

ON DELIGNE'S CONJECTURE FOR SYMMETRIC FIFTH L -FUNCTIONS AND QUADRUPLE PRODUCT L -FUNCTIONS OF MODULAR FORMS

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1. INTRODUCTION AND MAIN RESULTS

This report is based on a talk given at the RIMS conference "Automorphic form, automorphic L -functions and related topics" which was held online in January, 2022.

In [Del79], Deligne proposed a remarkable conjecture on the algebraicity of critical values of L -functions of motives, in terms of the periods obtained by comparing the Betti and de Rham realizations of the motives. As special cases, we consider the conjecture for symmetric power L -functions and tensor product L -functions of modular forms.

1.1. Symmetric power L -functions. Let

$$f(\tau) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_{\kappa}(N, \omega), \quad q = e^{2\pi\sqrt{-1}\tau}$$

be a normalized elliptic modular newform of weight $\kappa \geq 2$, level N , and nebentypus ω . For each prime $p \nmid N$, denote by α_p, β_p the Satake parameters of f at p and put

$$A_p = \begin{pmatrix} \alpha_p & 0 \\ 0 & \beta_p \end{pmatrix}.$$

Recall that α_p, β_p are the roots of the Hecke polynomial $X^2 - a_f(p)X + p^{\kappa-1}\omega(p)$. For $n \geq 1$, the symmetric n -th power L -function $L(s, \text{Sym}^n(f))$ is defined by an Euler product

$$L(s, \text{Sym}^n(f)) = \prod_p L_p(s, \text{Sym}^n(f)), \quad \text{Re}(s) > 1 + \frac{n(\kappa-1)}{2}.$$

Here the Euler factors are defined by

$$L_p(s, \text{Sym}^n(f)) = \det(\mathbf{1}_{n+1} - \text{Sym}^n(A_p) \cdot p^{-s})^{-1}$$

for $p \nmid N$, where $\text{Sym}^n : \text{GL}_2(\mathbb{C}) \rightarrow \text{GL}_{n+1}(\mathbb{C})$ is the symmetric n -th power representation. By the result of Barnet-Lamb, Geraghty, Harris, and Taylor [BLGHT11, Theorem B], the symmetric power L -functions admit meromorphic continuation to the whole complex plane and satisfy functional equations relating $L(s, \text{Sym}^n(f))$ to $L(1 + n(\kappa - 1) - s, \text{Sym}^n(f^\vee))$, where $f^\vee \in S_{\kappa}(N, \omega^{-1})$ is the normalized newform dual to f . The archimedean local factors are defined by

$$L_{\infty}(s, \text{Sym}^n(f)) = \begin{cases} \Gamma_{\mathbb{R}}(s - \frac{n(\kappa-1)}{2}) \prod_{i=0}^{r-1} \Gamma_{\mathbb{C}}(s - i(\kappa - 1)) & \text{if } n = 2r \text{ and } r(\kappa - 1) \text{ is even,} \\ \Gamma_{\mathbb{R}}(s - \frac{n(\kappa-1)}{2} + 1) \prod_{i=0}^{r-1} \Gamma_{\mathbb{C}}(s - i(\kappa - 1)) & \text{if } n = 2r \text{ and } r(\kappa - 1) \text{ is odd,} \\ \prod_{i=0}^r \Gamma_{\mathbb{C}}(s - i(\kappa - 1)) & \text{if } n = 2r + 1. \end{cases}$$

Here

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(\frac{s}{2}), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s).$$

A critical point for $L(s, \text{Sym}^n(f))$ is an integer m such that $L_{\infty}(s, \text{Sym}^n(f))$ and $L_{\infty}(1+n(\kappa-1)-s, \text{Sym}^n(f^\vee))$ are holomorphic at $s = m$. Associated to f , we have a pure motive M_f over \mathbb{Q} of rank 2 with coefficients in $\mathbb{Q}(f)$, which was constructed by Deligne [Del71] and Scholl [Sch90], such that

$$L(M_f, s) = (L(s, \sigma f))_{\sigma: \mathbb{Q}(f) \rightarrow \mathbb{C}}.$$

We have the Hodge decomposition

$$H_B(M_f) \otimes_{\mathbb{Q}} \mathbb{C} = H_B^{0, \kappa-1}(M_f) \oplus H_B^{\kappa-1, 0}(M_f)$$

as well as the Hodge filtration

$$H_{dR}(M_f) = F^0(M_f) \supsetneq F^{\kappa-1}(M_f) \supsetneq 0.$$

The comparison isomorphism

$$I_\infty : H_B(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H_{dR}(M_f) \otimes_{\mathbb{Q}} \mathbb{C}$$

induces

$$I_\infty^\pm : H_B^\pm(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \hookrightarrow H_B(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H_{dR}(M_f) \otimes_{\mathbb{Q}} \mathbb{C} \rightarrow H_{dR}(M_f)/F^{\kappa-1}(M_f) \otimes_{\mathbb{Q}} \mathbb{C}.$$

The Deligne's periods of M_f are elements in $(\mathbb{Q}(f) \otimes_{\mathbb{Q}} \mathbb{C})^\times / \mathbb{Q}(f)^\times$ defined by

$$\delta(M_f) := \det(I_\infty), \quad c^\pm(M_f) := \det(I_\infty^\pm),$$

where the determinants are computed with respect to $\mathbb{Q}(f)$ -rational bases on both sides. Consider the symmetric power motive $\text{Sym}^n(M_f)$. We have

$$L(\text{Sym}^n(M_f), s) = (L(s, \text{Sym}^n(\sigma f)))_{\sigma: \mathbb{Q}(f) \rightarrow \mathbb{C}}.$$

In [Del79, Proposition 7.7], Deligne computed the periods of $\text{Sym}^n(M_f)$. More precisely, we have

$$c^\pm(\text{Sym}^n(M_f)) = \begin{cases} \delta(M_f)^{r(r\pm 1)/2} (c^+(M_f) c^-(M_f))^{r(r+1)/2} & \text{if } n = 2r, \\ \delta(M_f)^{r(r+1)/2} c^\pm(M_f)^{(r+1)(r+2)/2} c^\mp(M_f)^{r(r+1)/2} & \text{if } n = 2r + 1. \end{cases}$$

As a special case of the conjecture in [Del79, Conjecture 2.8], we have the following:

Conjecture 1.1 (Deligne). *Let $m \in \mathbb{Z}$ be a critical point for $\text{Sym}^n(M_f)$. We have*

$$\frac{L(\text{Sym}^n(M_f), m)}{(2\pi\sqrt{-1})^{d^{(-1)^m}(\text{Sym}^n(M_f))m} \cdot c^{(-1)^m}(\text{Sym}^n(M_f))} \in \mathbb{Q}(f),$$

where $d^+(\text{Sym}^n(M_f)) = r + 1$, $d^-(\text{Sym}^n(M_f)) = r$ if $n = 2r$, and $d^\pm(\text{Sym}^n(M_f)) = r + 1$ if $n = 2r + 1$.

The conjecture holds if f is a CM-form. For general f , as explained in [Del79, § 7], the conjecture is known if $n = 1$. It was then considered by various authors when $n = 2, 3, 4, 6$ listed as follows:

- $n = 2$: Sturm [Stu80], [Stu89].
- $n = 3$: Garrett–Harris [GH93] and C.- [Che21a].
- $n = 4, 6$: Morimoto [Mor21] and C.- [Che21b], [Che21c].

In these cases, the conjecture was proved using the integral representations of automorphic L -functions and their algebraic/cohomological interpretations. When $n = 2$, we have the integral representation discovered by Shimura [Shi75]. When $n = 3$, the symmetric cube L -function appears as a factor of the triple product L -function $L(s, f \otimes f \otimes f)$ for which we have the integral representation due to Garrett [Gar87]. For $n = 2, 3$, the ideas for the proof of algebraicity of these integral representations are similar to the ones in the pioneering work of Shimura [Shi76]. The authors consider holomorphic Eisenstein series integrated against complex conjugation of elliptic modular forms. In [Mor21], Morimoto observed that (twisted) symmetric even power L -functions are factors of adjoint L -functions of unitary groups. In [GL21], Grobner and Lin proved a period relation between the Betti–Whittaker periods of cohomological conjugate self-dual cuspidal automorphic representations of GL_N over CM-fields and certain special values of adjoint L -functions of unitary groups. On the other hand, we have the result of Raghuram [Rag10], [Rag16] which expressed the algebraicity of critical values of Rankin–Selberg L -functions for $\text{GL}_N \times \text{GL}_{N-1}$ in terms of product of Betti–Whittaker periods. Therefore, Conjecture 1.1 for $n = 4, 6$ (under some assumptions) then follows from the algebraicity results of Morimoto [Mor14], [Mor18] for $\text{GSp}_4 \times \text{GL}_2$ and Garrett–Harris [GH93] for $\text{GL}_2 \times \text{GL}_2 \times \text{GL}_2$. In [Che21b], based on the same idea, we show that Conjecture 1.1 holds for $n = 4$ when $\kappa \geq 3$ by generalizing and refining the results of Grobner–Lin [GL21] to essentially conjugate self-dual representations in the case $\text{GL}_3 \times \text{GL}_2$. In [Che21c], we show that Conjecture 1.1 holds for $n = 6$ when $\kappa \geq 6$. We extend the result of Morimoto based on a different approach. The observation is that the (twisted) symmetric sixth power L -function is a factor of the adjoint L -function of the Kim–Ramakrishnan–Shahidi lift of f to GSp_4 . We define the de Rham–Whittaker periods associated to globally generic cohomological cuspidal automorphic representations of GSp_4 . In the case of the Kim–Ramakrishnan–Shahidi lift, we establish some periods relations between the de Rham–Whittaker periods and powers of Petersson norms of f . The conjecture then follows from our previous results [CI19], [Che22a]. Following is our main result for $n = 5$ (see also Remark 1.3 for higher n):

Theorem 1.2 ([Che22c]). *If $\kappa \geq 6$, then Conjecture 1.1 holds.*

Remark 1.3. Recently, we have proved Conjecture 1.1 in [Che22b, Theorem 5.11] when n is odd, κ is odd, and $\kappa \geq 5$. It's an ongoing project of the author to prove Conjecture 1.1 when n is even under the same assumptions on κ .

1.2. Quadruple product L -functions. As another example of Deligne's conjecture, we consider quadruple product L -functions of modular forms. Let $f_i \in S_{\kappa_i}(N_i, \omega_i)$ be normalized elliptic newform for $i = 1, 2, 3, 4$. Define the quadruple product L -function $L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4)$ by an Euler product

$$L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4) = \prod_p L_p(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4), \quad \operatorname{Re}(s) > 1 + \sum_{i=1}^4 \frac{\kappa_i - 1}{2}.$$

Here the Euler factors are given by

$$L_p(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4) = \det(\mathbf{1}_{16} - A_{1,p} \otimes A_{2,p} \otimes A_{3,p} \otimes A_{4,p} \cdot p^{-s})^{-1}$$

for $p \nmid N_1 N_2 N_3 N_4$. By the results of Jacquet–Shalika [JS81a], [JS81b] and Ramakrishnan [Ram00], the quadruple product L -function admits meromorphic continuation to the whole complex plane and satisfies a functional equation relating $L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4)$ to $L(1 + \sum_{i=1}^4 (\kappa_i - 1) - s, f_1^\vee \otimes f_2^\vee \otimes f_3^\vee \otimes f_4^\vee)$. For $1 \leq i \leq 4$, let $G(\omega_i)$ be the Gauss sum of ω_i and $\|f_i\|$ the Petersson norm of f_i defined by

$$\|f_i\| = \operatorname{vol}(\Gamma_0(N_i) \backslash \mathfrak{H})^{-1} \int_{\Gamma_0(N_i) \backslash \mathfrak{H}} |f_i(\tau)|^2 y^{\kappa_i - 2} d\tau.$$

Assume $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq \kappa_4$. We have three types of critical ranges:

$$\begin{cases} \kappa_1 + \kappa_4 - 1 > \kappa_1 - \kappa_4 + 1 > \kappa_2 + \kappa_3 - 1 > \kappa_2 - \kappa_3 + 1 & \text{Case 1,} \\ \kappa_1 + \kappa_4 - 1 > \kappa_2 + \kappa_3 - 1 > \kappa_1 - \kappa_4 + 1 > \kappa_2 - \kappa_3 + 1 & \text{Case 2,} \\ \kappa_2 + \kappa_3 - 1 > \kappa_1 + \kappa_4 - 1 > \kappa_1 - \kappa_4 + 1 > \kappa_2 - \kappa_3 + 1 & \text{Case 3.} \end{cases}$$

In [Bla87], Blasius explicitly computed Deligne's periods of tensor product motives for GL_2 . In particular, we have the following refinement of Deligne's conjecture for the quadruple product L -function:

Conjecture 1.4 (Blasius). *Let $m \in \mathbb{Z}$ be a critical point for $L(s, f_1 \otimes f_2 \otimes f_3 \otimes f_4)$. We have*

$$\sigma \left(\frac{L(m, f_1 \otimes f_2 \otimes f_3 \otimes f_4)}{(2\pi\sqrt{-1})^{8m} \cdot c(f_1 \otimes f_2 \otimes f_3 \otimes f_4)} \right) = \frac{L(m, \sigma f_1 \otimes \sigma f_2 \otimes \sigma f_3 \otimes \sigma f_4)}{(2\pi\sqrt{-1})^{8m} \cdot c(\sigma f_1 \otimes \sigma f_2 \otimes \sigma f_3 \otimes \sigma f_4)}, \quad \sigma \in \operatorname{Aut}(\mathbb{C}).$$

Here

$$c(f_1 \otimes f_2 \otimes f_3 \otimes f_4) = (2\pi\sqrt{-1})^{4 \sum_{i=1}^4 (1 - \kappa_i)} \cdot \prod_{i=1}^4 G(\omega_i)^4 \cdot (\pi \cdot \|f_i\|)^{t_i}$$

with

$$(t_1, t_2, t_3, t_4) = \begin{cases} (4, 0, 0, 0) & \text{Case 1,} \\ (3, 1, 1, 1) & \text{Case 2,} \\ (2, 2, 2, 0) & \text{Case 3.} \end{cases}$$

When two of the f_i 's are CM by the same imaginary quadratic extension, the quadruple product L -function decomposes into product of triple product L -functions. In this special case, Conjecture 1.4 reduces to Deligne's conjecture for triple product L -functions. For the general case, recently we were able to prove the conjecture under certain parity and regularity conditions on the weights. Following theorem is a special case of [Che22b, Theorem 5.8] ($n = 4$):

Theorem 1.5. *Conjecture 1.4 holds under the following conditions:*

- (1) $\kappa_1 + \kappa_2 + \kappa_3 + \kappa_4$ is even.
- (2) $|\sum_{i=1}^4 (\varepsilon_i - \varepsilon'_i)(\kappa_i - 1)| \geq 6$ for all $(\varepsilon_1, \dots, \varepsilon_4)$ and $(\varepsilon'_1, \dots, \varepsilon'_4)$ in $\{\pm 1\}^4$.

2. SKETCH OF PROOF

2.1. Sketch of proof of Theorem 1.2. Let Π be an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, where \mathbb{A} denotes the ring of adèles of \mathbb{Q} . We say Π is regular algebraic if the infinitesimal character of Π_∞ is regular and belongs to $(\mathbb{Z} + \frac{n+1}{2})^n$. We say Π is tamely isobaric if it is isobaric and the exponents of the summands are the same. First we recall the following theorem which is a consequence of (a variant of) the result of Raghuram [Rag10]. It is an algebraicity result on the ratio of product of critical values of Rankin–Selberg L -functions of regular algebraic tamely isobaric automorphic representations.

Theorem 2.1. *Let Σ, Σ' (resp. Π, Π') be regular algebraic tamely isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ (resp. $\mathrm{GL}_{n'}(\mathbb{A})$) satisfying the following conditions:*

- (1) Σ and Σ' are cuspidal.
- (2) $n' = n - 1$ and $(\Sigma_\infty, \Pi_\infty)$ is balanced.
- (3) $\Sigma_\infty = \Sigma'_\infty$ and $\Pi_\infty = \Pi'_\infty$.

Let $m_0 \in \mathbb{Z} + \frac{n+n'}{2}$ be a critical point for $L(s, \Sigma \times \Pi)$ such that $L(m_0, \Sigma \times \Pi') \cdot L(m_0, \Sigma' \times \Pi) \neq 0$. Then, for $\sigma \in \mathrm{Aut}(\mathbb{C})$, we have

$$\sigma \left(\frac{L(m_0, \Sigma \times \Pi) \cdot L(m_0, \Sigma' \times \Pi')}{L(m_0, \Sigma \times \Pi') \cdot L(m_0, \Sigma' \times \Pi)} \right) = \frac{L(m_0, {}^\sigma \Sigma \times {}^\sigma \Pi) \cdot L(m_0, {}^\sigma \Sigma' \times {}^\sigma \Pi')}{L(m_0, {}^\sigma \Sigma \times {}^\sigma \Pi') \cdot L(m_0, {}^\sigma \Sigma' \times {}^\sigma \Pi)}.$$

Remark 2.2. In practice, conditions (1) and (2) are too strong for application. In [Che22b, Theorem 1.2], we remove conditions (1) and (2). Instead, we impose some parity and regularity conditions on Σ_∞ and Π_∞ .

Back to our normalized newform $f \in S_\kappa(N, \omega)$. We may assume that f is not a CM-form. Let $\Pi(f)$ be a regular algebraic cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by f (it is unique up to twisting by integral powers of the adelic absolute value $|\cdot|_{\mathbb{A}}$). For $n \geq 1$, let $\mathrm{Sym}^n \Pi(f)$ be the functorial lift of $\Pi(f)$ to $\mathrm{GL}_{n+1}(\mathbb{A})$ with respect to the symmetric n -th power representation of $\mathrm{GL}_2(\mathbb{C})$. The existence of the lifts was proved by Newton and Thorne [NT21a], [NT21b] (see also [GJ72], [KS02], [Kim03], [CT15], [CT17] for $n \leq 8$). It is easy to see that $\mathrm{Sym}^n \Pi(f)$ is regular algebraic and tamely isobaric. Since we assumed that f is not a CM-form, $\mathrm{Sym}^n \Pi(f)$ is cuspidal. To prove Conjecture 1.1 for $n = 5$, we apply Theorem 2.1 in the case $\mathrm{GL}_4 \times \mathrm{GL}_3$. More precisely, let Σ and Π be regular algebraic cuspidal automorphic representations of $\mathrm{GL}_4(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$ respectively defined by

$$\Sigma = \mathrm{Sym}^3 \Pi(f), \quad \Pi = \mathrm{Sym}^2 \Pi(f).$$

One can verify easily that $(\Sigma_\infty, \Pi_\infty)$ is balanced (cf. [Rag10, Theorem 5.3]). For a cuspidal automorphic representation τ of $\mathrm{GL}_2(\mathbb{A})$, let $\Pi(f) \boxtimes \tau$ be the functorial lift of the Rankin–Selberg convolution of $\Pi(f)$ and τ to $\mathrm{GL}_4(\mathbb{A})$. The existence of the lift was proved by Ramakrishnan in [Ram00]. We assume further τ is chosen so that:

- τ is regular algebraic and non-CM.
- $(\Pi(f)_\infty \boxtimes \tau_\infty) \otimes |\cdot|_\infty^{-1/2} = \Sigma_\infty$.

We also choose an algebraic Hecke character of \mathbb{A}^\times such that

$$(\tau_\infty \otimes |\cdot|_\infty^{-1/2}) \boxplus \chi_\infty = \Pi_\infty.$$

Let Σ' and Π' be isobaric automorphic representations of $\mathrm{GL}_4(\mathbb{A})$ and $\mathrm{GL}_3(\mathbb{A})$ respectively defined by

$$\Sigma' = (\Pi(f) \boxtimes \tau) \otimes |\cdot|_{\mathbb{A}}^{-1/2}, \quad \Pi' = (\tau \otimes |\cdot|_{\mathbb{A}}^{-1/2}) \boxplus \chi.$$

By our assumptions on τ and χ , it is easy to see that Σ' (resp. Π') is regular algebraic and cuspidal (resp. tamely isobaric), and $\Sigma_\infty = \Sigma'_\infty$, $\Pi_\infty = \Pi'_\infty$. Therefore, by Theorem 2.1, for all non-central critical points $m + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ for $L(s, \Sigma \times \Pi)$, we have

$$(2.1) \quad L(m + \frac{1}{2}, \Sigma \times \Pi) \sim \frac{L(m + \frac{1}{2}, \Sigma \times \Pi') \cdot L(m + \frac{1}{2}, \Sigma' \times \Pi)}{L(m + \frac{1}{2}, \Sigma' \times \Pi')}.$$

Here \sim means the ratio of left-hand side by right-hand side is equivariant under $\text{Aut}(\mathbb{C})$. On the other hand, we have the following factorizations of L -functions:

$$\begin{aligned}
(2.2) \quad & L(s, \Sigma \times \Pi) = L(s, \text{Sym}^5 \Pi(f)) \cdot L(s, \text{Sym}^3 \Pi(f) \otimes \omega_{\Pi(f)}) \cdot L(s, \Pi(f) \otimes \omega_{\Pi(f)}^2), \\
& L(s, \Sigma \times \Pi') = L(s - \frac{1}{2}, \text{Sym}^3 \Pi(f) \times \tau) \cdot L(s, \text{Sym}^3 \Pi(f) \otimes \chi), \\
& L(s, \Sigma' \times \Pi) = L(s - \frac{1}{2}, \text{Sym}^3 \Pi(f) \times \tau) \cdot L(s - \frac{1}{2}, \Pi(f) \times \tau \otimes \omega_{\Pi(f)}), \\
& L(s, \Sigma' \times \Pi') = L(s - 1, \Pi(f) \times \tau \times \tau) \cdot L(s - \frac{1}{2}, \Pi(f) \times \tau \otimes \chi).
\end{aligned}$$

Here $\omega_{\Pi(f)}$ is the central character of $\Pi(f)$. By the result of Shimura [Shi76], Deligne's conjecture holds for $L(s, \Pi(f) \times \tau \otimes \omega_{\Pi(f)})$ and $L(s, \Pi(f) \times \tau \otimes \chi)$. By the results of Garrett–Harris [GH93] and the author [Che21a], Deligne's conjecture holds for the triple product L -function $L(s, \Pi(f) \times \tau \times \tau)$. When $\kappa \geq 3$, Deligne's conjecture also holds for $L(s, \text{Sym}^3 \Pi(f) \otimes \omega_{\Pi(f)})$ (cf. [Che21a, Theorem 1.6]). Consider the descent of $\text{Sym}^3 \Pi(f)$ to $\text{GSp}_4(\mathbb{A})$. By the results of Morimoto [Mor14], [Mor18] and the author [Che21b], we see that Deligne's conjecture holds for $L(s, \text{Sym}^3 \Pi(f) \times \tau)$ when $\kappa \geq 6$. We then conclude from (2.1) that Conjecture 1.1 for $n = 5$ holds for non-central critical points. Indeed, it is easy to deduce from (2.1) and Deligne's conjecture for the L -functions on the right-hand sides of (2.2) (except for $\text{Sym}^5 \Pi(f)$) that $L^{(\infty)}(m + \frac{1}{2}, \text{Sym}^5 \Pi(f))$ is equivalent to some integral powers of $2\pi\sqrt{-1}$, $\delta(f)$, and $c^\pm(f)$. A straightforward computation shows that the exponents do coincide with the expected ones. For the central critical point, Conjecture 1.1 follows from the non-central critical points together with the result of Harder–Raghuram [HR20].

2.2. Sketch of proof of Theorem 1.5. The idea of the proof is similar as above. We apply Theorem 2.1 in the case $\text{GL}_4 \times \text{GL}_4$ (cf. Remark 2.2). Let Π_i be a regular algebraic cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ generated by f_i for $i = 1, 2, 3, 4$. Let Σ and Π be the regular algebraic tamely isobaric automorphic representations of $\text{GL}_4(\mathbb{A})$ defined by

$$\Sigma = (\Pi_1 \boxtimes \Pi_2) \otimes | \cdot |_{\mathbb{A}}^{1/2}, \quad \Pi = (\Pi_3 \boxtimes \Pi_4) \otimes | \cdot |_{\mathbb{A}}^{-1/2}.$$

Let $\Pi'_1, \Pi'_2, \Pi'_3, \Pi'_4$ be auxiliary regular algebraic cuspidal automorphic representations of $\text{GL}_2(\mathbb{A})$ such that

$$\Pi'_{1,\infty} \boxplus \Pi'_{2,\infty} = \Sigma_\infty, \quad \Pi'_{3,\infty} \boxplus \Pi'_{4,\infty} = \Pi_\infty.$$

Let Σ' and Π' be the regular algebraic tamely isobaric automorphic representations of $\text{GL}_4(\mathbb{A})$ defined by

$$\Sigma' = \Pi'_1 \boxplus \Pi'_2, \quad \Pi' = \Pi'_3 \boxplus \Pi'_4.$$

The assumptions (1) and (2) in Theorem 1.5 then implies that Theorem 2.1 holds in our case. Therefore, we have

$$(2.3) \quad L(m, \Sigma \times \Pi) \sim \frac{L(m, \Sigma \times \Pi') \cdot L(m, \Sigma' \times \Pi)}{L(m, \Sigma' \times \Pi')}$$

for all critical points $m \in \mathbb{Z}$ for $L(s, \Sigma \times \Pi)$. On the other hand, we have the following factorizations of L -functions:

$$\begin{aligned}
(2.4) \quad & L(s, \Sigma \times \Pi) = L(s, \Pi_1 \times \Pi_2 \times \Pi_3 \times \Pi_4), \\
& L(s, \Sigma \times \Pi') = L(s + \frac{1}{2}, \Pi_1 \times \Pi_2 \times \Pi'_3) \cdot L(s + \frac{1}{2}, \Pi_1 \times \Pi_2 \times \Pi'_4), \\
& L(s, \Sigma' \times \Pi) = L(s - \frac{1}{2}, \Pi_3 \times \Pi_4 \times \Pi'_1) \cdot L(s - \frac{1}{2}, \Pi_3 \times \Pi_4 \times \Pi'_2), \\
& L(s, \Sigma' \times \Pi') = L(s, \Pi'_1 \times \Pi'_3) \cdot L(s, \Pi'_1 \times \Pi'_4) \cdot L(s, \Pi'_2 \times \Pi'_3) \cdot L(s, \Pi'_2 \times \Pi'_4).
\end{aligned}$$

By the result of Shimura [Shi76], we know that Deligne's conjecture holds for the Rankin–Selberg L -functions for $\text{GL}_2 \times \text{GL}_2$. Therefore, by (2.3), we are reduced to show that Deligne's conjecture holds for the triple product L -functions appear on the right-hand sides of (2.4). For these triple product L -functions, we can play the same trick as above. This time apply Theorem 2.1 in the case $\text{GL}_4 \times \text{GL}_2$.

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