PROBLEMS ON INVARIANTS OF KNOTS

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1. Problems

Problem 1. At the moment, there are three constructions that take as input a modular linear q-difference equation (eg one for each knot) and give as output an analytic function on $\mathbb{C} \setminus \mathbb{R}$ which extends to $\mathbb{C} \setminus (-\infty, 0]$.

These three constructions are

(a) an analytic one described in [GZ]
(b) an asymptotic series one described in [GGMnW] and in [GGMn]
(c) an arithmetic one described in [GZ].

(b,c) are conjectural, meaning their definition is not given, as well as the equality of (a),(b),(c).

The problem is to understand these three constructions, which are realizations of the same concept in three different categories.

Problem 2. In the paper [GK] we introduced the descendant Kashaev invariants of a knot K, which is a sequence $DJ^K_m(q)$ (indexed by m in the integers) of elements of the Habiro ring with the following properties

(a) $DJ^K_0(q)$ is the Kashaev invariant.
(b) the sequence $DJ^K_m(q)$ for $m \in \mathbb{Z}$ determines and is determined by the colored Jones polynomial of a knot,
(c) it leads to a 3-variable (t, x, q) version of the colored Jones polynomial.

The problem is to determine the loop expansion of the (t, x, q)-version of the colored Jones polynomial in terms of the loop expansion of the original colored Jones polynomial. This is theoretically possible, but an algorithm has not been written down.

Problem 3. In the paper [DG13] Dimofte and G. constructed a formal power series $\Phi_{\mathcal{T}}(h)$ using as input the Neumann–Zagier datum of an ideal triangulation $\mathcal{T}$ of a hyperbolic knot K, that is the Neumann–Zagier matrices, a solution z of the Neumann–Zagier equations corresponding to the discrete faithful representation, and a flatenning. The definition is rigorous using formal Gaussian integration, and each coefficient of the formal power series $\Phi_{\mathcal{T}}(h)$ is a finite sum of contributions of Feynmann diagrams, and lies in the trace field of K. The coefficient of h^ℓ in $\Phi_{\mathcal{T}}(h)$ is defined to be the $\ell + 1$-loop invariant. The completion $\hat{\Phi}_{\mathcal{T}}(h) = e^{V/h} h^{3/2} \Phi_{\mathcal{T}}(h)$ of the power series (where V is the complexified volume of the knot)

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$\Phi_{\mathcal{T}}(h)$ is supposed to be the asymptotic expansion of the Kashaev invariant to all orders in $h = 2\pi i/N$, according to the refined volume conjecture. In particular, the series $\Phi_{\mathcal{T}}(h)$ is a topological invariant of a knot. Its constant term (the 1-loop invariant) was shown to be independent of the 2–3 Pachner moves in [DG13] and hence a topological invariant. Show that the series $\Phi_{\mathcal{T}}(h)$ is invariant under 2–3 Pachner moves (hence a topological invariant), and show that the few knots that satisfy the volume conjecture actually satisfy the refined volume conjecture.

Problem 4. Prove that the 1-loop invariant, i.e., the constant term of the power series $\Phi_{\mathcal{T}}(h)$, equals to the adjoint torsion of the hyperbolic knot with respect to the medirian. This was conjectured in [DG13] and checked for tens of thousands of hyperbolic knots, both sides of the equality being elements of the trace field of the hyperbolic knot.

Problem 5. In the subsequent paper [DG18] we defined a formal power series $\Phi_{\mathcal{T},\alpha}(h)$ where $\alpha \in \mathbb{Q}/\mathbb{Z}$ (so that $\Phi_{\mathcal{T},0}(h) = \Phi_{\mathcal{T}}(h)$) which is conjectured to appear in the refined quantum modularity conjecture of [GZ]. Show that the series $\Phi_{\mathcal{T},\alpha}(h)$ is invariant under 2–3 Pachner moves and hence a topological invariant.

Problem 6. In a paper with Seokbeom Yoon [GY] we defined a polynomial invariant of a hyperbolic knot with coefficients in the trace field of the knot that determines the 1-loop invariant of all cyclic coverings of a hyperbolic knot. The invariant is defined using a twisted Neumann–Zagier datum, and we proved that the polynomial is independent of 2–3 Pachner moves and hence a topological invariant. Prove that it equals to the twisted Alexander polynomial of the adjoint of the geometric representation.

References

[DG13] Tudor Dimofte and Stavros Garoufalidis, *The quantum content of the gluing equations*, Geom. Topol. **17** (2013), no. 3, 1253–1315.

[DG18] ———, *Quantum modularity and complex Chern-Simons theory*, Commun. Number Theory Phys. **12** (2018), no. 1, 1–52.

[GGMn] Stavros Garoufalidis, Jie Gu, and Marcos Mariño, *Peacock patterns and resurgence in complex Chern-Simons theory*, arXiv:2012.00062, Preprint 2020, https://arxiv.org/abs/2012.00062.

[GGMnW] Stavros Garoufalidis, Jie Gu, Marcos Mariño, and Campbell Wheeler, *Resurgence of Chern–Simons theory at the trivial flat connection*, arXiv:2111.04763, Preprint 2021, https://arxiv.org/abs/2111.04763.

[GK] Stavros Garoufalidis and Rinat Kashaev, *The descendant colored Jones polynomials*, arXiv:2108.07553, Preprint 2021, https://arxiv.org/abs/2108.07553.

[GY] Stavros Garoufalidis and Seokbeom Yoon, *Twisted Neumann–Zagier matrices*, arXiv:2109.00379, Preprint 2021, https://arxiv.org/abs/2109.00379.

[GZ] Stavros Garoufalidis and Don Zagier, *Knots, perturbative series and quantum modularity*, arXiv:2111.06645, Preprint 2021, arXiv:2111.06645. https://arxiv.org/abs/2111.06645.

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