Cut-Elimination for Cyclic Proof Systems with Inductively Defined Propositions

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Abstract. Cyclic proof systems are extensions of the sequent-calculus style proof systems for logics with inductively defined predicates. In cyclic proof systems, inductive reasoning is realized as cyclic structures in proof trees. It has been already known that the cut-elimination property does not hold for the cyclic proof systems of some logics such as the first-order predicate logic and the separation logic. In this paper, we consider the cyclic proof systems with inductively defined propositions (that is, nullary predicates), and prove that the cut-elimination holds for the propositional logic, and it does not hold for the bunched logic.

1 Introduction

Cyclic proof systems [4] are extensions of the sequent-calculus style proof systems with inductively defined predicates and they allow cyclic structures in proof trees that represent the induction. The cyclic proof systems are proposed for many logics such as the first-order logic [4], the bunched logic [1], the separation logic [2], the linear logic [5], and the linear temporal logic [5]. The cyclic proof systems are considered to be suitable for (semi-)automatic inductive reasoning [3], since we do not have to fix the proposition to which the induction principle is applied a priori.

However, it has been already known that the cut-elimination property does not hold for the cyclic proof systems of some logics such as the first-order predicate logic [7], the separation logic [6], and the bunched logic [8]. The cut-elimination property is expected in automatic reasoning since to find cut formulas requires some heuristics.

In this paper, we consider the cyclic proof systems with inductively defined propositions (that is, nullary predicates), and prove that the cut-elimination holds for the ordinary propositional logic, and it does not hold for the bunched logic.

The cyclic proof system $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ is the propositional restriction of the cyclic proof system CLKID_ω in [4] for the first-order predicate logic. For the full system, Masuoka et al. showed that the cut-elimination fails. In this paper, we prove the cut-free completeness of $\mathrm{CLKID}_\omega^{\mathrm{prop}}$.

The bunched logic is logic to express quantitative properties of resources, and it contains both the multiplicative conjunction and the ordinary (classical) conjunction. The bunched logic was proposed for applications of program verification. The assertion logic of the separation logic, which is an extension of Hoare logic for pointer programs, is a variant of the bunched logic. The cyclic proof system $CLBI_{ID}^\omega$ of the bunched logic with inductive predicates was proposed in [1]. In this paper, we prove that the cut-elimination does not hold for $CLBI_{ID}^\omega$. This part has been published in [8], and we give a summary of the result in this paper.

2 Cyclic Proof System $\mathrm{CLKID}_\omega^{\mathrm{prop}}$

We will use a vector notation, like $\vec{X}$, to mean a sequence $X_1, \dots, X_n$ of syntactical objects, and write $|\vec{X}|$ for the length of the sequence $\vec{X}$. The set of finite subsets of a set S is written as $\mathcal{P}_{\mathrm{fin}}(S)$. For a natural number n, we write $[n]$ to mean $\{0, \dots, n-1\}$.

2.1 Syntax of $\mathrm{CLKID}_\omega^{\mathbf{prop}}$

Definition 1 (Languages). A language $\mathcal{L}$ consists of a finite set **PropSym** of (non-inductive) propositional symbols and a finite set $\{P_0, \ldots, P_N\}$ of inductive propositional symbols. The metavariables Q and P are used for non-inductive and inductive propositional symbols, respectively. The metavariable R is used for either non-inductive or inductive propotional symbols.

Definition 2 (Formulas). The *formulas* (denoted by $\varphi, \psi, \ldots$) in $\mathcal{L}$ are inductively defined by

$$\varphi ::= R \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi.$$

An atomic formula is a formula of the form R. The set of formulas and the set of atomic formulas are written as **Fml** and **Atom**, respectively.

For a finite set $X = \{\varphi_i \mid i \in [n]\}$ of formulas, $\bigwedge X$ and $\bigvee X$ are abbreviations of $\varphi_0 \wedge \cdots \wedge \varphi_{n-1}$ and $\varphi_0 \vee \cdots \vee \varphi_{n-1}$, respectively.

Definition 3 (Inductive definitions). A production rule for P_j has the form

$$(P_j, \{Q_i \in \mathbf{PropSym} \mid i \in [m]\} \cup \{P_{h_i} \mid i \in [n]\}).$$

It is often written as

$$\frac{Q_0 \quad \cdots \quad Q_{m-1} \quad P_{h_0} \quad \cdots \quad P_{h_{n-1}}}{P_j}\,.$$

A finite set of production rules is called an *inductive definition set.*

In the following, let $(P_j, \Phi_{j,i})$ be the i-th production rule for P_j, and Φ be the inductive definition set $\{(P_j, \Phi_{j,i})\}_{i,j}$

Definition 4 (Sequents). A sequent of CLKID_ω has the form $\Gamma \vdash \Delta$, where Γ and Δ are finite sets of formulas. The left-hand side and the right-hand side of $\vdash$ in a sequent are called the *antecedent* and the *succedent* of the sequent, respectively. Let S be a sequent. Then the antecedent and the succedent are denoted by $L(S)$ and $R(S)$, respectively.

We write $\varphi_1, \ldots, \varphi_m \vdash \phi_1, \ldots, \phi_n$ instead of $\{\varphi_1, \ldots, \varphi_m\} \vdash \{\phi_1, \ldots, \phi_n\}$. We also write Γ, φ and Γ_1, Γ_2 instead of $\Gamma \cup \{\varphi\}$ and $\Gamma_1 \cup \Gamma_2$, respectively.

A sequent $\Gamma \vdash \Delta$ is called *normal* if $\Gamma \cup \Delta \subseteq$ **Atom**. A *strongly normal* sequent is a normal sequent and its antecedent consists of non-inductive propositional symbols.

2.2 Semantics

Let $\mathbf{0}$, $\mathbf{1}$ and $\mathbf{2}$ be the empty set $\emptyset$, a singleton set $\{\emptyset\}$, and $\{\mathbf{0}, \mathbf{1}\}$, respectively. A *valuation* (denoted by v) is a function from **PropSym** to $\mathbf{2}$. The interpretation $[\![\varphi]\!]_v^\Phi$ and the approximating interpretation $[\![R]\!]_{v,n}^\Phi$ are defined as the least element in $\mathbf{2}$ that satisfies the following:

$$\begin{aligned}
[\![Q]\!]_v^\Phi = [\![Q]\!]_{v,n}^\Phi &= v(Q),\\
[\![\varphi_0 \wedge \varphi_1]\!]_v^\Phi &= [\![\varphi_0]\!]_v^\Phi \cap [\![\varphi_1]\!]_v^\Phi,\\
[\![\varphi_0 \vee \varphi_1]\!]_v^\Phi &= [\![\varphi_0]\!]_v^\Phi \cup [\![\varphi_1]\!]_v^\Phi,\\
[\![\neg\varphi]\!]_v^\Phi &= \mathbf{1} \setminus [\![\varphi]\!]_v^\Phi,\\
[\![P_j]\!]_v^\Phi &= \bigcup_n [\![P_j]\!]_{v,n}^\Phi,\\
[\![P_j]\!]_{v,0}^\Phi &= \mathbf{0},\\
[\![P_j]\!]_{v,k+1}^\Phi &= \bigcup_i \bigcap_{R \in \Phi_{j,i}} [\![R]\!]_{v,k}^\Phi
\end{aligned}$$

We often omit the superscript Φ and write $[\![\varphi]\!]_v$. We write $v \models \varphi$ when $[\![\varphi]\!]_v = \mathbf{1}$.

Definition 5 (Validity). *A sequent S is valid if and only if $v \models \bigvee R(S)$ holds for any valuation v such that $v \models \bigwedge L(S)$.*

$$\frac{}{\Gamma \vdash \Delta}\ (\mathrm{Ax}) \quad \text{where } \Gamma \cap \Delta \neq \emptyset \qquad \frac{\Gamma \vdash \Delta}{\Gamma' \vdash \Delta'}\ (\mathrm{Wk}) \quad \text{where } \Gamma \subseteq \Gamma' \text{ and } \Delta \subseteq \Delta'$$

$$\frac{\Gamma \vdash \Delta, \varphi \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta}\ (\mathrm{Cut})$$

$$\frac{\Gamma, \varphi_i \vdash \Delta}{\Gamma, \varphi_0 \wedge \varphi_1 \vdash \Delta}\ (\wedge\mathrm{L}) \quad i = 0, 1 \qquad \frac{\Gamma \vdash \Delta, \varphi_0 \quad \Gamma \vdash \Delta, \varphi_1}{\Gamma \vdash \Delta, \varphi_0 \wedge \varphi_1}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma, \varphi_0 \vdash \Delta \quad \Gamma, \varphi_1 \vdash \Delta}{\Gamma, \varphi_0 \vee \varphi_1 \vdash \Delta}\ (\vee\mathrm{L}) \qquad \frac{\Gamma \vdash \Delta, \varphi_i}{\Gamma \vdash \Delta, \varphi_0 \vee \varphi_1}\ (\vee\mathrm{R}) \quad i = 0, 1$$

$$\frac{\Gamma \vdash \Delta, \varphi}{\Gamma, \neg\varphi \vdash \Delta}\ (\neg\mathrm{L}) \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta, \neg\varphi}\ (\neg\mathrm{R})$$

Fig. 1. Inference rules of $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ (except (UL) and (UR))

2.3 Cyclic Proof System $\mathbf{CLKID}_\omega^{\mathbf{prop}}$

The cyclic proof system $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ is a sequent calculus style proof system for the classical propositional logic with inductive definitions.

The right unfolding rule and the left unfolding rule for Φ are defined as follows.

Definition 6 (Right unfolding rule). Let $\{Q_0, \ldots, Q_{m-1}, P_{h_0}, \ldots, P_{h_{n-1}}\}$ be $\Phi_{j,i}$. The right unfolding rule $(\mathrm{UR})_{j,i}$ is the following:

$$\frac{\Gamma \vdash \Delta, Q_0 \quad \cdots \quad \Gamma \vdash \Delta, Q_{m-1} \quad \Gamma \vdash \Delta, P_{h_0} \quad \cdots \quad \Gamma \vdash \Delta, P_{h_{n-1}}}{\Gamma \vdash \Delta, P_j}\ (\mathrm{UR})_{j,i}.$$

Definition 7 (Left unfolding rule). For each set $\{\Phi_{j,i}\}_{0 \leq i \leq K_j}$ of production rules of $\{P_j\}$ in Φ, the left unfolding rule $(\mathrm{UL})_j$ is the following:

$$\frac{\Gamma, \Phi_{j,0} \vdash \Delta \quad \cdots \quad \Gamma, \Phi_{j,K_j} \vdash \Delta}{\Gamma, P_j \vdash \Delta}\ (\mathrm{UL}).$$

We often omit the subscripts j, i in $(\mathrm{UR})_{j,i}$ and j in $(\mathrm{UL})_j$ when the indexes are not important or are obvious from context.

The inference rules of $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ are the rules shown in Figure 1 and (UL) and (UR).

Definition 8 (Derivation and pre-proof of $\mathbf{CLKID}_\omega^{\mathbf{prop}}$). Let S be a sequent of $\mathrm{CLKID}_\omega^{\mathrm{prop}}$. A *derivation* (denoted by $\mathcal{D}$) of S is a finite derivation tree constructed by using the inference rules of $\mathrm{CLKID}_\omega^{\mathrm{prop}}$, and the sequent at the root is S. The sequent at a node w of $\mathcal{D}$ is written $\mathcal{D}(w)$. A leaf node of $\mathcal{D}$ is called a *bud* if it does not appear at the conclusion position of (Ax). An internal node C of $\mathcal{D}$ is called a *companion* of a bud B if the sequent at C is the same as that of B.

Let $\mathcal{B}$ be the set of buds in $\mathcal{D}$ and X be a subset of $\mathcal{B}$. A pair $(\mathcal{D}, \mathcal{R})$ is called a *pre-proof of S with open buds X* when $\mathcal{D}$ is a derivation of S and $\mathcal{R}$ is a function that assigns each bud in $\mathcal{B} \setminus X$ to its companion. If X is the emptyset, $(\mathcal{D}, \mathcal{R})$ is called a *pre-proof* of S.

A *trace* and the global trace condition in a pre-proof $(\mathcal{D}, \mathcal{R})$ with X is defined in a similar way to the trace defined in [4].

Definition 9 (Cyclic proof of $\mathbf{CLKID}_\omega^{\mathbf{prop}}$). Let $(\mathcal{D}, \mathcal{R})$ be a pre-proof of S with X. It is called a *cyclic proof of S with open buds X* if it satisfies the global trace condition. If X is the emptyset, $(\mathcal{D}, \mathcal{R})$ is called a *cyclic proof* of S.

Theorem 1 (Soundness of $\mathbf{CLKID}_{\omega}^{\mathbf{prop}}$). *Let $(\mathcal{D}, \mathcal{R})$ be a cyclic proof of S. Then all sequents in $\mathcal{D}$ are valid.*

Proof. It is shown in a similar way to the soundness theorem of CLKID_{ω} [4].

3 Cut-Elimination of Cyclic Proof for Classical Propositional Logic

This section shows that $\mathrm{CLKID}_{\omega}^{\mathrm{prop}}$ enjoys the cut-elimination property.

By the soundness theorem of $\mathrm{CLKID}_{\omega}^{\mathrm{prop}}$, any sequent S of $\mathrm{CLKID}_{\omega}^{\mathrm{prop}}$ that has a cyclic proof is valid. We show the following cut-free completeness to show the cut-elimination property for $\mathrm{CLKID}_{\omega}^{\mathrm{prop}}$.

Theorem 2 (Cut-free completeness of $\mathbf{CLKID}_{\omega}^{\mathbf{prop}}$). *Let S be a sequent of $CLKID_{\omega}^{prop}$. If S is valid, then S has a cut-free cyclic proof.*

This theorem will be shown by the following three steps:

Claim 1 A valid strongly normal sequent, namely sequents of the form $\overrightarrow{Q} \vdash \overrightarrow{Q'}, \overrightarrow{P'}$, has a cut-free non-cyclic proof (Proposition 1).
Claim 2 A valid normal sequent, namely sequents of the form $\overrightarrow{Q}, \overrightarrow{P} \vdash \overrightarrow{Q'}, \overrightarrow{P'}$, has a cut-free cyclic proof (Proposition 3).
Claim 3 A valid sequent S has a cut-free cyclic proof (Theorem 2).

First we show **Claim 1**.

Definition 10 (Unfolding tree). An unfolding tree $\mathrm{uTree}(P)$ of an inductive proposition P is inductively defined as follows.
$\mathrm{uTree}(P_j, 0) = \emptyset$,
$\mathrm{uTree}(P_j, n+1) = \left\{ \dfrac{Q_0 \quad \cdots \quad Q_s \quad T_0 \quad \cdots \quad T_r}{P_j}\ (P_j, l) \;\middle|\; \begin{array}{l} \Phi_{j,l} = \{Q_0, \ldots, Q_s, P_{j_0}, \ldots, P_{j_r}\}, \\ T_i \in \mathrm{uTree}(P_{j_i}, n) \text{ for } 0 \leq i \leq r \end{array} \right\}$,
$\mathrm{uTree}(P_j) = \bigcup_{n \geq 0} \mathrm{uTree}(P_j, n)$.

For each $T \in \mathrm{uTree}(P)$, we define $\mathrm{Lvs}(T)$ by the set of (non-inductive) propositions at the leaf nodes in T.

From the definition it is immediately followed that $\mathrm{uTree}(P_j, n) \subseteq \mathrm{uTree}(P_j, n+1)$ for any n and P_j.

Lemma 1. *$v \models P_j$ if and only if there exists $T \in \mathrm{uTree}(P_j)$ such that $v \models \bigwedge \mathrm{Lvs}(T)$.*

Proof. To show the only-if part, we show the claim that, for any n, $[\![P_j]\!]_v^n = \mathbf{1}$ implies $v \models \bigwedge \mathrm{Lvs}(T)$ for some $T \in \mathrm{uTree}(P_j)$ by induction on n. The case of $n = 0$ is trivially shown by $[\![P_j]\!]_v^0 = \emptyset$. Assume that $n > 0$ and $[\![P_j]\!]_v^n = \mathbf{1}$. Then there exists l and $[\![R]\!]_v^{n-1} = \mathbf{1}$ for all $R \in \Phi_{j,l}$. By the induction hypothesis, for each $P_k \in \Phi_{j,l}$, there exists n_k and $T_k \in \mathrm{uTree}(P_k, n_k)$ such that $v \models \bigwedge \mathrm{Lvs}(T_k)$. Let n be the maximum number of n_k's. Then $T_k \in \mathrm{uTree}(P_k, n)$ holds for any k. Define T by

$$T = \frac{Q_1 \quad \cdots \quad Q_s \quad T_1 \quad \cdots \quad T_r}{P_j}\ (P_j, l).$$

Hence we have $T \in \mathrm{uTree}(P_j, n+1) \subseteq \mathrm{uTree}(P_j)$. We also have $v \models \mathrm{Lvs}(T)$ since $\mathrm{Lvs}(T) = \{Q \mid Q \in \Phi_{j,l}\} \cup \bigcup_k \mathrm{Lvs}(T_k)$. Therefore the only-if part is immediately obtained from the claim.

For showing the if-part, we prove the claim that, for any n, k and T, if $T \in \mathrm{uTree}(P_k, n)$ and $v \models \mathrm{Lvs}(T)$, then $v \models P_k$. This claim is shown by induction on n. The case of $n = 0$ is trivially shown by $\mathrm{uTree}(P_k, 0) = \emptyset$. Assume that $n > 0$, $T \in \mathrm{uTree}(P_k, n)$ and $v \models \mathrm{Lvs}(T)$. Then there exist $\Phi_{k,l} = \{Q_1, \ldots, Q_s, P_{k_1}, \ldots, P_{k_r}\}$ and $T_i \in \mathrm{uTree}(P_{k_i}, n-1)$ such that

$$t = \frac{Q_1 \quad \cdots \quad Q_s \quad T_1 \quad \cdots \quad T_r}{P_k}\ (P_k, l).$$

For each i, by $v \models \mathrm{Lvs}(T_i)$ and the induction hypothesis, we have $v \models P_{k_i}$. Hence $[\![\bigwedge \Phi_{k,l}]\!]_v = \mathbf{1}$ holds by using $v \models Q$ for all $Q \in \Phi_{k,l}$. Therefore $[\![P_k]\!]_v = \bigcup\{[\![\bigwedge \Phi_{k,l}]\!]_v \mid 1 \leq l \leq M_k\} = \mathbf{1}$. The if-part is immediately obtained from the claim.

Proposition 1. *Any valid strongly normal sequent has a cut-free non-cyclic proof.*

Proof. Let S be a valid strongly normal sequent $\overrightarrow{Q} \vdash \overrightarrow{Q'}, P_{j_1}, \ldots, P_{j_k}$. Define the valuation v by $v(Q) = \mathbf{1}$ if $Q \in \overrightarrow{Q}$, $v(Q) = \emptyset$ otherwise. Then $v \models \bigvee \overrightarrow{Q'}$ or $v \models P_{j_i}$ for some i since $v \models \bigwedge \overrightarrow{Q}$ and S is valid. If $v \models \bigvee \overrightarrow{Q'}$, there exists $Q' \in \overrightarrow{Q'}$ such that $v \models Q'$. So $Q' \in \overrightarrow{Q}$ by the definition of v. Thus we have a cut-free non-cyclic proof of S by using (Ax) and (Wk). Otherwise $v \models P_{j_i}$ for some i. Then, by Lemma 1, there exists $T \in \mathrm{uTree}(P_{j_i})$ such that $v \models \bigwedge \mathrm{Lvs}(T)$. Now we consider the following claim: if $T' \in \mathrm{uTree}(P_k, n)$ and $v \models \bigwedge \mathrm{Lvs}(T')$, then $\overrightarrow{Q} \vdash P_k$ has a cut-free non-cyclic proof. It is enough to show this claim, since a cut-free non-cyclic proof of $\overrightarrow{Q} \vdash \overrightarrow{Q'}, P_{j_1}, \ldots, P_{j_k}$ is constructed by applying (Wk) to the cut-free non-cyclic proof of $\overrightarrow{Q} \vdash P_{j_i}$ obtained from the claim.

The claim is shown by induction on n. The case of $n = 0$ is trivially shown. Suppose that $n > 0$, $T' \in \mathrm{uTree}(P_k, n)$ and $v \models \bigwedge \mathrm{Lvs}(T')$. Then there exist $\Phi_{k,l} = \{Q''_1, \ldots, Q''_s, P_{k_1}, \ldots, P_{k_r}\}$ and $T_i \in \mathrm{uTree}(P_{k_i}, n-1)$ such that

$$T' = \frac{Q''_1 \quad \cdots \quad Q''_s \quad T_1 \quad \cdots \quad T_r}{P_k} \ (P_k, l).$$

By $v \models \bigwedge \mathrm{Lvs}(T_i)$ and the induction hypothesis, $\overrightarrow{Q} \vdash P_{k_i}$ has a cut-free non-cyclic proof. We also have cut-free non-cyclic proof of $\overrightarrow{Q} \vdash Q''_i$ for all $1 \leq i \leq s$, since $v \models Q''_i$. Therefore we have a cut-free non-cyclic proof of $\overrightarrow{Q} \vdash P_k$:

$$\frac{\overrightarrow{Q} \vdash Q''_1 \quad \cdots \quad \overrightarrow{Q} \vdash Q''_s \quad \overrightarrow{Q} \vdash P_{k_1} \quad \cdots \quad \overrightarrow{Q} \vdash P_{k_r}}{\overrightarrow{Q} \vdash P_k} \ (\mathrm{UR})_{k,l}$$

Next we show **Claim 2**. Before that, we need to discuss the finiteness of sequents that can appear in a cut-free cyclic proof for a given sequent, which is our key observation in this section.

Definition 11 (Extended subformulas). The set of extended subformulas $\mathrm{exSub}(\varphi)$ of φ is inductively defined as follows:

$$\begin{aligned}
\mathrm{exSub}(Q) &= \{Q\},\\
\mathrm{exSub}(\neg\varphi) &= \{\neg\varphi\} \cup \mathrm{exSub}(\varphi),\\
\mathrm{exSub}(\varphi_1 \Box \varphi_2) &= \{\varphi_1 \Box \varphi_2\} \cup \mathrm{exSub}(\varphi_1) \cup \mathrm{exSub}(\varphi_2) \quad (\Box \text{ is } \wedge, \vee),\\
\mathrm{exSub}(P_j, 0) &= \{P_j\},\\
\mathrm{exSub}(P_j, n+1) &= \{Q \mid Q \in \bigcup \Phi_j\} \cup \bigcup\{\mathrm{exSub}(P_k, n) \mid P_k \in \bigcup \Phi_j\},\\
\mathrm{exSub}(P_j) &= \bigcup_{n \geq 0} \mathrm{exSub}(P_j, n).
\end{aligned}$$

For a set X of formulas, $\mathrm{exSub}(X)$ is defined by $\bigcup\{\mathrm{exSub}(\varphi) \mid \varphi \in X\}$.

Lemma 2. *The following claims hold.*
(1) If φ is a subformula of ψ, then $\varphi \in \mathrm{exSub}(\psi)$ and $\mathrm{exSub}(\varphi) \subseteq \mathrm{exSub}(\psi)$.
(2) If $P \in \bigcup \Phi_j$, then $\mathrm{exSub}(P) \subseteq \mathrm{exSub}(P_j)$.
(3) $P_j \in \mathrm{exSub}(\varphi)$ implies $\mathrm{exSub}(P_j) \subseteq \mathrm{exSub}(\varphi)$.
(4) $\mathrm{exSub}(\varphi)$ is finite for any formula φ.

Proof. (1) is shown by induction on ψ. To show (2), assume $P \in \bigcup \Phi_j$. Then $\mathrm{exSub}(P, n) \subseteq \mathrm{exSub}(P_j, n+1) \subseteq \mathrm{exSub}(P_j)$ for any n. Hence we have $\mathrm{exSub}(P) \subseteq \mathrm{exSub}(P_j)$. (3) is shown by induction on φ using (2). We only consider the case of P_j. This case is immediately obtained from the claim: $P_k \in \mathrm{exSub}(P_u, n)$ implies

$\mathrm{exSub}(P_k) \subseteq \mathrm{exSub}(P_u)$ for any n, k, u. This claim is shown by induction on n. The case of $n = 0$ is easily shown since $P_k = P_u$. Assume $n > 0$ and $P_k \in \mathrm{exSub}(P_u, n)$. Then $P_k \in \mathrm{exSub}(P_s, n-1) \subseteq \mathrm{exSub}(P_s)$ for some $P_s \in \bigcup \Phi_u$. Hence, by (2), we have $\mathrm{exSub}(P_k) \subseteq \mathrm{exSub}(P_s) \subseteq \mathrm{exSub}(P_u)$. For (4), the finiteness of $\mathrm{exSub}(\varphi)$ is shown by induction on φ since the number of proposition symbols is finite.

Let X be a set of formulas. We define $\mathrm{Seq}(X)$ by $\{\Gamma \vdash \Delta \mid \Gamma \cup \Delta \subseteq \mathrm{exSub}(X)\}$. $\mathrm{Seq}(\Gamma \vdash \Delta)$ is also defined by $\mathrm{Seq}(\Gamma \cup \Delta)$. By the above lemma, $\mathrm{Seq}(X)$ is a finite set if X is finite. Hence $\mathrm{Seq}(S)$ is finite for any sequent S since both the antecedent and the succedent of S are finite.

The finiteness of cut-free cyclic proofs in $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ is stated as follows.

Proposition 2 (Finiteness of cut-free proofs in $\mathbf{CLKID}_\omega^{\mathbf{prop}}$). *We have the following claims.*

(1) Let $S \in \mathrm{Seq}(X)$ *and* $\frac{S_1 \ \cdots \ S_n}{S}$ *be an instance of an inference rule* $R \neq (Cut)$ *in* $CLKID_\omega^{prop}$*. Then* $S_i \in \mathrm{Seq}(X)$ *for all* $1 \leq i \in n$*.*

(2) Let $\mathcal{D}$ *be a cut-free derivation of* S *with open buds* $\mathcal{B}$ *in* $CLKID_\omega^{prop}$*. Then all sequents in* $\mathcal{D}$ *belong to* $\mathrm{Seq}(S)$*.*

Proof. The claim (1) is shown by case analysis of the inference rules in $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ by using Lemma 2. To show the cases (UL) and (UR), it is enough to prove the following fact: if $P_j \in \mathrm{exSub}(X)$, then $\bigcup \Phi_j \subseteq \mathrm{exSub}(X)$. Assume $P_j \in \mathrm{exSub}(X)$ and $\Phi_{j,l} = \{Q_1, \ldots, Q_k, P_{j_1}, \ldots, P_{j_m}\}$ taking arbitrary $l \in \{1, \ldots, K_j\}$. Then $\mathrm{exSub}(P_j) \subseteq \mathrm{exSub}(X)$ holds by Lemma 2 (3). Thus we have $Q_i \in \mathrm{exSub}(P_j, 1) \subseteq \mathrm{exSub}(P_j) \subseteq \mathrm{exSub}(X)$. We also have $P_{j_i} \in \mathrm{exSub}(P_{j_i}) \subseteq \mathrm{exSub}(P_j) \subseteq \mathrm{exSub}(X)$ by Lemma 2 (2). Therefore we obtain $\bigcup \Phi_j \subseteq \mathrm{exSub}(X)$ since $\Phi_{j,l} \subseteq \mathrm{exSub}(X)$ holds for any $1 \leq l \leq M_j$. The claim (2) is shown by induction on $\mathcal{D}$ using (1).

Definition 12. Let S be a normal sequent $\overrightarrow{Q}, P_{j_1}, \ldots, P_{j_m} \vdash \Xi$. The set

$$\mathrm{Ldec}(S) \stackrel{\mathrm{def}}{=} \left\{ \overrightarrow{Q}, \Phi_{j_1,l_1}, \ldots, \Phi_{j_m,l_m} \vdash \Xi \mid 1 \leq l_i \leq M_{j_i} \text{ for all } i \in \{1, \ldots, m\} \right\}$$

is called the *left decomposition* of S.

Let X and Y be sets of sequents. A sequent S is said to be *cut-free derivable from* X (denoted by $X \rhd S$) if S has a cut-free derivation with open buds $\mathcal{B}$ and all sequents at $\mathcal{B}$ are in X. We write $X \rhd Y$ if $X \rhd S$ for all $S \in Y$. We note that the relation $\rhd$ is transitive, namely $X \rhd Y$ and $Y \rhd Z$ implies $X \rhd Z$.

Lemma 3. $\mathrm{Ldec}(S) \rhd S$ *holds for any normal sequent* S*.*

Proof. Assume that S is $\overrightarrow{Q}, P_{j_1}, \ldots, P_{j_m} \vdash \Xi$. Define $\mathrm{Ldec}'(S, \{j_1, \ldots, j_k\})$ by

$$\left\{ \overrightarrow{Q}, P_{j_1}, \ldots, P_{j_k}, \Phi_{j_{k+1},l_{k+1}}, \ldots, \Phi_{j_m,l_m} \vdash \Xi \mid 1 \leq l_i \leq M_{j_i} \text{ for all } i \in \{k+1, \ldots, m\} \right\}.$$

We note that $\mathrm{Ldec}'(S, \emptyset) = \mathrm{Ldec}(S)$ and $\mathrm{Ldec}'(S, \{j_1, \ldots, j_m\}) = \{S\}$. It is enough to show the claim $\mathrm{Ldec}'(S, \{j_1, \ldots, j_k\}) \rhd \mathrm{Ldec}'(S, \{j_1, \ldots, j_{k+1}\})$ for any $k \in \{0, \ldots, m-1\}$, since

$$\mathrm{Ldec}(S) = \mathrm{Ldec}'(S, \emptyset) \rhd \mathrm{Ldec}'(S, \{j_1\}) \rhd \cdots \rhd \mathrm{Ldec}'(S, \{j_1, \ldots, j_m\}) = \{S\},$$

holds by the claim, and then $\mathrm{Ldec}(S) \rhd S$ is obtained from the transitivity of $\rhd$.

In order to show the claim, take arbitrary $S \in \mathrm{Ldec}'(S, \{j_1, \ldots, j_{k+1}\})$. Then S has the form $\overrightarrow{Q}, \overrightarrow{P}, P_{j_{k+1}}, \overrightarrow{\Phi} \vdash \Xi$, where $\overrightarrow{P}$ and $\overrightarrow{\Phi}$ are $P_{j_1}, \ldots, P_{j_k}$ and $\Phi_{j_{k+2},l_{k+2}}, \ldots, \Phi_{j_m,l_m}$, respectively. We have

$$\frac{\overrightarrow{Q}, \overrightarrow{P}, \Phi_{j_{k+1},1}, \overrightarrow{\Phi} \vdash \Xi \quad \cdots \quad \overrightarrow{Q}, \overrightarrow{P}, \Phi_{j_{k+1},M_{j_{k+1}}}, \overrightarrow{\Phi} \vdash \Xi}{\overrightarrow{Q}, \overrightarrow{P}, P_{j_{k+1}}, \overrightarrow{\Phi} \vdash \Xi} \,(\mathrm{UL}),$$

which is a cut-free derivation of S from $\mathrm{Ldec}'(S, \{j_1, \ldots, j_k\})$. Thus $\mathrm{Ldec}'(S, \{j_1, \ldots, j_k\}) \rhd S$ holds for any $\mathrm{Ldec}'(S, \{j_1, \ldots, j_{k+1}\})$. Hence $\mathrm{Ldec}'(S, \{j_1, \ldots, j_k\}) \rhd \mathrm{Ldec}'(S, \{j_1, \ldots, j_{k+1}\})$.

Input S_0: normal sequent
Output $(\mathcal{D}, \mathcal{R}, \mathcal{B})$: a cut-free $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$ proof $(\mathcal{D}, \mathcal{R})$ of S_0 with open buds $\mathcal{B}$

```
D := S_0                                   (single node (only root) derivation of S_0 with bud S_0)
A := {root}                                                           (set of current bud nodes)
B := ∅
R := ∅
while A ≠ ∅ do
    Take w ∈ A and let S be D(w).
    A := A \ {w}
    if S is strongly normal then
        B := {w} ∪ B;  continue                                                              (a)
    if an internal node v is on a path from root to w s.t. D(v) = D(w) then
        R := {(w, v)} ∪ R;  continue                                                         (b)
    Update D replacing S at the node w of D by (S_1 ··· S_n / S) Ldec                         (c)
    A := {w_1, ..., w_n} ∪ A,
    where w_1, ..., w_n are new children of w for S_1, ..., S_n, respectively.
done
return (D, B, R)
```

Fig. 2. Algorithm: `normProof`

The derivation of S from $\mathrm{Ldec}(S)$ constructed in the above proof is a multiple times application of the (UL) rule. We note that, for any $S_i \in \mathrm{Ldec}(S)$, all inductive predicates in $L(S)$ are eventually unfolded by the rule instances of the (UL)-rule in the path from S to S_i, that is, any trace following this path contains a progressing point.

We consider an inference rule (Ldec) of the form

$$\frac{S_1 \quad \cdots \quad S_n}{S}\ (\mathrm{Ldec}), \text{where } \{S_1, \ldots, S_n\} = \mathrm{Ldec}(S),$$

which is admissible in $\mathrm{CLKID}_\omega^{\mathrm{prop}}$.

Definition 13. $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$ is a proof system whose inference rules are obtained from those of $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ replacing the (UL)-rule by the (Ldec)-rule. A preproof in $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$ (with open buds) is defined in a similar way to that of $\mathrm{CLKID}_\omega^{\mathrm{prop}}$. A cyclic proof in $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$ (with open buds) is defined by a preproof in which any infinite path passes through an infinite number of rule instances of the (Ldec)-rule.

Lemma 4. *If there is a cut-free cyclic proof of S in $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$ with open buds $\mathcal{B}$, then there is a cut-free cyclic proof of S in $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ with open buds $\mathcal{B}$.*

Proof. Assume that $\mathcal{P}'$ is a cut-free cyclic proof of S in $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$ with open buds $\mathcal{B}$. Let $\mathcal{P} = ((N, l, r), \mathcal{R})$ be a cut-free preproof of S in $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ with open buds $\mathcal{B}$ obtained by replacing each rule instance of (Ldec) by a multiple application of (UL) as constructed in the proof of Lemma 3. Take an infinite path π of $\mathcal{P}$. Then define an infinite path π' of $\mathcal{P}'$ obtained from π by replacing each subsequence $(v_1, \ldots, v_m)$ of π by (v_1, v_m), where $l(v_1)$ is not a premise of a rule instance of (UL), $r(v_i) = $ (UL) for each $i \in \{1, \ldots, m-1\}$, and $r(v_m) \neq$ (UL). Then π' contains infinite number of rule instances of (Ldec) since $\mathcal{P}'$ is a cyclic proof of $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$. Hence any trace following π has infinite number of progressing points as we mentioned before. Therefore $\mathcal{P}$ is a cut-free $\mathrm{CLKID}_\omega^{\mathrm{prop}}$ cyclic proof of S with open buds $\mathcal{B}$.

The algorithm `normProof` given in Figure 2 constructs a $\mathrm{CLKID}_\omega^{\mathrm{prop}\prime}$ cyclic proof of a normal sequent with open buds of strongly normal sequents.

Lemma 5. `normProof` *terminates for any input.*

Proof. Assume that `normProof`(S) does not terminate for some S. We show a contradiction. Consider the non-terminating run of `normProof`(S). Let $\mathcal{D}_k$ be the $\mathcal{D}$ after the k-th while-loop in the run, $\mathcal{A}_k$ be the $\mathcal{A}$ after the k-th while-loop, and S_k be the sequent S taken in the k-th while-loop. We note that, in each loop, either line (a), (b), or (c) is executed, and line (b) is executed infinitely many times in the run since otherwise, from some k_0, the numbers of elements in $\mathcal{A}_k$ ($k \geq k_0$) strictly decrease and the run eventually terminates. If (b) is executed in the k-th loop, then $\mathcal{D}_k$ is strict extension of $\mathcal{D}_{k-1}$, since S_k is not a strongly normal and at least one (UL) is done in Ldec(S_k). Define $\mathcal{D}_\infty := \bigcup_{k\geq 0} \mathcal{D}_k$. Then $\mathcal{D}_\infty$ has infinite nodes with finite branches. Hence, by König's lemma, there is an infinite path $\pi = (\mathtt{root}, w_1, w_2, \ldots)$ in $\mathcal{D}_\infty$. For each w_j, there is unique n_j such that w_j is added to $\mathcal{A}$ in the n_j-th loop. In each n_j-th loop, (c) is executed and the sequent S_{n_j} does not appear in $\{S_0, S_{n_1}, \ldots, S_{n_{j-1}}\}$, since (b) is skipped in the loop. Let M be $|\mathrm{Seq}(S)|$ and X be $\{S_0, S_{n_1}, \ldots, S_{n_M}\}$. Hence $|X| = M+1$, but it contradicts $X \subseteq \mathrm{Seq}(S)$ obtained from Proposition (2).

Lemma 6. *Let S be a valid normal sequent, and $(\mathcal{D}, \mathcal{R}, \mathcal{B})$ be the output of* `normProof`(S)*. Then $(\mathcal{D}, \mathcal{R})$ is a cut-free CLKID$_\omega^{\mathrm{prop}'}$ cyclic proof of S with open buds $\mathcal{B}$. Moreover, the sequents that appear at nodes in $\mathcal{B}$ are strongly normal and valid.*

Proof. By the previous lemma, the run of `normProof`(S) terminates in K-times loop. Let $\mathcal{D}_k$, $\mathcal{R}_k$, $\mathcal{A}_k$, and $\mathcal{B}_k$ be the $\mathcal{D}$, the $\mathcal{R}$, the $\mathcal{A}$, and the $\mathcal{B}$ after the k-th loop in the run, respectively.

We show the claim that $(\mathcal{D}_k, \mathcal{R}_k)$ is a cut-free cyclic proof of S with open buds $\mathcal{A}_k \cup \mathcal{B}_k$ by induction on k. The case of $k = 0$ is easily shown, since $\mathcal{D}_0 = \{S\}$, $\mathcal{R}_0 = \emptyset$, and $\mathcal{A}_0 \cup \mathcal{B}_0 = \{\mathtt{root}\}$. Suppose $k > 0$. Let w be the node taken from $\mathcal{A}_{k-1}$ in the k-th loop, and S_k be $\mathcal{D}_{k-1}(w)$. The k-th loop executes either (a), (b), or (c). In the cases (a) and (c), we have the expected result by the induction hypothesis. In the case (b), there is an internal node v between `root` and w in $\mathcal{D}_{k-1}$ such that $\mathcal{D}_{k-1}(v) = S_k$. We have $\mathcal{D}_k = \mathcal{D}_{k-1}$, $\mathcal{R}_k = \{(w, v)\} \cup \mathcal{R}_{k-1}$, and $\mathcal{A}_k \cup \mathcal{B}_k = (\mathcal{A}_{k-1} \cup \mathcal{B}_{k-1}) \setminus \{w\}$. Then the only new infinite path in $\mathcal{D}_k$ is $\pi = (\mathtt{root}, \ldots, v, \ldots, w, v, \ldots, w, \ldots)$. We note that all inductive predicates in the antecedent of $\mathcal{D}_k(v)$ are unfolded before reaching $\mathcal{D}_k(w)$, since Ldec is applied at least once between v and w. The other infinite paths in $\mathcal{D}$ also pass through Ldec by the induction hypothesis. Hence $(\mathcal{D}_k, \mathcal{R}_k)$ is a CLKID$_\omega^{\mathrm{prop}'}$ cyclic proof of S with $\mathcal{A}_k \cup \mathcal{B}_k$.

By the claim, $(\mathcal{D}, \mathcal{R})$ is a cut-free CLKID$_\omega^{\mathrm{prop}'}$ cyclic proof of S with open buds $\mathcal{B}$, since $(\mathcal{D}, \mathcal{R}) = (\mathcal{D}_K, \mathcal{R}_K)$ and $\mathcal{B} = \mathcal{B}_K$ and $\mathcal{A}_K = \emptyset$.

We can easily check that, for any k, $\mathcal{A}_k \cup \mathcal{B}_k$ is a set of nodes whose sequents are strongly normal. Hence so is $\mathcal{B}$. We claim that $\mathcal{D}$ is constructed by using only the (UL)-rule. Thus all sequents in $\mathcal{D}$ are valid because all assumptions of the (UL)-rule are valid if its conclusion is valid.

The cut-free provability of valid normal sequents is obtained by combining the previous results.

Proposition 3. *Any valid normal sequent has a cut-free cyclic proof in CLKID$_\omega^{\mathrm{prop}}$.*

Proof. Let S be a valid normal sequent, and $(\mathcal{D}, \mathcal{R}, \mathcal{B})$ be the output of `normProof`(S). Then, by Lemma 6, $(\mathcal{D}, \mathcal{R})$ is a cut-free CLKID$_\omega^{\mathrm{prop}'}$ cyclic proof of S with open buds $\mathcal{B}$ and the sequents on $\mathcal{B}$ are strongly normal and valid. By Lemma 4 there is a cut-free CLKID$_\omega^{\mathrm{prop}}$ cyclic proof of S with open buds $\mathcal{B}$. The sequents on $\mathcal{B}$ have cut-free non-cyclic proofs by Proposition 1. Hence, by combining them, we can obtain a cut-free cyclic proof of S in CLKID$_\omega^{\mathrm{prop}}$.

Next, we prove the cut-free provability of valid sequents. It is shown by the fact that a valid sequent has a cut-free derivation with open buds of valid normal sequents. The cut-free derivation is constructed by the algorithm `normalization` given in Figure 3.

Define $|S|$ by the total number of the logical connectives in a sequent S. We note that `normalization`(S) terminates for any S, since, if `normalization`(S) has an infinite run, then there is a constructed derivation $\mathcal{D}$ with a path of length $|S|+1$ (say $(\mathtt{root}, w_1, \ldots, w_{|S|})$), and then we have a contradiction from $|S| = |\mathcal{D}(\mathtt{root})| > |\mathcal{D}(w_1)| > \cdots > |\mathcal{D}(w_{|S|})| \geq 0$. Hence, for any S, `normalization`(S) returns $(\mathcal{D}, \mathcal{B})$, where $\mathcal{D}$ is a derivation of S with open buds $\mathcal{B}$. Also we can easily see that all sequents on nodes in $\mathcal{B}$ are normal.

We now show the cut-free completeness theorem of CLKID$_\omega^{\mathrm{prop}}$.

Proof (Proof of Theorem 2). Let S be a valid sequent and $(\mathcal{D}, \mathcal{B})$ be the result of `normalization`(S). Then $\mathcal{D}$ is a derivation of S with open buds $\mathcal{B}$ of normal sequents. Let $\{w_1, \ldots, w_n\}$ be $\mathcal{B}$ and S_j be $\mathcal{D}(w_j)$ for $1 \leq j \leq n$. We claim that, in each case (a)–(f) of the while-loop, $\mathcal{D}$ is extended keeping validity of sequents,

Input S_0: sequent
Output $(\mathcal{D}, \mathcal{B})$, where $\mathcal{D}$ is a cut-free derivation of S_0 with open buds $\mathcal{B}$

```
D := S_0                                    (single node (only root) derivation of S_0 with bud S_0)
G := {root}                                                              (set of unfinished nodes)
B := ∅
while G ≠ ∅ do
    Take w ∈ G and let S be D(w).
    G := G \ {w}
    if S is normal then
        B := {w} ∪ B;  continue
    if S = Γ, φ1 ∧ φ2 ⊢ Δ then                                                                (a)
        let S' be Γ, φ1, φ2 ⊢ Δ; update D replacing S at w by S' / S (∧L) [double line];
        G := {w'} ∪ G, where w' is the new node for S';  continue
    if S = Γ, φ1 ∨ φ2 ⊢ Δ then                                                                (b)
        let S'1 and S'2 be Γ, φ1 ⊢ Δ and Γ, φ2 ⊢ Δ, respectively;
        update D replacing S at w by S'1 S'2 / S (∨L);
        G := {w'1, w'2} ∪ G, where w'1 and w'2 are the new nodes for S'1 and S'2;  continue
    if S = Γ, ¬φ ⊢ Δ then                                                                     (c)
        let S' be Γ ⊢ Δ, φ;
        update D replacing S at w by S' / S (¬L);
        G := {w'} ∪ G, where w' is the new node for S';  continue
    if S = Γ ⊢ Δ, φ1 ∨ φ2 then                                                                (d)
        let S' be Γ ⊢ Δ, φ1, φ2;
        update D replacing S at w by S' / S (∨R) [double line];
        G := {w'} ∪ G, where w' is the new node for S';  continue
    if S = Γ ⊢ Δ, φ1 ∧ φ2 then                                                                (e)
        let S'1 and S'2 be Γ ⊢ Δ, φ1 and Γ ⊢ Δ, φ2, respectively;
        update D replacing S at w by S'1 S'2 / S (∧R);
        G := {w'1, w'2} ∪ G, where w'1 and w'2 are the new nodes for S'1 and S'2;  continue
    if S = Γ ⊢ Δ, ¬φ then                                                                     (f)
        let S' be Γ, φ ⊢ Δ;
        update D replacing S at w by S' / S (¬R);
        G := {w'} ∪ G, where w' is the new node for S';  continue
done
return (D, B)
```

Fig. 3. Algorithm: `normalization`

that is, if a bud node of $\mathcal{D}$ is valid, then the additional nodes are also valid. Hence $S_1, \ldots, S_n$ are valid normal sequents. By Proposition 3, S_j has a cut-free cyclic proof for any j. Therefore S has a cut-free cyclic proof.

Theorem 3 (Cut-elimination property of $\mathbf{CLKID}_\omega^{\mathbf{prop}}$). *Any provable sequent in $CLKID_\omega^{prop}$ has a cut-free cyclic proof.*

Proof. Let S be a provable sequent in $\mathrm{CLKID}_\omega^{\mathrm{prop}}$. Then it is valid by soundness. Hence S has a cut-free cyclic proof by Theorem 2.

4 Failure of Cut-Elimination for $\mathbf{CLBI}_{\mathbf{ID0}}^{\boldsymbol{\omega}}$

This section is a summary of the result in [8].

In this section, we show the cut-elimination fails for the cyclic proof system of the bunched logic $\mathrm{CLBI}_{\mathrm{ID0}}^\omega$, which is a core subsystem of the logic in [1].

4.1 Core Bunched Logic $\mathbf{BI}_{\mathbf{ID0}}$

As with $\mathrm{CLKID}_\omega^{\mathrm{prop}}$, we fix a signature consisting of non-inductive and inductive propositional symbols. In this section, we use metavariables A, B,... for non-inductive propositions and P, Q,... for inductive propositions.

Definition 14 (Formulas of $\mathbf{BI}_{\mathrm{ID0}}$). *Let I and $\top$ be propositional constants. The* formulas *of BI_{ID0}, denoted by ϕ, ψ,..., are defined as*

$$\phi ::= I \mid \top \mid A \mid P \mid \phi * \phi \mid \phi \wedge \phi.$$

In this paper, $*$ and $\wedge$ are treated as left-associative operators, that is, we write $\phi_1 * \phi_2 * \phi_3$ for $(\phi_1 * \phi_2) * \phi_3$. The notation A^n denotes $A * \cdots * A$ where the number of A's is n. We also use the notation $P * A^n$ for $P * A * \cdots * A$, namely $(\cdots((P * A) * A) \cdots) * A)$.

Definition 15 (Bunch). *The* bunches*, denoted by $\Gamma, \Delta, \ldots$, are defined as*

$$\Gamma, \Delta ::= \phi \mid \Gamma, \Gamma \mid \Gamma; \Gamma.$$

We write $\Gamma(\Delta)$ to mean that Γ of which Δ is a subtree. For a bunch $\Gamma(\Delta)$, $\Gamma(\Delta')$ is a bunch obtained by replacing the subtree Δ of Γ by Δ'.

The labels "," and ";" intuitively mean $*$ and $\wedge$, respectively. For a bunch Γ, we define the bunch formula ϕ_Γ as the formula defined as:

$$\begin{aligned} \phi_\Gamma &= \Gamma, && (\Gamma \text{ is a formula}); \\ \phi_{\Gamma_1,\Gamma_2} &= \phi_{\Gamma_1} * \phi_{\Gamma_2}; \\ \phi_{\Gamma_1;\Gamma_2} &= \phi_{\Gamma_1} \wedge \phi_{\Gamma_2}. \end{aligned}$$

Definition 16 (Equivalence of bunches). *Define the bunch equivalence $\equiv$ as the least equivalence relation satisfying:*

- *commutative monoid equations for ',' and I;*
- *commutative monoid equations for ';' and $\top$;*
- *congruence: if $\Delta \equiv \Delta'$ then $\Gamma(\Delta) \equiv \Gamma(\Delta')$.*

We use the notation $|\phi|$ and $|\Gamma|$ for the sizes of the formulas and the bunches, which are defined as usual.

In the case of the bunched logic, the inductive propositions are defined by not only the ordinary conjunctions but also multiplicative conjunctions, so the rules of inductive definitions are slightly generalized as follows.

Definition 17 (Inductive definition). *An* inductive definition clause *of P is of the form $P := \phi$. For a set Φ of inductive definition clauses of inductive propositions, we define $\Phi_P = \{\phi \mid P := \phi \in \Phi\}$. We say that P is defined by $P := \phi_1 \mid \cdots \mid \phi_k$ in Φ if and only if $\Phi_P = \{\phi_1, \cdots, \phi_k\}$.*

Definition 18 ($\mathbf{BI}_{\mathrm{ID0}}$ sequent). *Let Γ be a bunch and ϕ be a formula. $\Gamma \vdash \phi$ is called a* $\mathrm{BI}_{\mathrm{ID0}}$ sequent*. Γ is called the* antecedent *of $\Gamma \vdash \phi$ and ϕ is called the* succedent *of $\Gamma \vdash \phi$. We define $L(\Gamma \vdash \phi) = \Gamma$ and $R(\Gamma \vdash \phi) = \phi$.*

The *standard models* of $\mathrm{BI}_{\mathrm{ID0}}$ are defined in [1]. However, in the following we need only a particular class of the standard models, called the *multiset models*. For the set of atomic propositions $\{A_1, \ldots, A_n\}$, the multiset model M_{multi} for Σ is the tuple $\langle R_{\mathrm{multi}}, \uplus, \emptyset \rangle$ such that

- R_{multi} is the set of multisets consisting of $\mathsf{a}_1, \ldots, \mathsf{a}_n$;
- $\uplus$ is the merging operation of two multisets;

– the satisfaction relation $r \models \phi$ for $r \in R_{\text{multi}}$ is given as:

$$\begin{aligned}
r \models \top &\quad \text{always holds} \\
r \models I &\iff r = \emptyset \\
r \models A_i &\iff r = \{\mathsf{a}_i\} \text{ (for an atomic proposition } A_i) \\
r \models P &\iff r \models P^{(m)} \text{ for some } m \\
r \models P^{(0)} &\quad \text{never holds} \\
r \models P^{(m+1)} &\iff r \models \phi[P_1^{(m)}, \ldots, P_k^{(m)}/P_1, \ldots, P_k] \\
&\quad\ \text{for some } \phi \in \Phi_P \text{ containing inductive propositions } P_1, \ldots, P_k \\
r \models \phi_1 \wedge \phi_2 &\iff r \models \phi_1 \text{ and } r \models \phi_2 \\
r \models \phi_1 * \phi_2 &\iff r = r_1 \circ r_2 \text{ and } r_1 \models \phi_1 \text{ and } r_2 \models \phi_2 \\
&\quad\ \text{for some } r_1, r_2 \in R\ .
\end{aligned}$$

For example, $\{\mathsf{a}_1\} \models A_1$, $\{\mathsf{a}_1, \mathsf{a}_2\} \models A_1 * A_2$, and $\{\mathsf{a}_1, \mathsf{a}_1\} \models A_1 * A_1 * I$ are true, and $\{\mathsf{a}_1\} \models A_2$ and $\{\mathsf{a}_1\} \models A_1 * A_1$ are false.

The cyclic proof system $\mathrm{CLBI}^{\omega}_{\mathrm{ID0}}$ for the core bunched logic is defined as with $\mathrm{CLKID}^{\mathrm{prop}}_{\omega}$ by the following inference rules.

Definition 19. *The inference rules of $CLBI^{\omega}_{\mathrm{ID0}}$ are the following.*

$$\frac{}{\phi \vdash \phi}\ (Ax) \qquad \frac{\Gamma \vdash \phi \quad \Delta(\phi) \vdash \psi}{\Delta(\Gamma) \vdash \psi}\ (Cut),$$

$$\frac{\Gamma(\Delta) \vdash \phi}{\Gamma(\Delta; \Delta') \vdash \phi}\ (W) \qquad \frac{\Gamma(\Delta; \Delta) \vdash \phi}{\Gamma(\Delta) \vdash \phi}\ (C) \qquad \frac{\Gamma \vdash \phi}{\Delta \vdash \phi}\ (E) \quad (\Delta \equiv \Gamma),$$

$$\frac{\Gamma(\phi, \psi) \vdash \chi}{\Gamma(\phi * \psi) \vdash \chi}\ (*L) \qquad \frac{\Gamma \vdash \phi \quad \Delta \vdash \psi}{\Gamma, \Delta \vdash \phi * \psi}\ (*R) \qquad \frac{\Gamma(\phi; \psi) \vdash \chi}{\Gamma(\phi \wedge \psi) \vdash \chi}\ (\wedge L) \qquad \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ (\wedge R).$$

$$\frac{\Gamma(\phi_1) \vdash \phi \quad \cdots \quad \Gamma(\phi_n) \vdash \phi}{\Gamma(P) \vdash \phi}\ (UL) \qquad \frac{\Gamma \vdash \phi_i}{\Gamma \vdash P}\ (UR) \quad (1 \le i \le n),$$

where the inductive predicate P is defined by $P := \phi_1 \mid \ldots \mid \phi_n$. (UL) and (UR) are called unfolding rules.

The soundness is proved in [1].

Theorem 4 (Soundness $\mathbf{CLBI}^{\omega}_{\mathrm{ID0}}$). *If $\Gamma \vdash \phi$ is provable in $CLBI^{\omega}_{\mathrm{ID0}}$, then $\Gamma \vdash \phi$ is valid.*

4.2 Proof Unrolling

Our proof of the failure of the cut-elimination relies on a new technique, called *proof unrolling*: for a given cyclic proof of $\Gamma \vdash \phi$ and a bunch Γ' that is obtained by completely unfolding the inductive predicates in Γ, we can construct a non-cyclic proof of $\Gamma' \vdash \phi$ by unrolling the cycles in the given cyclic proof.

For example, consider two inductive propositions P_A and P_{AA}, whose inductive definitions are:

$$\mathrm{P}_A := I \mid \mathrm{P}_A * A \qquad\qquad \mathrm{P}_{AA} := I \mid \mathrm{P}_{AA} * A * A.$$

For these inductive propositions, the sequent $\mathrm{P}_{AA} \vdash \mathrm{P}_A$ is provable in $\mathrm{CLBI}^{\omega}_{\mathrm{ID0}}$ as Figure 4. The sequents marked (†) are corresponding bud and companion. The numbers (1), (2), ... are identifiers of sequents.

From this cyclic proof, we can construct a non-cyclic proof of $I * A * A * A * A \vdash P_A$ for $I * A * A * A * A \in \mathrm{Unf}(P_{AA})$ by the proof unrolling as Figure 5. The identifiers of sequents indicate the corresponding nodes in the cyclic proof, where we unroll the cycle at (†) twice, and for (UL) in the cyclic proof, we choose the right premise twice at (3) and the left premise at (2).

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          \dfrac{
            \dfrac{\mathrm{P}_{AA} \vdash \mathrm{P}_A (8)(\dagger) \qquad \dfrac{}{A \vdash A(9)}\,(Ax)}{\mathrm{P}_{AA}, A \vdash \mathrm{P}_A * A(7)}\,(*R)
          }{\mathrm{P}_{AA}, A \vdash \mathrm{P}_A(6)}\,(UR)
          \qquad \dfrac{}{A \vdash A(10)}\,(Ax)
        }{(\mathrm{P}_{AA}, A), A \vdash \mathrm{P}_A * A(5)}\,(*R)
      }{(\mathrm{P}_{AA}, A), A \vdash \mathrm{P}_A(4)}\,(UR)
    }{\mathrm{P}_{AA} * A, A \vdash \mathrm{P}_A(3)}\,(*L)
  }{\mathrm{P}_{AA} * A * A \vdash \mathrm{P}_A}\,(*L)
  \qquad
  \dfrac{\dfrac{}{I \vdash I(2)}\,(Ax)}{I \vdash \mathrm{P}_A}\,(UR)
}{\mathrm{P}_{AA} \vdash \mathrm{P}_A(1)(\dagger)}\,(UL)
$$

Fig. 4. Cyclic proof of $\mathrm{P}_{AA} \vdash \mathrm{P}_A$

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                      \dfrac{\dfrac{}{I \vdash I(2)}\,(Ax')}{I \vdash \mathrm{P}_A(8)}\,(UR)
                    }{I, A \vdash \mathrm{P}_A * A(7)}\,(*L)
                  }{I, A \vdash \mathrm{P}_A(6)}\,(UR)
                  \qquad \dfrac{}{A \vdash A(10)}\,(Ax')
                }{(I, A), A \vdash \mathrm{P}_A * A(5)}\,(*L)
              }{(I, A), A \vdash \mathrm{P}_A(4)}\,(UR)
            }{I * A, A \vdash \mathrm{P}_A(3)}\,(*L)
          }{I * A * A \vdash \mathrm{P}_A(8)}\,(*L)
          \qquad \dfrac{}{A \vdash A(9)}\,(Ax')
        }{I * A * A, A \vdash \mathrm{P}_A * A(7)}\,(*R)
      }{I * A * A, A \vdash \mathrm{P}_A(6)}\,(UR)
      \qquad \dfrac{}{A \vdash A(10)}\,(Ax')
    }{(I * A * A, A), A \vdash \mathrm{P}_A * A(5)}\,(*R)
  }{(I * A * A, A), A \vdash \mathrm{P}_A(4)}\,(UR)
}{
  \dfrac{I * A * A * A, A \vdash \mathrm{P}_A(3)}{I * A * A * A * A \vdash \mathrm{P}_A(1)}\,(*L)
}\,(*L)
$$

Fig. 5. Non-cyclic proof of $I * A * A * A * A \vdash \mathrm{P}_A$ constructed by proof unrolling

4.3 Failure of Cut-Elimination

We give a counterexample of the cut-elimination property in $\mathrm{CLBI}^{\omega}_{\mathrm{ID0}}$. We fix the language Σ consisting of the atomic propositions A and B, and the inductive propositions P_{AB}, P_{BA}, P_{A}, and P_{B}. We also fix the set Φ of inductive definitions for P_{AB}, P_{BA}, P_{A}, and P_{B} defined by:

$$\begin{array}{ll} \mathrm{P}_{AB} := \mathrm{P}_{B} \mid \mathrm{P}_{AB} * A; & \qquad \mathrm{P}_{A} := I \mid \mathrm{P}_{A} * A; \\ \mathrm{P}_{BA} := \mathrm{P}_{A} \mid \mathrm{P}_{BA} * B; & \qquad \mathrm{P}_{B} := I \mid \mathrm{P}_{B} * B. \end{array}$$

Intuitively, P_{A} and P_{B} mean $I * A^n$ and $I * B^m$ with arbitrary $n, m \geq 0$, respectively. P_{AB} and P_{BA} mean $(I * B^m) * A^n$ and $(I * A^m) * B^n$ with arbitrary $n, m \geq 0$, respectively. We note that P_{AB} and P_{BA} are logically equivalent in the standard models since the separating conjunction $*$ and the formula I are interpreted as a commutative monoid operator and the unit of it, respectively.

The intention of the name P_{AB} is that, during the unfolding of P_{AB}, A's appear first, and then B's appear in the unfolding of P_{B}. P_{BA} is also named with a similar intention.

Then, we show that the entailment $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ is a counterexample for the cut-elimination.

First, a cyclic proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ in $\mathrm{CLBI}^{\omega}_{\mathrm{ID0}}$ with (Cut) is given in Figure 6.

$$\dfrac{\dfrac{\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}(@) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\mathrm{P}_{A} \vdash \mathrm{P}_{A}}(Ax) \quad \dfrac{}{A \vdash A}(Ax)}{\mathrm{P}_{A}, A \vdash \mathrm{P}_{A} * A}(*R)}{\mathrm{P}_{A}, A \vdash \mathrm{P}_{A}}(UR)}{\mathrm{P}_{A}, A \vdash \mathrm{P}_{BA}}(UR) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathrm{P}_{BA}, A \vdash \mathrm{P}_{BA}(\#) \quad \dfrac{}{B \vdash B}(Ax)}{(\mathrm{P}_{BA}, A), B \vdash \mathrm{P}_{BA} * B}(*R)}{(\mathrm{P}_{BA}, B), A \vdash \mathrm{P}_{BA} * B}(E)}{(\mathrm{P}_{BA}, B), A \vdash \mathrm{P}_{BA}}(UR)}{\mathrm{P}_{BA} * B, A \vdash \mathrm{P}_{BA}}(*L)}{\mathrm{P}_{BA}, A \vdash \mathrm{P}_{BA}(\#)}(UL)}{\mathrm{P}_{AB}, A \vdash \mathrm{P}_{BA}}(Cut)}{\mathrm{P}_{AB} * A \vdash \mathrm{P}_{BA}\ (1)}(*L)$$

is the subproof of the following proof figure:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{I \vdash I}(Ax)}{I \vdash \mathrm{P}_{A}}(UR)}{I \vdash \mathrm{P}_{BA}}(UR) \qquad \dfrac{\dfrac{\dfrac{\mathrm{P}_{B} \vdash \mathrm{P}_{BA}(\dagger) \quad \dfrac{}{B \vdash B}(Ax)}{\mathrm{P}_{B}, B \vdash \mathrm{P}_{BA} * B}(*R)}{\mathrm{P}_{B}, B \vdash \mathrm{P}_{BA}}(UR)}{\mathrm{P}_{B} * B \vdash \mathrm{P}_{BA}}(*L)}{\mathrm{P}_{B} \vdash \mathrm{P}_{BA}(\dagger)}(UL) \qquad \begin{array}{c} \vdots \text{ the above proof figure} \\ \mathrm{P}_{AB} * A \vdash \mathrm{P}_{BA}\ (1) \end{array}}{\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}(@)}(UL) \quad ,$$

Each bud marked (†), (@), or (#) has its companion with the same mark.

Fig. 6. $\mathrm{CLBI}^{\omega}_{\mathrm{ID0}}$ proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$

Proposition 4. $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$ *is not cut-free provable in* $\mathit{CLBI}^{\omega}_{\mathrm{ID0}}$.

Here we give a proof sketch. We assume the existence of a cut-free cyclic proof of $\mathrm{P}_{AB} \vdash \mathrm{P}_{BA}$. By the proof unrolling, we can construct non-cyclic proofs of $\phi \vdash \mathrm{P}_{BA}$ for any unfolded formula ϕ of P_{AB}. Hence we have proofs of $I * A^n \vdash \mathrm{P}_{BA}$ for arbitrary n. We consider parts of the proofs of $I * A^n \vdash \mathrm{P}_{BA}$ which contain the conclusion and do not contain the rule (UR). We call such parts the *proof segments*. In such a proof segment, $\{\mathsf{a}^n\} \in M_{\mathrm{multi}}$ satisfies every antecedent. Then, $\{\mathsf{a}^n\}$ also satisfies every antecedent in the corresponding part of the cyclic proof. Since the cyclic proof is finite, for a sufficiently large n, the antecedents cannot contain A^n, but they must contain either P_{AB} or $\top$, and then both $\{\mathsf{a}^n\}$ and $\{\mathsf{a}^n, \mathsf{b}\}$ satisfy the antecedents. On the other hand, since the proof segment does not contain (UR), every succedent is P_{BA}. When we unfold P_{BA}, we have to decide either P_{A} or $\mathrm{P}_{BA} * B$. However, neither of them can be satisfied by both $\{\mathsf{a}^n\}$ and $\{\mathsf{a}^n, \mathsf{b}\}$.

Theorem 5 (Failure of cut-elimination in $\mathbf{CLBI}^{\omega}_{\mathrm{ID0}}$[8]). *$CLBI^{\omega}_{\mathrm{ID0}}$ does not enjoy the cut-elimination property.*

This result is easily extended to the original cyclic proof system $CLBI^{\omega}_{ID}$ in [1], which contains full logical connectives of the bunched logic and inductive predicates with arbitrary arity.

Corollary 1 (Failure of cut-elimination in $CLBI^{\omega}_{ID}$[8]). *$CLBI^{\omega}_{ID}$ does not enjoy cut-elimination property.*

5 Conclusion

We have considered two cyclic proofs with inductively defined propositions: $\mathrm{CLKID}^{\mathrm{prop}}_{\omega}$ for the ordinary propositional logic and $\mathrm{CLBI}^{\omega}_{\mathrm{ID0}}$ for the bunched logic. We have proved the cut-elimination holds for $\mathrm{CLKID}^{\mathrm{prop}}_{\omega}$, while fails for $\mathrm{CLBI}^{\omega}_{\mathrm{ID0}}$. In [8], it was also discussed that the proof of the failure of the cut-elimination can be applied to the (multiplicative) linear logic and the separation logic with nullary inductive predicates.

It is interesting for future work to investigate the reason for success or failure of the cut-elimination of the cyclic proof systems.

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