{e}) \\ f \mapsto \operatorname{tp}(f(M)/M) \mapsto \overline{f} = \pi(f) \end{gathered}$$

where M is a small model of T such that $\boldsymbol{e} \in \operatorname{dcl}(M)$, and $\pi : \operatorname{Aut}_{\boldsymbol{e}}(\mathcal{M}) \to \operatorname{Gal_L}(T, \boldsymbol{e})$ is the canonical projection.

The topology of $\operatorname{Gal_L}(T, \boldsymbol{e})$ is given by the topology induced by the quotient map ν, and it is independent of the choice of M.

Fact 8.

(1) $\operatorname{Gal_L}(T, \boldsymbol{e})$ *is a topological group.*

(2) *Let* $H \leq \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$ *and let* $H' = \pi(H) \leq \mathrm{Gal_L}(T, \boldsymbol{e})$. *Then* H' *is closed in* $\mathrm{Gal_L}(T, \boldsymbol{e})$ *and* $H = \pi^{-1}(H')$, *if and only if* $H = \mathrm{Aut}_{\boldsymbol{e}'\boldsymbol{e}}(\mathcal{M})$ *for some hyperimaginary* $\boldsymbol{e}' \in \mathrm{bdd}(\boldsymbol{e})$.

(3) *Let* $H' \leq \mathrm{Gal_L}(T, \boldsymbol{e})$ *be closed and* F *be an* $\emptyset$*-type-definable equivalence relation. Then for* $H = \pi^{-1}(H')$, $x_F \equiv^H_{\boldsymbol{e}} y_F$ *is equivalent to* $x_F \equiv_{\boldsymbol{e}'\boldsymbol{e}} y_F$ *for some hyperimaginary* $\boldsymbol{e}' \in \mathrm{bdd}(\boldsymbol{e})$, *and hence* $x_F \equiv^H_{\boldsymbol{e}} y_F$ *is an* $\boldsymbol{e}'\boldsymbol{e}$*-invariant type-definable bounded equivalence relation. Especially, if* $H' \trianglelefteq \mathrm{Gal_L}(T, \boldsymbol{e})$, *then* $x_F \equiv^H_{\boldsymbol{e}} y_F$ *is* $\boldsymbol{e}$*-invariant.*

Definition 9.

(1) $\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e})$ denotes the connected component of the identity in $\mathrm{Gal_L}(T, \boldsymbol{e})$.

(2) $\mathrm{Autf_s}\,(\mathcal{M}, \boldsymbol{e}) := \pi^{-1}(\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e}))$.

(3) Two hyperimaginaries b_F and c_F are said to have the same *Shelah-strong type* if there is $f \in \mathrm{Autf_s}\,(\mathcal{M}, \boldsymbol{e})$ such that $f(b_F) = c_F$, denoted by $b_F \equiv^{\mathrm{s}}_{\boldsymbol{e}} c_F$.

Remark 10. Note that $\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e})$ is a normal closed subgroup of $\mathrm{Gal_L}(T, \boldsymbol{e})$ ([4]) and $\equiv^{\mathrm{s}}_{\boldsymbol{e}}$ is the orbit equivalence relation $\equiv^{\mathrm{Autf_s}\,(\mathcal{M},\boldsymbol{e})}_{\boldsymbol{e}}$, thus $\equiv^{\mathrm{s}}_{\boldsymbol{e}}$ is type-definable over $\boldsymbol{e}$ by Fact 8(3). We denote

$$\mathrm{Gal_s}(T, \boldsymbol{e}) := \mathrm{Gal_L}(T, \boldsymbol{e}) / \mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e}) \cong \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M}) / \mathrm{Autf_s}\,(\mathcal{M}, \boldsymbol{e}).$$

Thus $\mathrm{Gal_s}(T, \boldsymbol{e})$ is a profinite (i.e. compact and totally disconnected) topological group. $\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e})$ is the intersection of all closed (normal) subgroups of finite indices in $\mathrm{Gal_L}(T, \boldsymbol{e})$, since such an intersection is the identity for a profinite group ([4]).

Proposition 11.

(1) $\mathrm{Autf_s}\,(\mathcal{M}, \boldsymbol{e}) = \mathrm{Aut}_{\mathrm{acl}(\boldsymbol{e})}(\mathcal{M})$.

(2) *Let* b_F, c_F *be hyperimaginaries. The following are equivalent.*
   (a) $b_F \equiv^{\mathrm{s}}_{\boldsymbol{e}} c_F$.
   (b) $b_F \equiv_{\mathrm{acl}(\boldsymbol{e})} c_F$.

Proof. (1). We claim first that

$$\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e}) = \bigcap \{\pi(\mathrm{Aut}_{d_L\boldsymbol{e}}(\mathcal{M})) : d_L \in \mathrm{acl}(\boldsymbol{e})\}.$$

Let $d_L \in \mathrm{acl}(\boldsymbol{e})$ where d_L is a hyperimaginary. Say $d^0_L (= d_L), \cdots, d^n_L$ are all the conjugates of d_L over $\boldsymbol{e}$. Then any $f \in \mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$ permutes the set $\{d^0_L, \cdots, d^n_L\}$. Hence it follows that $\mathrm{Aut}_{d_L\boldsymbol{e}}(\mathcal{M})$ has a finite index in $\mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$. Thus (due to Fact 8(2)) $\pi(\mathrm{Aut}_{d_L\boldsymbol{e}}(\mathcal{M}))$ is a closed subgroup of finite index in $\mathrm{Gal_L}(T, \boldsymbol{e})$. Then as in Remark 10, we have $\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e}) \leq \pi(\mathrm{Aut}_{d_L\boldsymbol{e}}(\mathcal{M}))$.

Conversely, given a normal closed subgroup $H' \leq \mathrm{Gal_L}(T, \boldsymbol{e})$ of finite index and $H := \pi^{-1}(H')$, Fact 8(2) says $H' = \pi(\mathrm{Aut}_{b_F\boldsymbol{e}}(\mathcal{M}))$ for some $b_F \in \mathrm{bdd}(\boldsymbol{e})$. But since H' is of finite index, the same holds for $H = \mathrm{Aut}_{b_F\boldsymbol{e}}(\mathcal{M})$ in $\mathrm{Aut}_{\boldsymbol{e}}(\mathcal{M})$, and we must have $b_F \in \mathrm{acl}(\boldsymbol{e})$. Thus the claim follows from Remark 10.

Therefore

$$\begin{aligned}\mathrm{Autf_s}\,(\mathcal{M}, \boldsymbol{e}) &= \pi^{-1}(\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e})) = \pi^{-1}(\bigcap \{\pi(\mathrm{Aut}_{d_L\boldsymbol{e}}(\mathcal{M})) : d_L \in \mathrm{acl}(\boldsymbol{e})\}) \\ &= \bigcap \{\mathrm{Aut}_{d_L\boldsymbol{e}}(\mathcal{M}) : d_L \in \mathrm{acl}(\boldsymbol{e})\} = \mathrm{Aut}_{\mathrm{acl}(\boldsymbol{e})}(\mathcal{M}),\end{aligned}$$

where the last equality follows by Remark & Definition 5(3).

(2) follows from (1). □

Recall that $\mathrm{acl}^{\mathrm{eq}}(\boldsymbol{e}) := \{\boldsymbol{e}\} \cup (\mathrm{acl}(\boldsymbol{e}) \cap \mathcal{M}^{\mathrm{eq}})$ is the *eq-algebraic closure* of $\boldsymbol{e}$, where as usual $\mathcal{M}^{\mathrm{eq}}$ is the set of all imaginary elements (equivalence classes of $\emptyset$-definable equivalence relations) of $\mathcal{M}$. Good summary of basic facts concerning imaginary elements can be found in [1, Chapter 1]. The following remark is proved using the proof of [9, Theorem 21].

Remark 12. For any small set A of imaginaries, $\mathrm{acl}^{\mathrm{eq}}(A)(= \mathrm{acl}(A) \cap \mathcal{M}^{\mathrm{eq}})$ is interdefinable with $\mathrm{acl}(A)$.

Proof. Recall that $\mathrm{Gal}^0_{\mathrm{L}}(T, A)$ is the intersection of all closed (normal) subgroups of finite indices in $\mathrm{Gal}_{\mathrm{L}}(T, A)$ (Remark 10). Let H' be a closed subgroup of finite index in $\mathrm{Gal}_{\mathrm{L}}(T, A)$. It suffices to show that $H' = \pi(\mathrm{Aut}_{\boldsymbol{b}A}(\mathcal{M}))$ for some $\boldsymbol{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$; by Fact 8(2), we have

$$\begin{aligned}\mathrm{Gal}^0_{\mathrm{L}}(T, \boldsymbol{e}) &= \bigcap\{H' : H' \text{ is a closed subgroup of finite index in } \mathrm{Gal}_{\mathrm{L}}(T, A)\}\\ &\subseteq \bigcap\{\pi(\mathrm{Aut}_{d_L A}(\mathcal{M})) : d_L \in \mathrm{acl}^{\mathrm{eq}}(A)\};\end{aligned}$$

thus if we show that $H' = \pi(\mathrm{Aut}_{\boldsymbol{b}A}(\mathcal{M}))$ for some $\boldsymbol{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$, then $\mathrm{Gal}^0_{\mathrm{L}}(T, A) = \bigcap\{\pi(\mathrm{Aut}_{d_L A}(\mathcal{M})) : d_L \in \mathrm{acl}^{\mathrm{eq}}(A)\}$. Taking π^{-1}, we get $\mathrm{Aut}_{\mathrm{acl}(A)}(\mathcal{M}) = \mathrm{Aut}_{\mathrm{acl}^{\mathrm{eq}}(A)}(\mathcal{M})$ (by a similar manner as in the last lines of the proof of Proposition 11(1)).

Since H is closed in $\mathrm{Gal}_{\mathrm{L}}(T, A)$, by Fact 8(3), $H = \pi(\mathrm{Aut}_{c_F A}(\mathcal{M}))$ for some hyperimaginary $c_F \in \mathrm{bdd}(A)$. But H has finite index in $\mathrm{Gal}_{\mathrm{L}}(T, A)$, hence (by Fact 8(2),) $c_F \in \mathrm{acl}(A)$. Say $\{c_F = c_0/F, \cdots, c_{n-1}/F\}$ is the set of all A-conjugates of c_F.

We may assume that F is closed under conjunction and all formulas in F are symmetric and reflexive. Note that by compactness, there is $\delta \in F$ such that for all $i < j < n$,

$$c_i c_j \not\models \exists z_0 z_1 z_2 (\delta(x, z_0) \wedge \delta(z_0, z_1) \wedge \delta(z_1, z_2) \wedge \delta(z_2, y)).$$

Let $\delta^4(x, y) \equiv \exists z_0 z_1 z_2 (\delta(x, z_0) \wedge \delta(z_0, z_1) \wedge \delta(z_1, z_2) \wedge \delta(z_2, y))$, and define $\delta^m(x, y)$ similarly for $m < \omega$. Note that in particular, $\delta(c_i, \mathcal{M})$'s are pairwise disjoint.

Let d be any realization of $\mathrm{tp}(c_0/A)$. Then $d \models \bigvee_{i<n} F(x, c_i)$, thus $d \models \bigvee_{i<n} \delta(x, c_i)$, implying that there is $\varphi(x) \in \mathrm{tp}(c_0/A)$ such that $\varphi(x) \models \bigvee_{i<n} \delta(x, c_i)$, that is, $\varphi(\mathcal{M})$ can be partitioned as $\{\varphi(\mathcal{M}) \cap \delta(c_i, \mathcal{M}) : i < n\}$. Note that we can say $\varphi(x)$ is A-invariant; this is possible because A is a set of imaginaries, not a hyperimaginary.

Claim 1. For any $a', a'' \models \varphi(x)$,
$a'a'' \models \delta^2(x, y)$ if and only if $a', a'' \in \varphi(\mathcal{M}) \cap \delta(c_i, \mathcal{M})$ for some $i < n$.

Proof. Assume $\models \delta^2(a', a'')$, hence there is some a^* such that $\models \delta(a', a^*) \wedge \delta(a^*, a'')$. Suppose a' and a'' belong to different components for a contradiction. Then

$$\models \delta(c_i, a') \wedge \delta(a', a^*) \wedge \delta(a^*, a'') \wedge \delta(a'', c_j)$$

for some $i \neq j < n$, implying $c_i c_j \models \delta^4(x, y)$, a contradiction.

For the converse, suppose $a', a'' \in \varphi(\mathcal{M}) \cap \delta(c_i, \mathcal{M})$ for some $i < n$. Then $\models \delta(a', c_i) \wedge \delta(c_i, a'')$. ☐

Now define

$$L(x, y) \equiv (\neg\varphi(x) \wedge \neg\varphi(y)) \vee (\varphi(x) \wedge \varphi(y) \wedge \delta^2(x, y)).$$

Since $\varphi(x)$ is A-invariant, L is an A-definable equivalence relation with finitely many classes, $\neg\varphi(\mathcal{M}), \varphi(\mathcal{M}) \cap \delta(c_0, \mathcal{M}), \cdots, \varphi(\mathcal{M}) \cap \delta(c_{n-1}, \mathcal{M})$. Note that some imaginary $\boldsymbol{b}(\in \mathrm{acl}(A))$ is interdefinable with c/L ([1, Lemma 1.10]).

Claim 2. c/F and $\boldsymbol{b}$ (or equivalently, c/L) are interdefinable over A.

Proof. Let $f \in \mathrm{Aut}_A(\mathcal{M})$. Then

$$\begin{aligned} f(c/F) = c/F \text{ iff } & F(f(c), c) \text{ holds iff } \models \delta^2(f(c), c) \\ & \text{iff } L(f(c), c) \text{ holds iff } f(c/L) = c/L, \end{aligned}$$

where the second logical equivalence follows since: Otherwise, $\models \delta^2(f(c), c)$ but $F(c_i, f(c))$ and $F(c, c_j)$ hold for some $i \neq j < n$. But then we have $\models \delta^4(c_i, c_j)$, a contradiction. □

By Claim 2, $H = \pi(\mathrm{Aut}_{c_F A}(\mathcal{M})) = \pi(\mathrm{Aut}_{\boldsymbol{b}A}(\mathcal{M}))$ where $\boldsymbol{b} \in \mathrm{acl}^{\mathrm{eq}}(A)$.

□

However, contrary to [5, Corollary 5.1.15], in general $\mathrm{acl}(\boldsymbol{e})$ and $\mathrm{acl}^{\mathrm{eq}}(\boldsymbol{e})$ *need not* be interdefinable; the error occurred there due to the incorrect proof of [5, 5.1.14(1) ⇒ (2)]. An example presented in [3] for another purpose supplies a counterexample. Consider the following 2-sorted model:

$M = ((M_1,\ S_1,\ \{g^1_{1/n} : n \geq 1\}),\ (M_2,\ S_2,\ \{g^2_{1/n} : n \geq 1\}),\ \delta)$ where

1. M_1 and M_2 are unit circles centered at origins of two disjoint (real) planes.
2. S_i is a ternary relation on M_i, defined by $S_i(b, c, d)$ holds if and only if b, c and d are in clockwise-order.
3. $g^i_{1/n}$ is a unary function on M_i such that $g^i_{1/n}(b)$ = rotation of b by $2\pi/n$-radians clockwise.
4. $\delta : M_1 \to M_2$ is the double covering, i.e. $\delta(\cos t,\ \sin t) = (\cos 2t,\ \sin 2t)$.
5. Let $\mathcal{M}$ be a monster model of $\mathrm{Th}(M)$ and $\mathcal{M}_1$, $\mathcal{M}_2$ be the two sorts of $\mathcal{M}$.

In [2, Theorems 5.8 and 5.9], it is shown that each $\mathrm{Th}(\mathcal{M}_i)$ has weak elimination of imaginaries (that is, for any imaginary element c, there is a finite real tuple b such that $c \in \mathrm{dcl}(b)$ and $b \in \mathrm{acl}(c)$), using the B. Poizat's notion of weak elimination of imaginaries ([7, Chapter 16.5]). The following fact is a folklore, whose explicit proof was observed in RIMS model theory workshop by I. Yoneda ([8]).

Fact 13. *A (complete) theory T has weak elimination of imaginaries if and only if every definable set has a smallest algebraically closed set over which it is definable.*

Remark & Definition 14.

1. For each element b of sort $i = 1, 2$, $g^i_r(b)$ means $(g^i_{1/n})^m(b)$ where r is a rational number m/n.
2. For each element b of sort 2, $\delta^{-1}(b) = \{c_0, c_1\}$, the δ-preimage of b.
3. For a set of elements $B = B_1 \cup B_2$ of $\mathcal{M}$ where each element of B_i is of sort i,

$$\begin{aligned} \mathrm{cl}(B) = & \{g^1_r(b) : r \in \mathbb{Q}, b \in B_1\} \cup \{\delta(g^1_r(b)) : r \in \mathbb{Q}, b \in B_1\} \\ & \cup \{g^2_r(b) : r \in \mathbb{Q}, b \in B_2\} \cup \bigcup_{r \in \mathbb{Q}, b \in B_2} \delta^{-1}(g^2_r(b)). \end{aligned}$$

4. Note that in the above item, the substructure generated by B is formed by omitting the last union: $\bigcup_{r \in \mathbb{Q}, b \in B_2} \delta^{-1}(g^2_r(b))$.

Lemma 15. *Let $B = \{b_0, \cdots, b_{n-1}\}$ be a subset of $\mathcal{M}$. Then*

$$\mathrm{acl}(B) = \mathrm{cl}(B).$$

Proof. Say $B = \{b_0, \cdots, b_{m-1}, b_m, \cdots, b_{n-1}\}$ where $b_0, \cdots, b_{m-1}$ are of sort 1 and the others are of 2. Choose any element b of sort 1. If

$$b \notin \{g^1_r(b_i) : r \in \mathbb{Q}, i < m\} \cup \bigcup_{r \in \mathbb{Q}, m \le i < n} \delta^{-1}(g^2_r(b_i)),$$

then $b \notin \mathrm{acl}(B)$ since there are infinitely many elements which are infinitesimally close to b and there is an B-automorphism mapping b to each such element.

Likewise, for an element b of sort 2, if

$$b \notin \{g^2_r(\delta(b_i)) : r \in \mathbb{Q}, i < m\} \cup \{g^2_r(b_i) : r \in \mathbb{Q}, m \le i < n\},$$

then $b \notin \mathrm{acl}(B)$. Thus $\mathrm{acl}(B) \subseteq \mathrm{cl}(B)$.

For the converse, it is easy to observe that

$$\{g^1_r(b_i) : r \in \mathbb{Q}, i < m\} \cup \{g^2_r(\delta(b_i)) : r \in \mathbb{Q}, i < m\}$$
$$\cup \{g^2_r(\delta(b_i)) : r \in \mathbb{Q}, m \le i < n\} \subseteq \mathrm{dcl}(B) \text{ and}$$
$$\bigcup_{r \in \mathbb{Q}, m \le i < n} \delta^{-1}(g^2_r(b_i)) \subseteq \mathrm{acl}(B)$$

since each $b \in \bigcup_{r \in \mathbb{Q}, m \le i < n} \delta^{-1}(g^2_r(b_i))$ has at most two B-automorphic images (has only one B-automorphic image if $m \neq 0$). □

Proposition 16. $\mathrm{Th}(\mathcal{M})$ *has weak elimination of imaginaries.*

Proof. Let $\varphi(x, y_0, \cdots, y_{n-1}) \in \mathcal{L}$ and $B = \{b_0, \cdots, b_{n-1}\} = \{b_0, \cdots, b_{m-1}\} \cup \{b_m, \cdots, b_{n-1}\}$ where $b_0, \cdots, b_{m-1}$ are of sort 1 and the others are of 2. According to Fact 13, it suffices to show that there is a smallest algebraically closed set over which $\varphi(\mathcal{M}, B) \equiv \varphi(\mathcal{M}, b_0, \cdots, b_{n-1})$ is definable.

Since there is some c_i such that $\delta(c_i) = b_i$ for each $i \in \{m, \cdots, n-1\}$, we may assume that every element of B is of sort 1. Choose $D = \{d_0, \cdots, d_{k-1}\} \subseteq B$ such that $\{g^1_r(d_i) : r \in \mathbb{Q}, i < k\} = \{g^1_r(b_i) : r \in \mathbb{Q}, i < n\}$ and $d_i \notin \mathrm{cl}(D) \setminus \{d_i\}$ for each $i < k$. Then $\varphi(\mathcal{M}, B)$ is definable over D and there is some minimal subset D' of D such that $\varphi(\mathcal{M}, B)$ is definable over $\mathrm{acl}(D')$ by Lemma 15. □

Now for $i = 1, 2$, we let $E_i(x, y)$ if and only if x and y in $\mathcal{M}_i$ are infinitesimally close, i.e.

$$E_i(x, y) := \bigwedge_{1<n} (S_i(x, y, g^i_{1/n}(x)) \vee S_i(y, x, g^i_{1/n}(y))),$$

which is an $\emptyset$-type-definable equivalence relation. Let $b \in \mathcal{M}_2$, $c, c' \in \mathcal{M}_1$ where $\delta(c) = \delta(c') = b$. Note that c, c' are antipodal to each other and c/E_1, c'/E_1 are conjugates over b/E_2, hence $c/E_1, c'/E_1 \in \mathrm{acl}(b/E_2)$.

Theorem 17. $\mathrm{acl}(b/E_2)$ *and* $\mathrm{acl}^{\mathrm{eq}}(b/E_2)$ *are* not *interdefinable.*

Proof. We prove following Claim and then conclude.

Claim. $\mathrm{acl}^{\mathrm{eq}}(b/E_2)$ is interdefinable with b/E_2.

Proof. To lead a contradiction, suppose that there are distinct imaginaries $d_1, d_2 \in \mathrm{acl}^{\mathrm{eq}}(b/E_2)$ such that $d_1 \equiv_{b/E_2} d_2$. Weak elimination of imaginaries of $\mathrm{Th}(\mathcal{M})$ (Proposition 16) implies that $\mathrm{acl}^{\mathrm{eq}}(d_1, d_2)$ and $D := \{d \in \mathcal{M} : d \in \mathrm{acl}^{\mathrm{eq}}(d_1, d_2)\}$ are interdefinable $(*)$. In particular, $D \subseteq \mathrm{acl}^{\mathrm{eq}}(b/E_2) \cap \mathcal{M}$. However, for any infinitesimally close $d, d' \in \mathcal{M}_i$ $(i = 1, 2)$, there is $f \in \mathrm{Aut}_{b/E_2}(\mathcal{M})$ sending d to d'. Hence indeed $D = \emptyset$, which contradicts $(*)$ (because $d_1 \equiv_{b/E_2} d_2$ and $d_1 \neq d_2 \in \mathrm{acl}^{\mathrm{eq}}(d_1, d_2)$). □

Now $c/E_1, c'/E_1 \in \mathrm{acl}(b/E_2) \setminus \mathrm{dcl}(b/E_2) = \mathrm{acl}(b/E_2) \setminus \mathrm{dcl}(\mathrm{acl}^{\mathrm{eq}}(b/E_2))$.

$\square$

References

[1] Enrique Casanovas, Simple theories and hyperimaginaries (Lecture Notes in Logic), Cambridge University Press, 2011.

[2] Jan Dobrowolski, Byunghan Kim and Junguk Lee, The Lascar groups and the first homology groups in model theory, Annals of Pure and Applied Logic **168**, (2017), 2129–2151.

[3] Jan Dobrowolski, Byunghan Kim, Alexei Kolesnikov and Junguk Lee, The relativized Lascar groups, type-amalgamation, and algebraicity, Journal of Symbolic Logic, **86**, (2021), 531–557.

[4] Karl H. Hofmann and Sidney A. Morris, The structure of compact groups, De Gruyter, 2013.

[5] Byunghan Kim, Simplicity theory, Oxford University Press, 2014.

[6] Hyoyoon Lee, Quotient groups of the Lascar group and strong types in the context of hyperimaginaries, PhD thesis, Yonsei University, in preparation.

[7] Bruno Poizat, A Course in Model Theory: An Introduction to Contemporary Mathematical Logic, Springer, 2000.

[8] Ikuo Yoneda, On two definitions of weak elimination of imaginaries, RIMS Model Theory Workshop 2021, December 2021.

[9] Martin Ziegler, Introduction to the Lascar group, Tits buildings and the model theory of groups, London Math. Lecture Note Series, **291**, Cambridge University Press, (2002), 279–298.

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