On Pop's conjecture
Elementary equivalence versus isomorphism

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Abstract Let K, L be finitely generated fields with $K \equiv L$. Is K isomorphic to L? In 2020, Dittmann and Pop [DP] solved this question affirmatively except the case of characteristic 2. We review their method. Finally, we consider analogous problem for infinite algebraic extensions of $\mathbb{Q}$.

1 Elementary equivalence versus isomorphism

Let $\mathcal{L}$ be a finite language and $\mathcal{C}$ be the class of finitely generated $\mathcal{L}$-structures. Elementary equivalence versus isomorphism problem (EEIP) is the following: for $A, B \in \mathcal{C}$,

$$\textit{Does } A \equiv B \textit{ imply } A \approxeq B \textit{ ?}$$

For $\mathcal{L}$=the ring-language, this question goes back to 1970s and seems to have first been posed explicitly by Pop [P1].

In 2020, Dittmann and Pop [DP] solved this question affirmatively except the case of characteristic 2.

(Characteristic 2 case requires the assumption that if transcendental degree > 3, the resolution of singularities above $\mathbb{F}_2$ holds.)

2 Quasi-finitely axiomatizability

Let $\mathcal{L}$ be a finite language and A be an infinite finitely generated $\mathcal{L}$-structure.

We say that A is quasi-finitely axiomatizable (QFA) if there is a $\mathcal{L}$-sentence σ satisfied by A such that every finitely generated $\mathcal{L}$-structure satisfying σ is isomorphic to A.

Remarks.

- This notion of QFA does not agree with the one commonly used in Zilber's program, there QFA means that its theory is axiomatizable by a finite number of axioms and the schema of infinity.

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- QFA implies the affirmative answer to Elementary equivalence versus isomorphism problem.
- In fact, Pop and Ditmann proved that every f.g. field is QFA (with the assumption above in the case of characteristic 2).

3 Examples of QFA-structures

Groups

- free nilpotent non-abelian groups (Oger and Sabbagh 2006)
- free metabelian group of finite rank ≥ 2 (Kherif 2015)

Rings

- $(\mathbb{Z}, +, \times)$ (Sabbagh 2004)
- f.g. commutative rings (Kherif 2007)

fields

- global fields (Rumely 1980)
- f.g. fields of Kronecker dimension < 3 (Pop 2017)
- f.g. fields with the assumption above in the case of characteristic 2 (Dittmann and Pop 2020)

The fact that f.g. commutative rings are QFA does not imply infinite f.g. fields being QFA. There is the fact that *if A is a f.g. field as a ring, then A is finite,* that is, if A is a f.g. field as a ring, A is the form of $\mathbb{F}_p[a]$ where a is algebraic over $\mathbb{F}_p$.

We note that f.g. as a ring is different from f.g. as a field.

A f.g field is of the form $\mathbb{Q}(t_1, \ldots, t_n, a_1, \ldots, a_m)$ or $\mathbb{F}_p(t_1, \ldots, t_n, a_1, \ldots, a_m)$ where t_i are transcendental over $\mathbb{Q}$ or $\mathbb{F}_p$ and a_j are algebraic over $\mathbb{Q}(t_1, \ldots, t_n)$ or $\mathbb{F}_p(t_1, \ldots, t_n)$ respectively.

A f.g. commutative ring is of the form $\mathbb{Z}[t_1, \ldots, t_n, a_1, \ldots, a_m]$ or of the form $(\mathbb{Z}/n\mathbb{Z})[t_1, \ldots, t_n, a_1, \ldots, a_m]$.

4 Bi-interpretability with $\mathbb{Z}$

For the definition of bi-interpretability, see [AKNS].

Bi-interpretability with $\mathbb{Z}$ has a simple feature.

Lemma (AKNS, Lemma 2.17). *$\mathfrak{A}$ is bi-interpretable with $\mathbb{Z}$ iff there are binary operations $\oplus$ and $\otimes$ such that $(\mathbb{Z}, +, \times) \approxeq (\mathfrak{A}, \oplus, \otimes)$ and $(\mathfrak{A}, \oplus, \otimes)$ is interdefinable with $\mathfrak{A} = (A, \ldots)$.*

We say that a structure with same universe as $\mathfrak{A}$ is <u>*interdefinable*</u> with $\mathfrak{A}$ if both structures have the same definable sets.

Propsition (AKNS, Proposition2.28)**.** *If $\mathfrak{A}$ is bi-interpretable with $\mathbb{Z}$, then $\mathfrak{A}$ is QFA.*

Kherif realized that one can use bi-interpretability of a f.g. structure $\mathfrak{A}$ with $(\mathbb{Z}, +, \times)$ as a general method to prove that $\mathfrak{A}$ is QFA. Somewhat later, Scanlon [S] independently used this method to show that each f.g. field is QFA. His proof has a gap, but his work appears to reduce the problem "definability of valuations" by which one gets the result that each f.g. field is bi-interpretable with $(\mathbb{Z}, +, \times)$, and Dittmann and Pop succeeded it.

Remark.
<u>*QFA does not imply Bi-interpretability with $\mathbb{Z}$.*</u>

For example, $\mathbb{Z} \times \mathbb{Z}$ is not bi-interpretable with $\mathbb{Z}$, but is QFA. On the other hand, every f.g. integral domain is bi-interpretable with $\mathbb{Z}$. Together with this fact and additional arguments, Kherif proved that every f.g. commutative ring is QFA.

Aschenbrenner, Kherif, Naziazeno, and Scanlon determined the necessary and sufficient condition for a f.g commutative ring to be bi-interpretable with $\mathbb{Z}$. See [AKNS].

5 Extended EEIP, extending the class of fields

We consider the class $\mathcal{C}$ of fields which are infinitely generated but of finite transcendency degree over its prime field. Without of that restriction, no infinitely generated infinite fields are QFA.

Therefore, we consider EEIP for fields in $\mathcal{C}$, especially, EEIP for subfields of $\mathbb{Q}^{alg}$.

Example.

$$\mathbb{Q}^{alg} \equiv (\mathbb{Q}(t))^{alg} \equiv (\mathbb{Q}(t, s))^{alg},$$

where, $s, , t \in \mathbb{C}$ are algebraically independent over $\mathbb{Q}$ and K^{alg} denotes algebraic closure of K in $\mathbb{C}$.

Hence, EEIP has a negative solution for algebraically closed fields $\mathbb{Q}^{alg}$.

Now, let $K = \mathbb{Q}(\{\cos(2\pi/p^n) : n \in \mathbb{N}\})$ where p is a prime integer. K is totally real and its ring of integers $\mathcal{O}_K$ is definable without parameters in K and $\mathbb{Z}$ is definable without parameters in $\mathcal{O}_K$. We are able to show that EEIP for K has a positive solution. Generally, we can prove

Propsition. *Let $K \subset \mathbb{C}$ be an infinite totally real extension of $\mathbb{Q}$ such that its ring of integers $\mathcal{O}_K$ is definable without parameters in K and $\mathbb{Z}$ is definable without parameters in $\mathcal{O}_K$. Then,*

$$L \in \mathcal{C}, L \equiv K \Longrightarrow L \approxeq K.$$

Hence, EEIP has a positive solution for such K. This follows from the fact that in a ring of totally real integers, one can define Gödel β function. (Details was published in RIMS Kôkyûroku. "Rings of totally real integers with N")

Questions

Let $K \subset \mathbb{Q}^{alg}$.

- Let K be decidable. Does EEIP have a negative solution for K?
- Let K interpret $\mathbb{Z}$. Does EEIP have a positive solution for K?

There are decidable infinite algebraic extensions of $\mathbb{Q}$ other than $\mathbb{Q}^{alg}$, for example, $\mathbb{Q}^{M}$, the field of totally M-adic algebraic integers, that is, those algebraic integers whose minimal polynomial over $\mathbb{Q}$ splits into linear factors over $\mathbb{Q}_p$ for every $p \in M$ where M is a finite nonempty set of prime integers. (Ershov [Er])

There are infinite algebraic extensions of $\mathbb{Q}$ in which $\mathbb{Z}$ is interpretable other than totally real ones, for example, $\mathbb{Q}(\{\zeta_{p^n} : n \in \mathbb{N}\})$ where p is a prime integer and ζ_{p^n} is a primitive p^n-th root of unity. (Videla [V])

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