

The Manin constant and p -adic bounds on denominators of the Fourier coefficients of newforms at cusps

Abhishek Saha
(Joint work with K Česnavičius and M Neururer)

Queen Mary University of London

Some facts on cusps for $\Gamma_0(N)$

- Any cusp $\mathfrak{c} \in X_0(N)(\mathbb{C})$ is equivalent to

$$\mathfrak{c} = \frac{a}{L}, \text{ for some } L|N, \gcd(a, L) = 1.$$

We call L the *denominator* of \mathfrak{c} . There are exactly $\phi(\gcd(L, N/L))$ cusps of denominator L .

- The *width* of a cusp $\mathfrak{c} = \frac{a}{L}$ equals

$$w(\mathfrak{c}) = \frac{N}{\gcd(L^2, N)}.$$

$w(\mathfrak{c})$ is the smallest integer w such that $\begin{pmatrix} a & * \\ L & * \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & * \\ L & * \end{pmatrix}^{-1} \in \Gamma_0(N)$.

- The *Atkin-Lehner involutions*: Let $\mathfrak{c} = \frac{a}{L}$ be a cusp. Then there exists an Atkin-Lehner involution taking \mathfrak{c} to a cusp of denominator L' iff $\text{val}_p(L') \in \{\text{val}_p(L), \text{val}_p(N) - \text{val}_p(L)\}$ for each $p|N$.

The main question

- Let $f = \sum_{n>0} a_f(n)q^n$, $q = e^{2\pi iz}$ be a holomorphic *newform* of weight k , level N , trivial character.
- Normalize $a_f(1) = 1$. Then it well-known that all $a_f(n) \in \overline{\mathbb{Z}}$.
- Fourier expansion at \mathfrak{c} : Let $\mathfrak{c} = \gamma\infty$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

$$(f|_k\gamma)(z) = \sum_{n \geq 0} a_f(n; \mathfrak{c}) q^{\frac{n}{w(\mathfrak{c})}}.$$

Note: $a_f(n; \mathfrak{c})$ only well-defined up to a $w(\mathfrak{c})$ 'th root of unity.

- What can we say about the “denominators” of $a_f(n; \mathfrak{c})$?

For a prime p , we are interested in good *lower* bounds for

$$\mathrm{val}_p(f|_{\mathfrak{c}}) := \inf_{n \geq 0} (\mathrm{val}_p(a_f(n; \mathfrak{c}))).$$

Here, $\mathrm{val}_p: \overline{\mathbb{Q}}_p \rightarrow \mathbb{Q} \cup \{\infty\}$ is the p -adic valuation with $\mathrm{val}_p(p) = 1$, extended to $\overline{\mathbb{C}}$ via any fixed choice of isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}}_p$.

Let f be a normalized newform for $\Gamma_0(N)$ of weight k .

Find good *lower* bounds for $\mathrm{val}_p(f|_{\mathfrak{c}}) := \inf_{n \geq 0} (\mathrm{val}_p(a_f(n; \mathfrak{c})))$.

- 1 Clearly, $\mathrm{val}_p(f|_{\infty}) = 0$.
- 2 *The q -expansion principle:* If the Fourier coefficients at infinity lie in a ring R , then the Fourier coefficients at any cusp lie in $R[1/N, e^{\frac{2\pi i}{N}}]$. In particular, $\mathrm{val}_p(f|_{\mathfrak{c}}) = 0$ if $p \nmid N$.
- 3 Suppose N is *squarefree* and $p|N$. Then using Atkin-Lehner operators, all cusps can be moved to ∞ . An easy calculation now shows that:

$$\mathrm{val}_p(f|_{\mathfrak{c}}) = \begin{cases} -\frac{k}{2} & \text{if } \mathrm{val}_p(L) = 0, \\ 0 & \text{if } \mathrm{val}_p(L) = 1. \end{cases}$$

- 4 Nothing much previously known for general N . Some generic bounds exist due to Conrad using intersection theory on regular stacky surfaces, but are quite weak and have other issues.
- 5 For the general case, it suffices (thanks to AL operators) to restrict to cusps of denominator L such that $L^2|N$.

Examples

$$N = 2^3 \cdot 3, k = 2, p = 2$$

$$f = q - q^2 + q^4 + q^5 + 2q^7 + \dots$$

$$f|_2 \left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix} \right) = \frac{1}{6} \left(iq^{\frac{1}{6}} + iq^{\frac{1}{2}} - 2iq^{\frac{5}{6}} + \dots \right).$$

$$\text{So } \text{val}_2(f|_{1/2}) = -1.$$

$$N = 2 \cdot 3^5, k = 2, p = 3$$

$$f = q - q^2 + q^4 + 3q^5 - 4q^7 + \dots$$

$$f|_2 \left(\begin{smallmatrix} 1 & -1 \\ 3 & -2 \end{smallmatrix} \right) = \frac{1}{54} \left(\zeta_{162}^{25} q^{\frac{1}{54}} + \zeta_{162}^{50} q^{\frac{2}{54}} + \zeta_{162}^{19} q^{\frac{4}{54}} + \dots \right)$$

$$f|_2 \left(\begin{smallmatrix} 1 & 1 \\ 9 & 10 \end{smallmatrix} \right) = \frac{1}{6} \left(\zeta_{54}^7 q^{\frac{1}{6}} + \zeta_{54}^{14} q^{\frac{1}{3}} + \zeta_{54}^4 q^{\frac{4}{6}} + \dots \right).$$

$$\text{So } \text{val}_3(f|_{1/3}) = -3, \text{val}_3(f|_{1/9}) = -1.$$

Examples (contd.)

$$N = 5^2, k = 4, p = 5$$

$$f = q + 4q^2 - 2q^3 + 8q^4 + \dots$$

$$f|_4 \left(\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix} \right) = \frac{1}{5} \left((-4\zeta_5^3 - 3\zeta_5 - 3) q + (-12\zeta_5^2 - 16\zeta_5 - 12) q^2 + \dots \right).$$

$$\text{val}_5(f|_{1/5}) = -1/2.$$

$$N = 7^2, k = 4, p = 7$$

$$f = q - 5q^2 + 17q^4 - 45q^8 + \dots$$

$$f|_4 \left(\begin{smallmatrix} 1 & 1 \\ 7 & 1 \end{smallmatrix} \right) = \frac{1}{7} \left((-2\zeta_7^5 - 4\zeta_7^4 - 6\zeta_7^3 - 8\zeta_7^2 - 3\zeta_7 - 5) q \right. \\ \left. + (-30\zeta_7^5 + 10\zeta_7^4 - 20\zeta_7^3 - 15\zeta_7^2 - 10\zeta_7 - 5) q^2 + \dots \right)$$

$$\text{val}_7(f|_{1/7}) = -1/6.$$

Examples (contd.)

$$N = 2^8 \cdot 3, k = 2, p = 2$$

$$f = q + q^3 + 4q^7 + \dots$$

$$f|_2 \left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix} \right) = \frac{1}{192} \left(\zeta_{128} q^{\frac{1}{192}} + \zeta_{128}^3 q^{\frac{3}{192}} + \dots \right)$$

$$f|_2 \left(\begin{smallmatrix} 1 & -1 \\ 4 & -3 \end{smallmatrix} \right) = \frac{1}{48} \left(\zeta_{64}^{15} q^{\frac{1}{48}} + \dots \right)$$

$$f|_2 \left(\begin{smallmatrix} 3 & 1 \\ 8 & 3 \end{smallmatrix} \right) = \frac{1}{12} \left(\zeta_{32}^5 q^{\frac{1}{12}} + \dots \right)$$

$$f|_2 \left(\begin{smallmatrix} 5 & -1 \\ 16 & -3 \end{smallmatrix} \right) = \frac{1}{3} \left(2\zeta_{16}^7 q^{\frac{2}{3}} + \dots \right)$$

$$\text{val}_7(f|_{1/2}) = -6, \text{val}_7(f|_{1/4}) = -4, \text{val}_7(f|_{3/8}) = -2, \text{val}_7(f|_{5/16}) = 1.$$

The main theorem

Theorem 1

For a newform f of weight k for $\Gamma_0(N)$, a prime p , and a cusp c of denominator L , the quantity $\text{val}_p(f|_c)$ depends only on f and $\text{val}_p(L)$.

For $0 \leq \text{val}_p(L) \leq \frac{\text{val}_p(N)}{2}$, we have the bounds $\text{val}_p(f|_c) \geq$

$$-\frac{k}{2} (\text{val}_p(N) - 2\text{val}_p(L)) + \begin{cases} 0 & \text{if } \text{val}_p(L) = 0, \\ 0 & \text{if } \text{val}_p(L) = 1, \text{val}_p(N) > 2, \\ -\frac{1}{2} & \text{if } \text{val}_p(L) = \frac{1}{2}\text{val}_p(N) = 1, \\ 1 - \frac{1}{2}\text{val}_p(L) & \text{otherwise.} \end{cases}$$

For $p = 2$, we get even stronger bounds...

The main theorem

Theorem 1 (contd...)

If $p = 2$ we have the additional stronger bounds.

$$\text{val}_2(f|_c) \geq -\frac{k}{2} (\text{val}_p(N) - 2\text{val}_p(L))$$
$$+ \begin{cases} 0 & \text{if } \text{val}_2(L) = \frac{1}{2}\text{val}_2(N) = 1, \\ \frac{k}{2} & \text{if } \text{val}_2(L) = \frac{1}{2}\text{val}_2(N) \in \{2, 3, 4\}, \\ \frac{k}{2} + 1 - \frac{1}{4}\text{val}_2(N) & \text{if } \text{val}_2(L) = \frac{1}{2}\text{val}_2(N) > 4, \\ 0 & \text{if } \text{val}_2(L) = 3, \text{val}_2(N) > 6. \end{cases}$$

- We have checked experimentally that our bounds are *sharp* for newforms associated to elliptic curves and $p \leq 17$.

An application to the Manin constant

The modularity theorem (Wiles-Taylor, B-C-D-T)

Given an elliptic curve E/\mathbb{Q} of conductor N ,

- (*E is modular*) There exists a newform f of weight 2 for $\Gamma_0(N)$ and with integral Fourier coefficients such that $a_f(p) = p + 1 - |E(\mathbb{F}_p)|$.
- (*E has a modular parametrization*) There is a surjection $\phi: X_0(N)_{\mathbb{Q}} \twoheadrightarrow E$.

Note: ϕ is not unique, so it is common to normalize ϕ to be *optimal*, that is, $\deg(\phi)$ to be the least possible.

- The *Manin constant* c_ϕ is defined by $\phi^*(\omega_E) = c_\phi \cdot \omega_f$ where ω_E is the Néron differential and $\omega_f = 2\pi i f(z) dz$.

Conjecture (Manin, 1972)

If ϕ is optimal then $c_\phi = \pm 1$.

Conjecture (Manin, 1972)

If ϕ is optimal then $c_\phi = \pm 1$.

- (Gabber in PhD studies; Edixhoven, 1991) c_ϕ is an integer.
- (Abbes–Ullmo, 1996): If ϕ is optimal and $p|c_\phi$, then $p|N$.
- Mazur, Raynaud, Agashe–Ribet–Stein,.....: Further improvements
- (Cremona): Computationally verified conjecture for all $N \leq 390000$.
- **(Česnavičius, 2018): If ϕ is optimal and $p|c_\phi$, then $p^2|N$. (This implies Manin's conjecture if N is squarefree)**

Recall: $v_2(N) \leq 8$, $v_3(N) \leq 5$, $v_p(N) \leq 2$ for $p > 3$.

Theorem 2

For $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$, every surjection $\phi: (X_\Gamma)_\mathbb{Q} \twoheadrightarrow E$ satisfies $c_\phi \mid 6 \cdot \deg(\phi)$, and if N is cube-free or $\Gamma = \Gamma_1(N)$, then even $c_\phi \mid \deg(\phi)$.

This is interesting because $\deg(\phi)$ has little in common with N . No apparent connection between the conditions $p^2|N$ and $p|\deg(\phi)$.

A very brief sketch of proof of Theorem 2:

- 1 Using Theorem 1, we show that

$$\omega_f \text{ lies in the } \mathbb{Z}\text{-lattice } H^0(X_0(N)_\mathbb{Z}, \Omega) \subset H^0(X_0(N)_\mathbb{Q}, \Omega^1), \quad (1)$$

where Ω denotes the relative dualizing sheaf. (Arithmetic geometric considerations reduce this to certain bounds on the p -adic valuations of the denominators of the Fourier coefficients of f at *all* the cusps of $X_0(N)_\mathbb{C}$. Theorem 1 gives much stronger bounds than needed.)

- 2 Using above, we show that ω_f lies in an even *a priori* smaller lattice $H^0(\mathcal{J}_0(N), \Omega^1)$ that seems otherwise inaccessible. Here $\mathcal{J}_0(N)$ is the Néron model of the Jacobian $J_0(N)$.
- 3 Now Theorem 2 follows from the fact that the composition $\pi \circ \pi^\vee: E \rightarrow J_0(N) \rightarrow E$ is multiplication by $\deg(\phi)$.

For the rest of this talk I will focus on the proof of Theorem 1.

Recall Theorem 1:

Theorem 1

For a newform f of weight k for $\Gamma_0(N)$, a prime p , and a cusp \mathfrak{c} of denominator L , the quantity $\text{val}_p(f|_{\mathfrak{c}})$ depends only on f and $\text{val}_p(L)$.

For $0 \leq \text{val}_p(L) \leq \frac{\text{val}_p(N)}{2}$, we have the bounds $\text{val}_p(f|_{\mathfrak{c}}) \geq$

$$-\frac{k}{2} (\text{val}_p(N) - 2\text{val}_p(L)) + \begin{cases} 0 & \text{if } \text{val}_p(L) = 0, \\ 0 & \text{if } \text{val}_p(L) = 1, \text{val}_p(N) > 2, \\ -\frac{1}{2} & \text{if } \text{val}_p(L) = \frac{1}{2}\text{val}_p(N) = 1, \\ 1 - \frac{1}{2}\text{val}_p(L) & \text{otherwise.} \end{cases}$$

with sharper bounds for $p = 2$.

Fourier expansions and Whittaker models

In order to prove Theorem 1, for a cusp $\mathfrak{c} = \gamma\infty$ and a prime p , we want to prove lower bounds on

$$\text{val}_p(f|_{\mathfrak{c}}) := \inf_{n \geq 0} (\text{val}_p(a_f(n; \mathfrak{c})))$$

where

$$(f|_k \gamma)(z) = \sum_{n \geq 0} a_f(n; \mathfrak{c}) q^{\frac{n}{w(\mathfrak{c})}}.$$

Fourier coefficients at general cusps are subtle: e.g., the coefficients $a_f(n; \mathfrak{c})$ are not multiplicative. One way to understand $a_f(n; \mathfrak{c})$ is via the Whittaker model.

- Let $\phi_f : \mathrm{GL}_2(\mathbb{A}) \rightarrow \mathbb{C}$ be the automorphic form associated to f via adelicization.
- ϕ_f generates a cuspidal automorphic representation $\pi = \otimes_v \pi_v$.
- The *global Whittaker newform* $W_f(g) = \int_{\mathbb{Q} \backslash \mathbb{A}} \phi_f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \psi(-x) dx$ packages together all Fourier coefficients at all cusps. In particular, $a_f(r; \mathfrak{c}) = W_f(g_{r, \mathfrak{c}})$ for some explicit $g_{r, \mathfrak{c}} \in \mathrm{GL}_2(\mathbb{A})$.
- On the other hand, $W_f(g) = \prod_v W_{\pi_v}(g_v)$, where $W_{\pi_v} : \mathrm{GL}_2(\mathbb{Q}_v) \rightarrow \mathbb{C}$ is the *local Whittaker newform* that depends only on the local representation π_v .

An explicit relation

For a newform f of weight k for $\Gamma_0(N)$, a prime p , and a matrix $\gamma = \begin{pmatrix} a & * \\ L & * \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $\mathfrak{c} = \frac{a}{L}$, with $L^2 | N$, up to a root of unity:

$$a_f(r; \mathfrak{c}) = a_f(r_0) \left(\frac{r}{r_0 w(\mathfrak{c})} \right)^{k/2} \prod_{q|N} W_{\pi_q} \left(\begin{pmatrix} 0 & \frac{r}{N} \\ 1 & \frac{u_q}{L} \end{pmatrix} \right).$$

where r_0 is the N -free part of r , and $u_q \in \mathbb{Z}_q^\times$.

- **Upshot:** Proving lower bounds for $\mathrm{val}_p(f|_{\mathfrak{c}})$ reduce to proving lower bounds for $\mathrm{val}_p \left(W_{\pi_q} \left(\begin{pmatrix} 0 & q^t \\ 1 & \frac{u_q}{q^\ell} \end{pmatrix} \right) \right)$ for primes p and q both dividing N , $t \in \mathbb{Z}$, $0 \leq \ell \leq \frac{c(\pi_q)}{2}$, $u_q \in \mathbb{Z}_q^\times$.
- Since $|x|_p = p^{-\mathrm{val}_p(x)}$, this is a *p-adic analogue* of the local sup-norm question of bounding $|W_{\pi_q}|_\infty$ in highly ramified cases. (Templier 2014, S. 2016, Assing 2019)
- The values of W_{π_q} at diagonal matrices are well-known, the key point is to access the non-diagonal elements.
- **Remark:** Any matrix g in $\mathrm{GL}_2(\mathbb{Q}_q)$ has a double coset representative in $N(F)gK_0(n)$ of the form $\begin{pmatrix} 0 & q^t \\ 1 & \frac{u}{q^\ell} \end{pmatrix}$ for $0 \leq \ell \leq n$; local Atkin–Lehner operators halve the range of ℓ .

To prove lower bounds for $\mathrm{val}_p \left(W_{\pi_q} \left(\begin{pmatrix} 0 & q^t \\ 1 & \frac{u}{q^\ell} \end{pmatrix} \right) \right)$ we refine and extend a method developed for the *sup-norm problem* (S. 2016- 2019, Assing 2018-2019, Assing–Corbett 2019,...).

The local functional equation (Jacquet–Langlands, 1972)

For a non-archimedean local field F , an infinite-dimensional representation π of $\mathrm{GL}_2(F)$, an element W in the local Whittaker model of π , and a character μ of F^\times , putting

$$Z(W, s, \mu) = \int_{F^\times} W\left(\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right) \mu(y) |y|^{s-\frac{1}{2}} d^\times y$$

$$\frac{Z(W, s, \mu)}{L(s, \pi \otimes \mu)} \varepsilon(s, \pi \otimes \mu) = \frac{Z\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot W, 1-s, \mu^{-1}\right)}{L(1-s, \pi \otimes \mu^{-1})}, \quad (2)$$

Above $\varepsilon(s, \pi)$ is the local GL_2 ϵ -factor (Jacquet–Langlands).

Using this, one can formulate a “basic identity” (S, 2016) that writes down $W_{\pi_q}(g_q)$ as an explicit linear combination of terms involving GL_2 and GL_1 ϵ -factors.

For example, if π is supercuspidal, the basic identity becomes

The basic identity for supercuspidal reps π

For a supercuspidal rep π of $\mathrm{PGL}_2(\mathbb{Q}_q)$, $u \in \mathbb{Z}_q^\times$, and $1 \leq \ell \leq \frac{c(\pi)}{2}$.

$$W_\pi\left(\begin{pmatrix} 0 & q^t \\ 1 & \frac{u}{q^\ell} \end{pmatrix}\right) = (1 - q^{-1})^{-1} q^{-\frac{\ell}{2}} \sum_{\substack{c(\mu)=\ell \\ c(\mu\pi)=-t}} \varepsilon(1/2, \mu) \varepsilon(1/2, \mu^{-1}\pi) \mu(u). \quad (3)$$

For other representations, the basic identity takes a similar (though slightly more complicated) shape. The resulting formulae were written by me in some cases (S, 2016 - 2018) and in all cases by Assing in his thesis (2019).

So we need to solve the problem of computing *p-adic valuations of ϵ -factors* of representations of $\mathrm{GL}_r(\mathbb{Q}_q)$ where $r = 1, 2$.

The case $q \neq p$

Theorem 3

For a finite extension F/\mathbb{Q}_q , an infinite-dimensional ramified representation π of $\mathrm{GL}_2(F)$ associated to a holomorphic newform, and a matrix $g \in \mathrm{GL}_2(F)$, we have $W_\pi(g) \in \overline{\mathbb{Z}} \left[\frac{1}{q} \right]$.
In particular, if $p \neq q$, then $\mathrm{val}_p(W_\pi(g)) \geq 0$.

This relies on a formula for the Whittaker newvector in terms of a family of nonarchimedean ${}_2F_1$ hypergeometric integrals (Assing 2019; also unpublished works of Templier (2012) and Hu (2016)).

Sketch of proof of Theorem 3 (assuming above-mentioned formula)

Suppose G compact group, $K \subseteq G$ of finite index, $\mathrm{vol}(K) \in R$. Let $f : G \rightarrow R$ be a right- K -invariant function. Then $\int_G f(g) dg \in R$.

So we are reduced to the case $q = p$.

The case $q = p$

So the next problem is: Let F be a finite extension of \mathbb{Q}_p . Understand the p -adic valuations of $\varepsilon(1/2, \mu)$ and $\varepsilon(1/2, \mu \otimes \pi)$ where μ is a finite order character of F^\times and π be an infinite-dimensional, irreducible, unitary representation of $\mathrm{PGL}_2(F)$.

- If π is principal series, we need to also assume that it comes from a global holomorphic newform (otherwise we cannot expect good results).
- Note: $\varepsilon(1/2, \mu)$ and $\varepsilon(1/2, \mu \otimes \pi)$ are **algebraic numbers of absolute value 1**, but are **not** necessarily roots of unity.

The case of GL_1

- The GL_1 ϵ -factors defined by Tate are closely related to classical Gauss sums.
- For a classical Gauss sum, there is a well-known result (Stickelberger's congruence) that gives its p -adic valuation.

Theorem 4

For a finite extension F/\mathbb{Q}_p , and a character $\chi: F^\times \rightarrow \mathbb{C}^\times$ of finite order,

- ① if $a(\chi) = 1$, then,

$$\text{val}_p(\epsilon(\frac{1}{2}, \chi)) = -\frac{[\mathbb{F}_F : \mathbb{F}_p]}{2} + \frac{s(\chi)}{p-1}, \quad 0 \leq s(\chi) \leq (p-1)[\mathbb{F}_F/\mathbb{F}_p];$$

- ② if $\chi^2 = 1$ or $a(\chi) > 1$, then $\epsilon(\frac{1}{2}, \chi)$ is a root of unity, and so

$$\text{val}_p(\epsilon(\frac{1}{2}, \chi)) = 0.$$

A classification of infinite-dimensional, irreducible, unitary representation of $GL_2(F)$ and trivial central character.

- ① Principal series representations
- ② Special representations (twists of Steinberg)
- ③ Supercuspidal representations:
 - a Dihedral supercuspidal
 - b *Non-dihedral supercuspidal* (can only occur if $p = 2$)

All other cases reduce to GL_1

In cases 1, 2 and 3a, one can write the GL_2 ϵ -factor in terms of GL_1 ϵ -factors. So the problem here reduces to one we have solved.

Analysis of non-dihedral representations

There are exactly 16 representations of Type 3b. Using the Local Langlands correspondence and the basic identity we write down $W_\pi(g)$ exactly in each case, from which the required bounds follow.

- We now know how to estimate the p -adic valuations of $GL_{r-\epsilon}$ -factors for $r = 1, 2$.
- The basic identity expresses $W_{\pi_p} \left(\begin{pmatrix} 0 & p^t \\ 1 & \frac{u_p}{p^\ell} \end{pmatrix} \right)$ as an explicit finite sum involving the above.
- This allows us to prove our main local result, which gives sharp p -adic bounds for $\text{val}_p(W_{\pi_p}(\cdot))$ in all cases.

Here is a special case of our main local result:

A special case of our local theorem

Theorem 5

Let p be odd and F/\mathbb{Q}_p a finite extension. Let π be a supercuspidal representation of $GL_2(F)$, with trivial central character and $c(\pi) = n > 2$. For $0 \leq \ell \leq n/2$ and $u \in \mathcal{O}_F$,

$$\text{val}_p(W_{\pi} \left(\begin{pmatrix} 0 & p^t \\ 1 & u p^{-\ell} \end{pmatrix} \right)) \geq \begin{cases} 0 & \text{if } \ell = 0, 1, \\ [\mathbb{F}_F : \mathbb{F}_p] \left(1 - \frac{\ell}{2}\right) & \text{otherwise.} \end{cases}$$

- Our main local theorem gives such bounds (with *lots* of subcases) covering all representations and conductors.
- If $p = 2$, we only do the case $F = \mathbb{Q}_2$.
- We get stronger bounds for \mathbb{Q}_2 by exploiting additional parity cancellation in sums of ϵ -factors.

Now, Theorem 1 follows as described earlier...

That is, we combine the local bounds on $\text{val}_p(W_\pi\left(\begin{pmatrix} 0 & p^t \\ 1 & up^{-\ell} \end{pmatrix}\right))$ given by (the general version of) Theorem 5 with

$$a_f(r; \mathfrak{c}) = a_f(r_0) \left(\frac{r}{r_0 w(\mathfrak{c})} \right)^{k/2} \prod_{q|N} W_{\pi_q} \left(\begin{pmatrix} 0 & \frac{r}{N} \\ 1 & \frac{uq}{L} \end{pmatrix} \right)$$

to obtain the sharp lower bounds for $\text{val}_p(f|_{\mathfrak{c}})$ for holomorphic newforms f at each cusp \mathfrak{c} , which is the content of Theorem 1.