CRYSTAL STRUCTURE ON LOCALIZED QUANTUM COORDINATE RINGS

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1. Introduction

The article is a summary of [12]. Let R be a quiver Hecke algebra associated with a simple Lie algebra $\mathfrak{g}$ and R-gmod the category of finite-dimensional graded R-modules. We set $\mathcal{K}(R\text{-gmod})$ to be the Grothendieck ring of R-gmod. It is well-known that the unipotent quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$ is categorified by $\mathcal{K}(R\text{-gmod})$. The basic theory of localization for the monoidal category $\widetilde{R}$-gmod of R-gmod is initiated by [5] and its Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ defines the localized (unipotent) quantum coordinate ring $\widetilde{\mathcal{A}_q(\mathfrak{n})}$. In [11], Lauda-Vazirani defined certain crystal structure on the family of simple modules of R-gmod and they have shown that this crystal is isomorphic to the crystal $B(\infty)$ of the nilpotent half of $U_q(\mathfrak{g})$. In this article, considering the family of self-dual simple module $\mathbb{B}(\widetilde{R}\text{-gmod})$ of the localized category $\widetilde{R}$-gmod, we define a crystal structure of $\widetilde{\mathcal{A}_q(\mathfrak{n})}$ and show that it is isomorphic to the cellular crystal $\mathbb{B}_{\mathbf{i}}$, which is defined to a reduced word for the longest Weyl group element w_0. This result can be seen as a localized version of the result by Lauda-Vazirani.

2. Preliminaries

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}_- = \langle e_i, h_i, f_i \rangle_{i\in I:=\{1,2,\cdots,n\}}$ be a simple Lie algebra associated with a Cartan matrix $A = (a_{ij})_{i,j\in I}$ where $\{e_i,\ f_i,\ h_i\}_{i\in I}$ are the standard Chevalley generators and $\mathfrak{n} = \langle e_i \rangle_{i\in I}$ (resp. $\mathfrak{t} = \langle h_i \rangle_{i\in I}$, $\mathfrak{n}_- = \langle f_i \rangle_{i\in I}$) is the positive nilpotent subalgebra (resp. the Cartan subalgebra, the negative nilpotent subalgebra).

Let $\{\alpha_i\}_{i\in I}$ be the set of simple roots of $\mathfrak{g}$ and $\langle\ \ ,\ \ \rangle$ a pairing on $\mathfrak{t} \times \mathfrak{t}^*$ satisfying $a_{ij} = (\langle h_i, \alpha_j \rangle)_{i,j\in I}$. We also define a symmetric bilinear form $(\ ,\ )$ on $\mathfrak{t}^*$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$ and $\langle h_i, \lambda \rangle = \frac{2(\alpha_i,\lambda)}{(\alpha_i,\alpha_i)}$ for $\lambda \in \mathfrak{t}^*$.

Let $P := \{\lambda \in \mathfrak{t}^* \,|\, \langle h_i, \lambda \rangle \in \mathbb{Z}$ for any $i \in I\}$ be the weight lattice and $P_+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0$ for any $i \in I\}$ the set of dominant weights. Set $Q := \oplus_{i\in I}\mathbb{Z}\alpha_i$ (resp. $Q_+ := \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$), which is called the root lattice (resp. positive root lattice). For an element $\beta = \sum_i m_i\alpha_i \in Q_+$ define $|\beta| = \sum_i m_i$, which is called the height of β. Let $W = \langle s_i \,|\, s_i \rangle_{i\in I}$ be the Weyl group associated with P, where s_i is the simple reflection defined by $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$ $(\lambda \in P)$.

We denote the dual weight lattice of P by $P^* := \{h \in \mathfrak{t} \,|\, \langle h, P \rangle \subset \mathbb{Z}\}$. Let $U_q(\mathfrak{g}) := \langle e_i,\ f_i,\ q^h \rangle_{i\in I, h\in P^*}$ be the quantum algebra associated with $\mathfrak{g}$ with the defining relations (see e.g.,[1, 2]) and $U_q^-(\mathfrak{g}) := \langle f_i \rangle_{i\in I}$ (resp. $U_q^+(\mathfrak{g}) := \langle e_i \rangle_{i\in I}$) the negative (resp. positive) nilpotent subalgebras of $U_q(\mathfrak{g})$. We also define the $\mathbb{Z}$-form $U^-_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{g})$ of $U_q^-(\mathfrak{g})$ as in [5]. Set $q_i := q^{(\alpha_i,\alpha_i)/2}$, $[n]_i = (q_i^n - q_i^{-1})/(q_i - q_i^{-1})$, $[n]_i! := \prod_{0\leq k\leq n}[k]_i$ and $X_i^{(n)} := X_i^n/[n]_i!$ for $X_i = f_i, e_i$ for $i \in I$, $n \in \mathbb{Z}_{\geq 0}$.

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Now, let us define the *(unipotent) quantum coordinate ring* $\mathcal{A}_q(\mathfrak{n})$ by

$$\mathcal{A}_q(\mathfrak{n}) = \bigoplus_{\beta\in Q_-} \mathcal{A}_q(\mathfrak{n})_\beta \qquad \mathcal{A}_q(\mathfrak{n})_\beta := \mathrm{Hom}_{\mathbb{Q}(q)}(U_q^+(\mathfrak{g})_{-\beta}, \mathbb{Q}(q))$$

Note that $U_q^-(\mathfrak{g}) \cong \mathcal{A}_q(\mathfrak{n})$ as a $\mathbb{Q}(q)$-algebra. The $\mathbb{Z}$-form $\mathcal{A}(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}$ is defined as in [5].

3. Crystal Bases and Crystals

3.1. **Crystal Base of $U_q^-(\mathfrak{g}) \cong \mathcal{A}_q(\mathfrak{n})$.** Let us define the crystal base $(L(\infty), B(\infty))$ of $U_q^-(\mathfrak{g})$([1]). For $i \in I$ the operator $e_i' \in \mathrm{End}(U_q^-(\mathfrak{g}))$ is defined by the formula

$$e_i'(PQ) = e_i'(P)Q + q_i^{\langle h_i,\beta\rangle} P e_i'(Q), \quad e_i'(f_j) = \delta_{i,j}, \quad e_i'(1) = 0,$$

for any $P \in U_q(\mathfrak{g})_\beta$, $Q \in U_q(\mathfrak{g})$, $i, j \in I$. By the fact that for $P \in U_q(\mathfrak{g})_\beta$, there exists the following unique decomposition

$$P = \sum_{k\geq 0} f_i^{(k)} P_n, \tag{3.1}$$

where $P_n \in \mathrm{Ker}(e_i') \cap U_q^-(\mathfrak{g})_{\beta+k\alpha_i}$. And define the operators $\tilde{e}_i, \tilde{f}_i \in \mathrm{End}(U_q^-(\mathfrak{g}))$ on $P \in U_q^-(\mathfrak{g})_\beta$ by using the decomposition (3.1)

$$\tilde{e}_i P = \sum_{k>0} f_i^{(k-1)} P_n, \qquad \tilde{f}_i P = \sum_{k\geq 0} f_i^{(k+1)} P_n,$$

which are called *Kashiwara operators*. Now, set

$$L(\infty) := \sum_{k\geq 0, i_1,\cdots,i_k\in I} \mathbb{A}\tilde{f}_{i_1}\cdots\tilde{f}_{i_k}u_\infty, \qquad B(\infty) = \{\tilde{f}_{i_1}\cdots\tilde{f}_{i_k}u_\infty \bmod qL(\infty) \,|\, k\geq 0, i_1,\cdots,i_k \in I\} \setminus \{0\},$$

$$\varepsilon_i(b) = \max\{k : \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle,$$

where $u_\infty = 1 \in U_q(\mathfrak{g})$ and $\mathbb{A} \subset \mathbb{Q}(q)$ is the local subring at $q = 0$.

Theorem 3.1 ([1])**.** A pair $(L(\infty), B(\infty))$ is a crystal base of $U_q^-(\mathfrak{g})$. Indeed, we obtain

$$\begin{aligned}
&\tilde{e}_i L(\infty) \subset L(\infty), \quad \tilde{f}_i L(\infty) \subset L(\infty),\\
&\tilde{e}_i B(\infty) \subset B(\infty) \sqcup \{0\}, \quad \tilde{f}_i B(\infty) \subset B(\infty) \sqcup \{0\},\\
&\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i \ \text{ for } b, \tilde{e}_i b \in B(\infty), \quad \mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i \ \text{ for } b, \tilde{f}_i b \in B(\infty),\\
&\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1 \ \ \varphi_i(\tilde{e}_i b) = \varepsilon_i(b) + 1, \ \text{ for } b, \tilde{e}_i b \in B(\infty),\\
&\varepsilon_i(\tilde{f}_i b) = \varphi(b) + 1 \ \ \varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1, \ \text{ for } b, \tilde{f}_i b \in B(\infty),\\
&\tilde{f}_i b = b' \iff \tilde{e}_i b' = b, \quad \text{ for } b,\ b' \in B(\infty)
\end{aligned}$$

3.2. **Crystals.** We shall introduce the notion *crystal* following [2], which is a combinatorial object obtained by abstracting the properties of crystal bases in Theorem 3.1.

Definition 3.2 ([2])**.** A 6-tuple $(B, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\tilde{e}_i\}, \{\tilde{f}_i\})_{i\in I}$ is a *crystal* if B is a set and there exists a certain special element 0 outside of B and maps:

$$\mathrm{wt}: B \to P, \quad \varepsilon_i : B \to \mathbb{Z} \sqcup \{-\infty\}, \quad \varphi_i : B \to \mathbb{Z} \sqcup \{-\infty\} \quad (i \in I), \tag{3.2}$$

$$\tilde{e}_i : B \sqcup \{0\} \to B \sqcup \{0\}, \quad \tilde{f}_i : B \sqcup \{0\} \to B \sqcup \{0\} \ (i \in I), \tag{3.3}$$

satisfying :

(1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$.
(2) If $b, \tilde{e}_i b \in B$, then $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$, $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$.
(3) If $b, \tilde{f}_i b \in B$, then $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$, $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.

(4) For $b, b' \in B$ and $i \in I$, one has $\tilde{f}_i b = b'$ iff $b = \tilde{e}_i b'$.
(5) If $\varphi_i(b) = -\infty$ for $b \in B$, then $\tilde{e}_i b = \tilde{f}_i b = 0$ and $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$.

Here, a c*crystal graph* of crystal B is a I-colored oriented graph defined by $b \xrightarrow{i} b' \Leftrightarrow \tilde{f}_i(b) = b'$ for $b, b' \in B$.

Definition 3.3 ([2])**.** For crystals B_1 and B_2, Ψ is a *strict embedding* (resp. *isomorphism*) from B_1 to B_2 if $\Psi : B_1 \sqcup \{0\} \to B_2 \sqcup \{0\}$ is an injective (resp. bijective) map satisfying that $\Psi(0) = 0$, $\mathrm{wt}(\Psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\Psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\Psi(b)) = \varphi_i(b)$ for any $b \in B_1$ and Ψ commutes with all $\tilde{e}_i$'s and $\tilde{f}_i$'s,.

We obtain the tensor structure of crystals as follows([1, 2]):

Proposition 3.4. For crystals B_1 and B_2, set

$$B_1 \otimes B_2 = \{b_1 \otimes b_2 := (b_1, b_2) \mid b_1 \in B_1,\ b_2 \in B_2\} (= B_1 \times B_2).$$

Then, $B_1 \otimes B_2$ becomes a crystal by defining:

$$wt(b_1 \otimes b_2) = wt(b_1) + wt(b_2), \tag{3.4}$$

$$\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1)\rangle), \tag{3.5}$$

$$\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2), \varphi_i(b_1) + \langle h_i, wt(b_2)\rangle), \tag{3.6}$$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \tag{3.7}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). \end{cases} \tag{3.8}$$

Example 3.5. For $i \in I$, set $B_i := \{(n)_i \,|\, n \in \mathbb{Z}\}$ and

$$\begin{aligned} &\mathrm{wt}((n)_i) = n\alpha_i,\ \ \varepsilon_i((n)_i) = -n,\ \ \varphi_i((n)_i) = n, \\ &\varepsilon_j((n)_i) = \varphi_j((n)_i) = -\infty\ \ (i \neq j), \\ &\tilde{e}_i((n)_i) = (n+1)_i, \quad \tilde{f}_i((n)_i) = (n-1)_i, \\ &\tilde{e}_j((n)_i) = \tilde{f}_j((n)_i) = 0\ \ (i \neq j). \end{aligned}$$

Then B_i $(i \in I)$ possesses a crystal structure. Note that as a set the crystal B_i can be identified with the set of integers $\mathbb{Z}$.

3.3. **Explicit structure of the crystal $B_{i_1} \otimes \cdots \otimes B_{i_m}$.** Here we shall describe an explicit structure of tensor product of B_i's. Fix a sequence of indices $\mathbf{i} = (i_1, \cdots, i_m) \in I^m$ and write

$$(x_1, \cdots, x_m) := \tilde{f}_{i_1}^{x_1}(0)_{i_1} \otimes \cdots \otimes \tilde{f}_{i_m}^{x_m}(0)_{i_m} = (-x_1)_{i_1} \otimes \cdots \otimes (-x_m)_{i_m},$$

where if $n < 0$, then $\tilde{f}_i^n(0)_i$ means $\tilde{e}_i^{-n}(0)_i$. Note that here we do not necessarily assume that $\mathbf{i}$ is a reduced word though later we will take $\mathbf{i}$ to be a reduced longest word. By the tensor structure of crystals in Proposition 3.4, for the sequence $\mathbf{i}$ as above, we can describe the explicit crystal structure on $\mathbb{B}_{\mathbf{i}} := B_{i_1} \otimes \cdots \otimes B_{i_m}$ as follows: For $x = (x_1, \cdots, x_m) \in \mathbb{B}_{\mathbf{i}}$, define

$$\sigma_k(x) := x_k + \sum_{j<k} \langle h_{i_k}, \alpha_{i_j}\rangle x_j$$

and for $i \in I$ define

$$\begin{aligned} &\widetilde{\sigma}^{(i)}(x) := \max\{\sigma_k(x) \,|\, 1 \leq k \leq m \text{ and } i_k = i\}, \\ &\widetilde{M}^{(i)} = \widetilde{M}^{(i)}(x) := \{k \,|\, 1 \leq k \leq m,\ i_k = i,\ \sigma_k(x) = \widetilde{\sigma}^{(i)}(x)\}, \\ &\widetilde{m}_f^{(i)} = \widetilde{m}_f^{(i)}(x) := \max \widetilde{M}^{(i)}(x), \quad \widetilde{m}_e^{(i)} = \widetilde{m}_e^{(i)}(x) := \min\, \widetilde{M}^{(i)}(x). \end{aligned}$$

Now, the actions of the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ and the functions ε_i, φ_i and wt are written explicitly:

$$\tilde{f}_i(x)_k := x_k + \delta_{k,\widetilde{m}_f^{(i)}}, \qquad \tilde{e}_i(x)_k := x_k - \delta_{k,\widetilde{m}_e^{(i)}}, \tag{3.9}$$

$$\mathrm{wt}(x) := -\sum_{k=1}^{m} x_k \alpha_{i_k}, \quad \varepsilon_i(x) := \widetilde{\sigma}^{(i)}(x), \quad \varphi_i(x) := \langle h_i, \mathrm{wt}(x)\rangle + \varepsilon_i(x). \tag{3.10}$$

Define the function $\beta_k^{(\mathbf{i})}$ on $\mathbb{B}_{\mathbf{i}}$ by :

$$\beta_k^{(\mathbf{i})}(x) := \sigma_{k^+}(x) - \sigma_k(x) = x_k + \sum_{k<j<k^+} \langle h_i, \alpha_{i_j}\rangle x_j + x_{k^+}, \tag{3.11}$$

for $x = (x_1, \cdots, x_m) \in \mathbb{B}_{\mathbf{i}}$, where for $k \in [1, N]$, k^+ (resp. k^-) is the minimum (resp. maximum) number $j \in [1, N]$ such that $k < j$ (resp. $l < k$) and $i_k = i_j$ if it exists, otherwise $N + 1$ (resp. 0). Here one knows that $\widetilde{m}_f^{(i)}(x)$ and $\widetilde{m}_e^{(i)}(x)$ are determined by $\{\beta_k^{(\mathbf{i})}(x) \,|\, 1 \le k \le N,\ i_k = i\}$.

3.4. **Braid-type isomorphism.** We shall introduce some isomorphism of crystals, called "braid-type isomorphism".
Set $c_{ij} := \langle h_i, \alpha_j\rangle\langle h_j, \alpha_i\rangle$, $c_1 := -\langle h_i, \alpha_j\rangle$ and $c_2 := -\langle h_j, \alpha_i\rangle$. In the sequel, for $x \in \mathbb{Z}$, put

$$x_+ := \begin{cases} x & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Proposition 3.6 ([13])**.** There exist the following isomorphisms of crystals $\phi_{ij}^{(k)}$ $(k = 0, 1, 2, 3)$

(1) If $c_{ij} = 0$,

$$\phi_{ij}^{(0)} : B_i \otimes B_i \xrightarrow{\sim} B_j \otimes B_i, \tag{3.12}$$

where $\phi_{ij}^{(0)}((x)_i \otimes (y)_j) = (y)_j \otimes (x)_i$.

(2) If $c_{ij} = 1$,

$$\phi_{ij}^{(1)} : B_i \otimes B_j \otimes B_i \xrightarrow{\sim} B_j \otimes B_i \otimes B_j, \tag{3.13}$$

where

$$\phi_{ij}^{(1)}((x)_i \otimes (y)_j \otimes (z)_i) = (z + (-x + y - z)_+)_j \otimes (x + z)_i \otimes (y - z - (-x + y - z)_+)_j.$$

(3) If $c_{ij} = 2$,

$$\phi_{ij}^{(2)} : B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i, \tag{3.14}$$

where $\phi_{ij}^{(2)}$ is given by the following: for $(x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j$ we set $(X)_j \otimes (Y)_i \otimes (Z)_j \otimes (W)_i := \phi_{ij}^{(2)}((x)_i \otimes (y)_j \otimes (z)_i \otimes (w)_j)$.

$$X = w + (-c_2 x + y - w + c_2(x - c_1 y + z)_+)_+, \tag{3.15}$$

$$Y = x + c_1 w + (-x + z - c_1 w + (x - c_1 y + z)_+)_+, \tag{3.16}$$

$$Z = y - (-c_2 x + y - w + c_2(x - c_1 y + z)_+)_+, \tag{3.17}$$

$$W = z - c_1 w - (-x + z - c_1 w + (x - c_1 y + z)_+)_+. \tag{3.18}$$

(4) If $c_{ij} = 3$, the map

$$\phi_{ij}^{(3)} : B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \xrightarrow{\sim} B_j \otimes B_i \otimes B_j \otimes B_i \otimes B_j \otimes B_i, \tag{3.19}$$

is defined by the following: for $(x)_i\otimes(y)_j\otimes(z)_i\otimes(u)_j\otimes(v)_i\otimes(w)_j$ we set $A := -x+c_1y-z$, $B := -y+c_2z-u$, $C := -z+c_1u-v$ and $D := -u+c_2v-w$. Then $(X)_j\otimes(Y)_i\otimes(Z)_j\otimes(U)_i\otimes(V)_j\otimes(W)_i := \phi^{(3)}_{ij}((x)_i\otimes(y)_j\otimes(z)_i\otimes(u)_j\otimes(v)_i\otimes(w)_j)$ is given by

$$\begin{aligned}
X &= w+(D+(c_2C+(2B+A_+)_+)_+)_+,\\
Y &= x+c_1w+(c_1D+(3C+(2c_1B+2A_+)_+)_+)_+,\\
Z &= y+u+w-X-V,\\
U &= x+z+v-Y-W,\\
V &= u-w-(2D+(2c_2C+(3B+c_2A_+)_+)_+)_+,\\
W &= v-c_1w-(c_1D+(2C+(c_1B+A_+)_+)_+)_+.
\end{aligned}$$

They also satisfy $\phi^{(k)}_{ij}\circ\phi^{(k)}_{ji} = \mathrm{id}$.

We call such isomorphisms of crystals *braid-type isomorphisms*.

We also define a *braid-move* on the set of reduced words of $w\in W$ to be a composition of the following transformations induced from braid relations:

$$\begin{aligned}
&\cdots ij\cdots\to\cdots ji\cdots(c_{ij}=0), \quad \cdots iji\cdots\to\cdots jij\cdots(c_{ij}=1),\\
&\cdots ijij\cdots\to\cdots jiji\cdots(c_{ij}=2), \quad \cdots ijijij\cdots\to\cdots jijiji\cdots(c_{ij}=3),
\end{aligned}$$

which are called by 2-move, 3-move, 4-move, 6-move respectively.

3.5. **Cellular Crystal $\mathbb{B}_{\mathbf{i}} = \mathbb{B}_{i_1i_2\cdots i_k} = B_{i_1}\otimes\cdots\otimes B_{i_k}$.** For a reduced word $\mathbf{i} = i_1i_2\cdots i_k$ of some Weyl group element, we call the crystal $\mathbb{B}_{\mathbf{i}} := B_{i_1}\otimes\cdots\otimes B_{i_k}$ a *cellular crystal* associated with a reduced word $\mathbf{i}$. Indeed, it is obtained by applying the tropicalization functor to the geometric crystal on the Langlands-dual Schubert cell ${}^LX_w$, where $w = s_{i_1}\cdots s_{i_k}$ is an element of the Well group W ([14]). It is immediate from the braid-type isomorphisms that for any $w\in W$ and its reduced words $i_1\cdots i_l$ and $j_1\cdots j_l$, we get the following isomorphism of crystals:

$$B_{i_1}\otimes\cdots\otimes B_{i_l}\cong B_{j_1}\otimes\cdots\otimes B_{j_l}. \tag{3.20}$$

3.6. **Half potential and the crystal $B(\infty)$.** For a Laurent polynomial $\phi(x_1,\cdots,x_n)$ with positive coefficients, the tropicalization of ϕ is denoted by $\widetilde{\phi} := \mathrm{Trop}(\phi)$, which is given by the rule: $\mathrm{Trop}(ax+by) = \min(x,y)$ with $a,b>0$, $\mathrm{Trop}(xy) = x+y$ and $\mathrm{Trop}(x/y) = x-y$ and $\mathrm{Trop}(c) = 0$ for $c>0$. In [10], the crystal $B(\infty)$ has been realized as a certain subset of $\mathbb{B}_{\mathbf{i}}$ defined as follows:

Theorem 3.7 ([10, Theorem 5.11])**.** Define the subset of $\mathbb{B}_{\mathbf{i}}$:

$$(\widetilde{\mathbb{B}^-_{w_0}})_{\Phi^{(+)},\Theta_{\mathbf{i}}} = \{x=(x_1,\cdots,x_N)\in\mathbb{B}_{\mathbf{i}} \mid \widetilde{\Phi}^{(+)}(x)\geq 0\},$$

where $\mathbb{B}^-_{w_0}$ is a certain geometric crystal, $\widetilde{\Phi}^{(+)}$ is a tropicalization of the half potential $\Phi^{(+)}$ which is a Laurent polynomial with positive coefficients in N variables and $\Theta_{\mathbf{i}}$ is a certain positive structure on the geometric crystal $\mathbb{B}^-_{w_0}$. Then, $(\widetilde{\mathbb{B}^-_{w_0}})_{\Phi^{(+)},\Theta_{\mathbf{i}}}\cong B(\infty)$.

Remark 3.8. To define the crystal structure on $(\widetilde{\mathbb{B}^-_{w_0}})_{\Phi^{(+)},\Theta_{\mathbf{i}}}$, it is supposed that if $\tilde{e}_ix\notin(\widetilde{\mathbb{B}^-_{w_0}})_{\Phi^{(+)},\Theta_{\mathbf{i}}}$, then $\tilde{e}_ix = 0$. Thus, in this sense, the embedding $B(\infty)\cong(\widetilde{\mathbb{B}^-_{w_0}})_{\Phi^{(+)},\Theta_{\mathbf{i}}}\hookrightarrow\mathbb{B}_{\mathbf{i}}$ is not a strict embedding. In [15, 14], it has been given the strict embedding of $B(\infty)\hookrightarrow\mathbb{B}_{\mathbf{i}}$, which is called "Kashiwara embedding" and the method to describe the image of this embedding, called "polyhedral realization".

3.7. **Subspace $\mathcal{H}_{\mathbf{i}}$.** The object $\mathcal{H}_{\mathbf{i}}$ will play a significant role for this article.
Fix a reduced longest word $\mathbf{i} = i_1 \cdots i_N$ and take the function $\beta_k^{(\mathbf{i})}(x) = x_k + \sum_{k<j<k^+}\langle h_{i_k}\alpha_{i_j}\rangle x_j + x_{k^+}$ $(1 \le k \le N)$ as in (3.11). In what follows, let us identify the $\mathbb{Z}$-lattice $\mathbb{Z}^N$ with $B_{\mathbf{i}}$ and then we define the summation of elements $x = (x_1, \cdots, x_N)$ and $y = (y_1, \cdots, y_N)$ by $x+y = (x_1+y_1, \cdots, x_N+y_N)$ as a standard one in $\mathbb{Z}^N$. Here, we define the subspace $\mathcal{H}_{\mathbf{i}} \subset \mathbb{Z}^N$ by

$$\mathcal{H}_{\mathbf{i}} := \{x \in \mathbb{Z}^N (= \mathbb{B}_{\mathbf{i}}) \,|\, \beta_k^{(\mathbf{i})}(x) = 0 \text{ for any } k \text{ such that } k^+ \le N\} \subset \mathbb{B}_{\mathbf{i}}. \tag{3.21}$$

The following result was presented in [10]:

Proposition 3.9 ([10])**.** For $\mathbf{i} = i_1 i_2 \cdots i_N$, $k = 1, 2, \cdots, N$ and a fundamental weight Λ_i, set

$$h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N}\Lambda_i\rangle \quad \text{and} \quad \mathbf{h}_i := (h_i^{(1)}, h_i^{(2)}, \ldots, h_i^{(N)}) \in \mathbb{B}_{\mathbf{i}} \tag{3.22}$$

Then, we obtain that $\{\mathbf{h}_1, \cdots, \mathbf{h}_N\}$ is a $\mathbb{Z}$-basis of $\mathcal{H}_{\mathbf{i}}$, namely,

$$\mathcal{H}_{\mathbf{i}} = \mathbb{Z}\mathbf{h}_1 \oplus \mathbb{Z}\mathbf{h}_2 \oplus \cdots \oplus \mathbb{Z}\mathbf{h}_n. \tag{3.23}$$

Example 3.10. In $\mathfrak{g} = G_2$-case. Set $a_{12} = -1$ and $a_{21} = -3$. Taking a reduced longest word $\mathbf{i} = 121212$, one has

$$\beta_1^{(\mathbf{i})}(x) = x_1 - x_2 + x_3, \quad \beta_2^{(\mathbf{i})}(x) = x_2 - 3x_3 + x_4, \quad \beta_3^{(\mathbf{i})}(x) = x_3 - x_4 + x_5, \quad \beta_4^{(\mathbf{i})}(x) = x_4 - 3x_5 + x_6.$$

By the formula (3.22), one gets

$$\mathbf{h}_1 = (1, 3, 2, 3, 1, 0), \qquad \mathbf{h}_2 = (0, 1, 1, 2, 1, 1).$$

Then the solution space $\mathcal{H}_{\mathbf{i}}$ of $\beta_1^{(\mathbf{i})}(x) = \beta_2^{(\mathbf{i})}(x) = \beta_3^{(\mathbf{i})}(x) = \beta_4^{(\mathbf{i})}(x) = 0$ is given by

$$\mathcal{H}_{\mathbf{i}} = \{c_1\mathbf{h}_1 + c_2\mathbf{h}_2 = (c_1, c_2 + 3c_1, c_2 + 2c_1, 2c_2 + 3c_1, c_2 + c_1, c_2) \mid c_1, c_2 \in \mathbb{Z}\}.$$

Lemma 3.11. The braid-type isomorphisms are well-defined on $\mathcal{H}_{\mathbf{i}}$, that is, $\phi_{ij}^{(k)}(\mathcal{H}_{\mathbf{i}}) = \mathcal{H}_{\mathbf{i}'}$, where $\mathbf{i}'$ is the reduced word obtained by applying the corresponding braid-moves. We also obtain the following formula:

(1) For any $h = (\cdots, x, y, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes \cdots \in \mathcal{H}_{\mathbf{i}}$, assume that $a_{ij} = a_{ji} = 0$. Applying the braid-type isomorphism $\phi_{ij}^{(0)}$ on (x, y) in h, we have

$$\phi_{ij}^{(0)}(h) = (\cdots, y, x, \cdots) = \cdots \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{\mathbf{i}'} \tag{3.24}$$

(2) For any $h = (\cdots, x, y, z, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes \cdots \in \mathcal{H}_{\mathbf{i}}$, assume that $a_{ij} = a_{ji} = -1$. Applying the braid-type isomorphism $\phi_{ij}^{(1)}$ on (x, y, z) in h, we have

$$\phi_{ij}^{(1)}(h) = (\cdots, z, y, x, \cdots) = \cdots \otimes (-z)_j \otimes (-y)_i \otimes (-x)_j \otimes \cdots \in \mathcal{H}_{\mathbf{i}'} \tag{3.25}$$

(3) For $h = (\cdots, x, y, z, w, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes (-w)_j \cdots \in \mathcal{H}_{\mathbf{i}}$, assume that $a_{ij} \cdot a_{ji} = 2$. Applying the braid-type isomorphism $\phi_{ij}^{(2)}$ on (x, y, z, w) in h, we have

$$\phi_{ij}^{(2)}(h) = (\cdots, w, z, y, x, \cdots) = \cdots \otimes (-w)_j \otimes (-z)_i \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{\mathbf{i}'} \tag{3.26}$$

(4) For $h = (\cdots, x, y, z, u, v, w, \cdots) = \cdots \otimes (-x)_i \otimes (-y)_j \otimes (-z)_i \otimes (-u)_j \otimes (-v)_i \otimes (-w)_j \cdots \in \mathcal{H}_{\mathbf{i}}$, assume that $a_{ij} \cdot a_{ji} = 3$. Applying the braid-type isomorphism $\phi_{ij}^{(3)}$ on (x, y, z, u, v, w) in h, we have

$$\phi_{ij}^{(3)}(h) = (\cdots, w, v, u, z, y, x, \cdots) = \cdots \otimes (-w)_j \otimes (-z)_i \otimes (-y)_j \otimes (-x)_i \otimes \cdots \in \mathcal{H}_{\mathbf{i}'} \tag{3.27}$$

In [10, Sect.8], we have shown the following statements under the condition "$\mathbf{H}_{\mathbf{i}}$", where we omit the explicit form of $\mathbf{H}_{\mathbf{i}}$ since we do not need it here. But, we succeed in showing the following proposition without the condition $\mathbf{H}_{\mathbf{i}}$ since in [10] we have shown that there exists a specific reduced longest word $\mathbf{i}_0$ satisfying the condition $\mathbf{H}_{\mathbf{i}_0}$ for each simple Lie algebra $\mathfrak{g}$ and we got Lemma 3.11.

Proposition 3.12. Let $\mathbf{i} = i_1 i_2 \cdots i_N$ be an arbitrary reduced longest word. Here if the crystal $B(\infty)$ is realized in $\mathbb{B}_{\mathbf{i}}$ as in 3.6, we shall denote it by $B(\infty)_{\mathbf{i}}$ to emphasize the word $\mathbf{i}$. For $h \in \mathcal{H}_{\mathbf{i}}$, define

$$B^h(\infty)_{\mathbf{i}} := \{x + h \in \mathbb{Z}^N (= \mathbb{B}_{\mathbf{i}}) \mid x \in B(\infty)_{\mathbf{i}}\} \subset \mathbb{B}_{\mathbf{i}}.$$

(1) For any $x + h \in B^h(\infty)_{\mathbf{i}}$ and $i \in I$, we obtain

$$\tilde{e}_i(x + h) = \tilde{e}_i(x) + h, \qquad \tilde{f}_i(x + h) = \tilde{f}_i(x) + h. \tag{3.28}$$

(2) For any $h \in \mathcal{H}_{\mathbf{i}}$, we have $B(\infty)_{\mathbf{i}} \cap B^h(\infty)_{\mathbf{i}} \neq \emptyset$.
(3)

$$\mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^h(\infty)_{\mathbf{i}}$$

Remark 3.13. In the setting of the half-potential method in [10], as mentioned in Remark 3.8, the crystal $B(\infty)$ is realized as a subset of $\mathbb{B}_{\mathbf{i}}$ and it is supposed that $\tilde{e}_i x = 0$ if $\tilde{e}_i x \notin (\widetilde{\mathbb{B}^-_{w_0}})_{\Phi^{(+)},\Theta_{\mathbf{i}}} \cong B(\infty)$. At the statement (2), since $x \in B(\infty)_{\mathbf{i}}$ is considered as an element of $\mathbb{B}_{\mathbf{i}}$, $\tilde{e}_i x$ is also considered as an element in $\mathbb{B}_{\mathbf{i}}$. That is, even if $\tilde{e}_i x \notin B(\infty)$, we consider that $\tilde{e}_i x \in \mathbb{B}_{\mathbf{i}}$ and then it never vanishes.

It is immediate from this proposition that one has the following theorem:

Theorem 3.14 ([10])**.** For any simple Lie algebra $\mathfrak{g}$ and any reduced word $i_1 i_2 \cdots i_k$, the cellular crystal $\mathbb{B}_{i_1 i_2 \cdots i_k} = B_{i_1} \otimes B_{i_2} \otimes \cdots \otimes B_{i_k}$ is connected as a crystal graph.

4. Quiver Hecke Algebra and its modules

In this section, we shall introduce the quiver Hecke algebra and its basic properties (see [4, 5, 7, 16]).

4.1. Definition of Quiver Hecke Algebra.

For a finite index set I and a field $\mathbf{k}$, let $(\mathcal{Q}_{i,j}(u, v))_{i,j \in I} \in \mathbf{k}[u, v]$ be polynomials satisfying:

(1) $\mathcal{Q}_{i,j}(u, v) = \mathcal{Q}_{j,i}(v, u)$ for any $i, j \in I$.
(2) $\mathcal{Q}_{i,j}(u, v)$ is in the form:

$$\mathcal{Q}_{i,j}(u, v) = \begin{cases} \displaystyle\sum_{a(\alpha_i,\alpha_i)+b(\alpha_j,\alpha_j)=-2(\alpha_i,\alpha_j)} t_{i,j;a,b} u^a v^b & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;-a_{ij},0} \in \mathbf{k}^\times$.

For $\beta = \sum_i m_i \alpha_i \in Q_+$ with $|\beta| := \sum_i m_i = m$, set $I^\beta := \{\nu = (\nu_1, \cdots, \nu_m) \in I^m \mid \sum_{k=1}^m \alpha_{\nu_k} = \beta\}$.

Definition 4.1. For $\beta \in Q_+$, the *quiver Hecke algebra* $R(\beta)$ associated with a Cartan matrix A and polynomials $\mathcal{Q}_{i,j}(u, v)$ is the $\mathbf{k}$-algebra generated by

$$\{e(\nu) | \nu \in I^\beta\}, \quad \{x_k | 1 \leq k \leq n\}, \quad \{\tau_i | 1 \leq i \leq n-1\}$$

with the following relations:

$$e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu\in I^\beta} e(\nu) = 1, \quad e(\nu)x_k = x_k e(\nu), \quad x_k x_l = x_l x_k,$$
$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k\tau_l = \tau_l\tau_k \ \text{ if } |k-l|>1,$$
$$\tau_k^2 e(\nu) = \mathcal{Q}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),$$
$$(\tau_k x_l - x_{s_k(l)}\tau_k)e(\nu) = \begin{cases} -e(\nu) & \text{if } l=k,\ \nu_k=\nu_{k+1},\\ e(\nu) & \text{if } l=k+1,\ \nu_k=\nu_{k+1},\\ 0 & \text{otherwise,}\end{cases}$$
$$(\tau_{k+1}\tau_k\tau_{k+1} - \tau_k\tau_{k+1}\tau_k)e(\nu) = \begin{cases} \overline{\mathcal{Q}}_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}, x_{k+2})e(\nu) & \text{if } \nu_k=\nu_{k+2},\\ 0 & \text{otherwise,}\end{cases}$$

where $\overline{\mathcal{Q}}_{i,j}(u,v,w) = \frac{\mathcal{Q}_{i,j}(u,v)-\mathcal{Q}_{i,j}(w,v)}{u-w} \in \mathbf{k}[u,v,w]$.

(1) The relations above are homogeneous if we define

$$\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau_l e(\nu)) = -(\alpha_{\nu_l}, \alpha_{\nu_{l+1}}).$$

Thus, $R(\beta)$ becomes a $\mathbb{Z}$-graded algebra. Here we define the *weight* of $R(\beta)$-module M as $\mathrm{wt}(M) = -\beta$.

(2) Let $M = \bigoplus_{k\in\mathbb{Z}} M_k$ be a $\mathbb{Z}$-graded $R(\beta)$-module. Define a *grading shift functor* q on the category of graded $R(\beta)$-modules $R(\beta)$-Mod by

$$qM := \bigoplus_{k\in\mathbb{Z}} (qM)_k, \quad \text{where } (qM)_k = M_{k-1}.$$

(3) For $M, N \in R(\beta)$-Mod, let $\mathrm{Hom}_{R(\beta)}(M,N)$ be the space of degree preserving morphisms and define $\mathrm{HOM}_{R(\beta)}(M,N) := \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{R(\beta)}(q^k M, N)$, which is a space of morphisms up to grading shift. We define $\deg(f) = k$ for $f \in \mathrm{Hom}_{R(\beta)}(q^k M, N)$.

(4) Let ψ be the anti-automorphism of $R(\beta)$ preserving all generators. For $M \in R(\beta)$-Mod, define $M^* := \mathrm{HOM}_{\mathbf{k}}(M, \mathbf{k})$ with the $R(\beta)$- module structure by $(r\cdot f)(u) := f(\psi(r)u)$ for $r \in R(\beta)$, $u \in M$ and $f \in M^*$, which is called a *dual module* of M. In particular, if $M \cong M^*$ we call M is *self-dual*.

(5) For $\beta, \gamma \in Q_+$, set $e(\beta,\gamma) = \sum_{\nu\in I^\beta, \nu'\in I^\gamma} e(\nu,\nu')$. We define an injective homomorphism $\xi_{\beta,\gamma}$: $R(\beta)\otimes R(\gamma) \to e(\beta,\gamma)R(\beta+\gamma)e(\beta,\gamma)$ by $\xi(\beta,\gamma)(e(\nu)\otimes e(\nu')) = e(\nu,\nu')$, $\xi(\beta,\gamma)(x_k e(\beta)\otimes 1) = x_k e(\beta,\gamma)$, $\xi(\beta,\gamma)(1\otimes x_k e(\gamma)) = x_{k+|\beta|}e(\beta,\gamma)$, $\xi(\beta,\gamma)(\tau_k e(\beta)\otimes 1) = \tau_k e(\beta,\gamma)$, $\xi(\beta,\gamma)(1\otimes \tau_k e(\gamma)) = \tau_{k+|\beta|}e(\beta,\gamma)$.

(6) For $M \in R(\beta)$-Mod and $N \in R(\gamma)$-Mod, define the *convolution product* $\circ$ by

$$M\circ N := R(\beta+\gamma)e(\beta,\gamma) \otimes_{R(\beta)\otimes R(\gamma)} (M\otimes N)$$

For simple $M \in R(\beta)$-Mod and simple $N \in R(\gamma)$-Mod, we say M and N *strongly commutes* if $M\circ N$ is simple and M is *real* if $M\circ M$ is simple.

(7) For $M \in R(\beta)$-Mod and $N \in R(\gamma)$-Mod, denote by $M\nabla N := \mathrm{hd}(M\circ N)$ the head of $M\circ N$ and $M\Delta N := \mathrm{soc}(M\circ N)$ the socle of $M\circ N$, where the head of module M is the quotient by its radical and the socle of module M is the summation of all simple submodules.

4.2. **Categorification of quantum coordinate ring $\mathcal{A}_q(\mathfrak{n})$.** Let $R(\beta)$-gmod be the full subcategory of $R(\beta)$-Mod whose objects are finite-dimensional graded $R(\beta)$-modules and set R-gmod$=\bigoplus_{\beta\in Q_+} R(\beta)$-gmod. Define the functors

$$E_i : R(\beta)\text{-gmod} \to R(\beta-\alpha_i)\text{-gmod}, \qquad F_i : R(\beta)\text{-gmod} \to R(\beta+\alpha_i)\text{-gmod}\ ,$$

by $E_i(M) := e(\alpha_i, \beta - \alpha_i)M, \quad F_i(M) = L(i) \circ M$, where $e(\alpha_i, \beta - \alpha_i) := \sum_{\nu \in I^\beta, \nu_1 = i} e(\nu)$ and $L(i) := R(\alpha_i)/R(\alpha_i)x_1$ is a 1-dimensional simple $R(\alpha_i)$-module. Let $\mathcal{K}(R\text{-gmod})$ be the Grothendieck ring of R-gmod and then $\mathcal{K}(R\text{-gmod})$ becomes a $\mathbb{Z}[q, q^{-1}]$-algebra with the multiplication induced by the convolution product and $\mathbb{Z}[q, q^{-1}]$-action induced by the grading shift functor q. Here, one obtain the following:

Theorem 4.2 ([4, 16])**.** As a $\mathbb{Z}[q, q^{-1}]$-algebra there exists an isomorphism

$$\mathcal{K}(R\text{-gmod}) \cong \mathcal{A}_q(\mathfrak{n})_{\mathbb{Z}[q,q^{-1}]}.$$

4.3. **Categorification of the crystal** $B(\infty)$ **by Lauda and Vazirani** [11]**.** The following lemma is given in [4]:

Lemma 4.3 ([4])**.** For any simple $R(\beta)$-module M, $\mathrm{soc}(E_iM)$, $\mathrm{hd}(E_iM)$ and $\mathrm{hd}(F_iM)$ are all simple modules. Here we also have that $\mathrm{soc}(E_iM) \cong \mathrm{hd}(E_iM)$ up to grading shift.

For $M \in R(\beta)$-gmod, define

$$\mathrm{wt}(M) = -\beta, \quad \varepsilon_i(M) = \max\{n \in \mathbb{Z} \,|\, E_i^n M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \langle h_i, \mathrm{wt}(M)\rangle, \tag{4.1}$$

$$\widetilde{E}_iM := q_i^{1-\varepsilon_i(M)}\mathrm{soc}(E_iM) \cong q_i^{\varepsilon_i(M)-1}\mathrm{hd}(E_iM), \qquad \widetilde{F}_iM := q_i^{\varepsilon_i(M)}\mathrm{hd}(F_iM). \tag{4.2}$$

Set $\mathbb{B}(R\text{-gmod}) := \{S \,|\, S$ is a self-dual simple module in R-gmod$\}$. Then, it follows from Lemma 4.3 that $\widetilde{E}_i$ and $\widetilde{F}_i$ are well-defined on $\mathbb{B}(R\text{-gmod})$.

Theorem 4.4 ([11])**.** The 6-tuple $(\mathbb{B}(R\text{-gmod}), \{\widetilde{E}_i\}, \{\widetilde{F}_i\}, \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\})_{i \in I}$ holds a crystal structure and there exists the following isomorphism of crystals:

$$\Psi : \mathbb{B}(R\text{-gmod}) \quad \overset{\sim}{\longrightarrow} \quad B(\infty).$$

Remark 4.5. Note that Lauda and Vasirani showed this theorem under more general setting that $\mathfrak{g}$ is arbitrary symmetrizable Kac-Moody Lie algebra. Here we assume that $\mathfrak{g}$ is a simple Lie algebra. The definition of $\widetilde{E}_i$ and $\widetilde{F}_i$ in (4.2) differs from the one in [11], which follows the one in [7].

5. Localization of monoidal category

Here we shall review the theory of localization for monoidal category following [5].

5.1. **Braiders and Real Commuting Family.** Let Λ be $\mathbb{Z}$-lattice and $\mathcal{T} = \oplus_{\lambda \in \Lambda}\mathcal{T}_\lambda$ be a $\mathbf{k}$-linear Λ-graded monoidal category with a data consisting of a bifunctor $\otimes : \mathcal{T}_\lambda \times \mathcal{T}_\mu \to \mathcal{T}_{\lambda+\mu}$, an isomorphism $a(X, Y, Z) : (X \otimes Y) \otimes Z \overset{\sim}{\longrightarrow} X \otimes (Y \otimes Z)$ satisfying $a(X, Y, Z \otimes W) \circ a(X \otimes Y, Z, W) = \mathrm{id}_X \otimes a(Y, Z, W) \circ a(X, Y \otimes Z, W) \circ a(X, Y, Z) \otimes \mathrm{id}_W$ and an object $\mathbf{1} \in \mathcal{T}_0$ endowed with an isomorphism $\epsilon : \mathbf{1} \otimes \mathbf{1} \overset{\sim}{\longrightarrow} \mathbf{1}$ such that the functor $X \mapsto X \otimes \mathbf{1}$ and $X \mapsto \mathbf{1} \otimes X$ are fully-faithful.

Definition 5.1 ([5])**.** Let q be the grading shift functor on $\mathcal{T}$. A *graded braider* is a triple (C, R_C, ϕ), where $C \in \mathcal{T}$, $\mathbb{Z}$-linear map $\phi : \Lambda \to \mathbb{Z}$ and a morphism:

$$R_C : C \otimes X \to q^{\phi(\lambda)} X \otimes C \quad (X \in \mathcal{T}_\lambda),$$

satisfying the following commutative diagram:

$$\begin{array}{ccc} C \otimes X \otimes Y & \xrightarrow{R_C(X) \otimes Y} & q^{\phi(\lambda)} \otimes X \otimes C \otimes Y \\ & \searrow^{R_C(X \otimes Y)} & \Big\downarrow X \otimes R_C(Y) \\ & & q^{\phi(\lambda+\mu)}(X \otimes Y) \otimes C \end{array} \qquad (X \in \mathcal{T}_\lambda, \ Y \in \mathcal{T}_\mu)$$

and being functorial, that is, for any $X, Y \in \mathcal{T}$ and $f \in \mathrm{Hom}_{\mathcal{T}}(X, Y)$ it satisfies the following commutative diagram:

$$\begin{array}{ccc} C \otimes X & \xrightarrow{\mathrm{id} \otimes f} & C \otimes Y \\ {\scriptstyle R_C(X)} \downarrow & & \downarrow {\scriptstyle R_C(Y)} \\ X \otimes C & \xrightarrow{f \otimes \mathrm{id}} & Y \otimes C \end{array}$$

Definition 5.2 ([5])**.** Let I be an index set and $(C_i, R_{C_i}, \phi_i)_{i \in I}$ a family of graded braiders in $\mathcal{T}$. We say that $(C_i, R_{C_i}, \phi_i)_{i \in I}$ is a *real commuting family of graded braiders* in $\mathcal{T}$ if

(1) $C_i \in \mathcal{T}_{\lambda_i}$ for some $\lambda_i \in \Lambda$, and $\phi_i(\lambda_i) = 0$, $\phi_i(\lambda_j) + \phi_j(\lambda_i) = 0$ for any $i, j \in I$.
(2) $R_{C_i}(C_i) \in \mathbf{k}^{\times} \mathrm{id}_{C_i \otimes C_i}$ for any $i \in I$.
(3) $R_{C_i}(C_j) \otimes R_{C_j}(C_i) \in \mathbf{k}^{\times} \mathrm{id}_{C_i \otimes C_j}$ for any $i, j \in I$.

Note that R_{C_i}'s satisfy so-called "Yang-Baxter equation", such as,

$$R_{C_i}(C_j) \circ R_{C_i}(C_k) \circ R_{C_j}(C_k) = R_{C_j}(C_k) \circ R_{C_i}(C_k) \circ R_{C_i}(C_j) \quad \text{on} \quad C_i \circ C_j \circ C_k.$$

For a finite index set I, set $\Gamma := \oplus_{i \in I} \mathbb{Z} e_i$ and $\Gamma_+ := \oplus_{i \in I} \mathbb{Z}_{\geq 0} e_i$.

Lemma 5.3 ([5])**.** Suppose that we have a real commuting family of graded braiders $(C_i, R_{C_i}, \phi_i)_{i \in I}$. We can choose a bilinear map $H : \Gamma \times \Gamma \to \mathbb{Z}$ such that $\phi_i(\lambda_j) = H(e_i, e_j) - H(e_j, e_i)$ and there exist

(1) an object C^{α} for any $\alpha \in \Gamma_+$.
(2) an isomorphism $\xi_{\alpha,\beta} : C^{\alpha} \otimes C^{\beta} \xrightarrow{\sim} q^{H(\alpha,\beta)} C^{\alpha+\beta}$ for any $\alpha, \beta \in \Gamma_+$

such that $C^0 = 1$ and $C^{e_i} = C_i$.

5.2. **Localization.** Let $\mathcal{T}$ and $(C_i, R_{C_i}, \phi_i)_{i \in I}$ be as above and $\{C^{\alpha}\}_{\alpha \in \Gamma_+}$ objects as in the previous lemma. We define a partial order $\preceq$ on Γ by

$$\alpha \preceq \beta \Longleftrightarrow \beta - \alpha \in \Gamma_+$$

For $\alpha_1, \alpha_2, \cdots \in \Gamma$, define

$$\mathcal{D}_{\alpha_1, \alpha_2, \cdots} := \{\delta \in \Gamma \,|\, \alpha_j + \delta \in \Gamma_+ \quad \text{for any } j = 1, 2, \cdots\}.$$

For $X \in \mathcal{T}_{\lambda}$, $Y \in \mathcal{T}_{\mu}$ and $\delta \in \mathcal{D}_{\alpha,\beta}$, set

$$H_{\delta}((X, \alpha), (Y, \beta)) := \mathrm{Hom}_{\mathcal{T}}(C^{\delta+\alpha} \otimes X, q^{P(\alpha,\beta,\delta,\mu)} Y \otimes C^{\beta+\delta}),$$

where a $\mathbb{Z}$-valued function $P(\alpha, \beta, \delta, \mu) := H(\delta, \beta - \alpha) + \phi(\delta + \beta, \mu)$ and the map $\phi : \Gamma \times \Lambda \to \mathbb{Z}$ is defined by $\phi(\alpha, L(\beta)) = H(\alpha, \beta) - H(\beta, \alpha)$ and $L : \Gamma \to \Lambda$ is defined by $L(e_i) = \lambda_i$ ([5]).

Lemma 5.4 ([5])**.** For $\delta \preceq \delta'$ there exists the map

$$\zeta_{\delta,\delta'} : H_{\delta}((X, \alpha), (Y, \beta)) \to H_{\delta'}((X, \alpha), (Y, \beta))$$

satisfying

$$\zeta_{\delta,\delta'} \circ \zeta_{\delta',\delta''} = \zeta_{\delta,\delta''} \text{for } \delta \preceq \delta' \preceq \delta''.$$

Therefore, we find that $\{H_{\delta}((X, \alpha), (Y, \beta))\}_{\delta \in \mathcal{D}_{\alpha,\beta}}$ becomes an inductive system.

Definition 5.5 (Localization [5])**.** We define the category $\widetilde{\mathcal{T}}$ by

$$\begin{aligned} &\mathrm{Ob}(\widetilde{\mathcal{T}}) := \mathrm{Ob}(\mathcal{T}) \times \Gamma, \\ &\mathrm{Hom}_{\widetilde{\mathcal{T}}}((X, \alpha), (Y, \beta)) := \varinjlim_{\substack{\delta \in \mathcal{D}(\alpha,\beta), \\ \lambda + L(\alpha) = \mu + L(\beta)}} H_{\delta}((X, \alpha), (Y, \beta)), \end{aligned}$$

where $X \in \mathcal{T}_{\lambda}$, $Y \in \mathcal{T}_{\mu}$ and the function $L : \Gamma \to \Lambda$ ($e_i \mapsto \lambda_i$) is as above. We call this $\widetilde{\mathcal{T}}$ a *localization* of $\mathcal{T}$ by $(C_i, R_{C_i}, \phi_i)_{i \in I}$ and denote it by $\mathcal{T}[C_i^{\otimes -1} \,|\, i \in I]$ when we emphasize $\{C_i \mid i \in I\}$.

Theorem 5.6 ([5])**.** $\widetilde{\mathcal{T}}$ becomes a monoidal category. Moreover, there exists a monoidal functor $\Upsilon : \mathcal{T} \to \widetilde{\mathcal{T}}$ such that

(1) $\Upsilon(C_i)$ is *invertible* in $\widetilde{\mathcal{T}}$ for any $i \in I$, namely, the functors $X \mapsto X \otimes \Upsilon(C_i)$ and $X \mapsto \Upsilon(C_i) \otimes X$ are equivalence of categories.
(2) For any $i \in I$ and $X \in \mathcal{T}$, $\Upsilon(R_{C_i}(X)) : \Upsilon(C_i \otimes X) \to \Upsilon(X \otimes C_i)$ is an isomorphism.
(3) The functor Υ holds the following universality: If there exists another monoidal category $\mathcal{T}'$ and a monoidal fucntor $\Upsilon' : \mathcal{T} \to \mathcal{T}'$ satisfying the above statements (1) and (2), then there exists a monoidal functor $F : \widetilde{\mathcal{T}} \to \mathcal{T}'$ (unique up to iso.) such that $\Upsilon' = F \circ \Upsilon$.

Proposition 5.7 ([5])**.** Under the setting above, we obtain

(1) $(X, \alpha+\beta) \cong q^{-H(\beta,\alpha)}(C^\alpha \otimes X, \beta)$, $(1, \beta) \otimes (1, -\beta) \cong q^{-H(\beta,\beta)}(1, 0)$ for $\alpha \in \Gamma_+, \beta \in \Gamma$ and $X \in \widetilde{\mathcal{T}}$.
(2) If $\mathcal{T}$ is an abelian category, then so is $\widetilde{\mathcal{T}}$.
(3) The functors $\Upsilon : \mathcal{T} \to \widetilde{\mathcal{T}}$ is exact.
(4) If the functor $-\otimes Y$ and $Y \otimes -$ are exact for any Y in $\mathcal{T}$, then the functors $\widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}$ ($X \mapsto X \otimes Y$ (resp. $X \to Y \otimes X$)) are exact for any Y in $\widetilde{\mathcal{T}}$.

6. Localization of the category R-gmod

In this section, we shall apply the method of localization to the category R-gmod.

6.1. **Determinantial Modules.** Here we just go back to the setting as in Sect.4. Let $L(i^n) := q_i^{\frac{n(n-1)}{2}} L(i)^{\circ n}$ be a simple $R(n\alpha_i)$-module satisfying $\mathrm{qdim}(L(i^n)) = [n]_i! := \prod_{k=1}^n \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}}$ $(q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}})$.

Definition 6.1 ([5, 7])**.** For $M \in R$-gmod, define

$$\widetilde{F}_i^n(M) := L(i^n)\nabla M.$$

For a Weyl group element w, let $s_{i_1} \cdots s_{i_l}$ be its reduced expression. For a dominant weight $\Lambda \in P_+$, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_l}\Lambda \rangle \qquad (k = 1, \cdots, l).$$

We define the *determinantial module* associated with w and Λ by

$$\mathbf{M}(w\Lambda, \Lambda) := \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_l}^{m_l}\mathbf{1},$$

where $\mathbf{1}$ is a trivial $R(0)$-module.

Note that in general, one can define determinantial modules $\mathbf{M}(w\Lambda, u\Lambda)$ $(w, u \in W)$ which corresponds to the generalized minor $\Delta_{w\Lambda,u\Lambda}$.

Now, let us see some similarity between the family of determinantial modules $\{\mathbf{M}(w_0\Lambda, \Lambda)\}_{\Lambda \in P_+}$ and the subspace $\mathcal{H}_{\mathbf{i}}$. As has seen above that for a reduced longest word $\mathbf{i} = i_1 \cdots i_N$, the subspace $\mathcal{H}_{\mathbf{i}} \subset \mathbb{B}_{\mathbf{i}}$ is presented by

$$\mathcal{H}_{\mathbf{i}} = \bigoplus_{i \in I} \mathbb{Z}\mathbf{h}_i, \quad \mathbf{h}_i = ((h_i^{(k)} := \langle h_{i_k}, s_{i_{k+1}} \cdots s_{i_N}\Lambda_i \rangle)_{k=1,\cdots,N}.$$

Furthermore, we also get

Proposition 6.2. For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$ and $\Lambda \in P_+$, set

$$m_k := \langle h_{i_k}, s_{i_{k+1}} s_{i_{k+2}} \cdots s_{i_l}\Lambda \rangle \quad (k = 1, 2, \cdots, N) \quad \text{and} \quad \mathbf{h}_\Lambda := (m_1, \cdots, m_N).$$

Then we obtain

$$\mathbf{h}_\Lambda = \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) = \tilde{f}_{i_1}^{m_1}(0)_{i_1} \otimes \tilde{f}_{i_2}^{m_2}(0)_{i_2} \otimes \cdots \otimes \tilde{f}_{i_N}^{m_N}(0)_{i_N} \in \mathcal{H}_{\mathbf{i}},$$

where note that for $\Lambda = \sum_i a_i\Lambda_i$, one has $\mathbf{h}_\Lambda = \sum_i a_i\mathbf{h}_{\Lambda_i}$.

By this proposition, one observes that there would exist a certain correspondence

$$\mathbf{M}(w_0\Lambda,\Lambda) = \widetilde{F}_{i_1}^{m_1}\cdots\widetilde{F}_{i_l}^{m_l}\mathbf{1} \quad\longleftrightarrow\quad \mathbf{h}_\Lambda = \tilde{f}_{i_1}^{m_1}\cdots\tilde{f}_{i_N}^{m_N}((0)_{i_1}\otimes(0)_{i_2}\otimes\cdots\otimes(0)_{i_N}). \tag{6.1}$$

Definition 6.3 ([5])**.** For $\beta\in Q_+$, define a central element in $R(\beta)$ by $\mathfrak{p}_i := \sum_{\nu\in I^\beta}\left(\prod_{a\in\{1,2,\cdots,\mathrm{ht}(\beta)\},\nu_a=i} x_a\right)e(\nu)\in R(\beta)$. For a simple $M\in R(\beta)$-gmod, define an *affinization* $\widehat{M}$ of M with degree d:

(1) There is an endomorphism $z:\widehat{M}\to\widehat{M}$ of degree $d>0$ such that $\widehat{M}$ is finitely generated free module of $\mathbf{k}[z]$ and $\widehat{M}/z\widehat{M}\cong M$.
(2) $\mathfrak{p}_i\widehat{M}\neq 0$ for any $i\in I$.

Theorem 6.4 ([5, Theorem 3.26])**.** For any $\Lambda\in P_+$ and $w\in W$, the determinantial module $\mathbf{M}(w\Lambda,\Lambda)$ is a real simple module and admits an affinization $\widehat{\mathbf{M}}(w\Lambda,\Lambda)$.

Note that indeed, if $\mathfrak{g}$ is simply-laced, then the affinization $\widehat{M}$ always exists for any simple $M\in R(\beta)$-gmod as ([3]),

$$\widehat{M} = \mathbf{k}[z]\otimes_{\mathbf{k}} M.$$

6.2. **Localization.**

Definition 6.5 ([5])**.** Let M be a simple R-module. A graded braider (M,R_M,ϕ) is *non-degenerate* if $R_M(L(i)):M\circ L(i)\to L(i)\circ M$ is a non-zero homomorphism.

For R-gmod, there exists a non-degenerate real commuting family of graded braiders $(C_i,R_{C_i},\phi_i)_{i\in I}$([5]). Set $C_\Lambda := \mathbf{M}(w_0\Lambda,\Lambda)$ and denote C_{Λ_i} by C_i.

Proposition 6.6 ([8])**.** For $\Lambda=\sum_i m_i\Lambda_i\in P_+$, we obtain the following isomorphism up to grading shift:

$$C_\Lambda := \mathbf{M}(w_0\Lambda,\Lambda)\cong C_1^{\circ m_1}\circ\cdots\circ C_n^{\circ m_n}. \tag{6.2}$$

Theorem 6.7 ([5, Proposition 5.1])**.** Define the function $\phi_i:Q\to\mathbb{Z}$ by

$$\phi_i(\beta) := -(\beta, w_0\Lambda_i+\Lambda_i).$$

Then there exists $\{(C_i,R_{C_i},\phi_i)\}_{i\in I}$ a non-degenerate real commuting family of graded braiders of the monoidal category R-gmod.

Now, we take $\Gamma = P = \bigoplus_i\mathbb{Z}\Lambda_i$ and $\Gamma_+ = P_+ = \bigoplus_i\mathbb{Z}_{\geq 0}\Lambda_i$. Here, we obtain the localization R-gmod$[C_i^{\circ -1}\,|\,i\in I]$ by $\{(C_i,R_{C_i},\phi_i)\}_{i\in I}$, which will be denoted by $\widetilde{R}$-gmod.

By the above Proposition, it holds the following properties:

Proposition 6.8 ([5])**.** Let $\Phi:R\text{-gmod}\to\widetilde{R}\text{-gmod}$ be the canonical functor. Then,

(1) $\widetilde{R}$-gmod is an abelian category and the functor Φ is exact.
(2) For any simple object $S\in R$-gmod, $\Phi(S)$ is simple in $\widetilde{R}$-gmod.
(3) $\widetilde{C}_i := \Phi(C_i)$ $(i\in I)$ is invertible central graded braider in $\widetilde{R}$-gmod.

For $\mu\in P$, define $\widetilde{C}_\mu$ such that $\widetilde{C}_\mu := \Phi(C_\mu)$ for $\mu\in P_+$, $\widetilde{C}_{-\Lambda_i} = C_i^{\circ -1}$ and $\widetilde{C}_{\lambda+\mu} = \widetilde{C}_\lambda\circ\widetilde{C}_\mu$ for $\lambda,\mu\in P$ up to grading shift.

(4) Any simple object in $\widetilde{R}$-gmod is isomorphic to $\widetilde{C}_\Lambda\circ\Phi(S)$ for some simple module $S\in R$-gmod and $\Lambda\in P$.

Note that in (4) $\Lambda\in P$ and $S\in R$-gmod are not necessarily unique.

Remark 6.9. In [5], the localization is applied to more general category $\mathscr{C}_w$, which is the full subcategory of R-gmod associated with a Weyl group element w. The category R-gmod here coincides with $\mathscr{C}_{w_0}$ associated with the longest element w_0 in W.

Definition 6.10. The category $\widetilde{R}$-gmod is abelian and monoidal. Therefore, its Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ holds a natural $\mathbb{Z}[q, q^{-1}]$-algebra structure, which defines a *localized quantum coordinate ring* $\widetilde{\mathcal{A}_q(\mathfrak{n})} := \mathbb{Q}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{K}(\widetilde{R}\text{-gmod})$.

Indeed, the Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ is described as follows:

Proposition 6.11 ([5, Corollary 5.4])**.** The Grothendieck ring $\mathcal{K}(\widetilde{R}\text{-gmod})$ is isomorphic to the left ring of quotients of the ring $\mathcal{K}(R\text{-gmod})$ with respect to the multiplicative set

$$\mathcal{S} := \{q^k \prod_{i\in I}[C_i]^{a_i} \mid k \in \mathbb{Z},\ (a_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}\},$$

that is, $\mathcal{K}(\widetilde{R}\text{-gmod}) \cong \mathcal{S}^{-1}\mathcal{K}(R\text{-gmod})$.

7. Crystal Structure on localized quantum coordinate rings

We shall mention the main theorem, crystal structure on localized quantum coordinate ring $\widetilde{\mathcal{A}_q(\mathfrak{n})}$. More precisely, we shall define a crystal structure on a family of self-dual simple objects in the category $\widetilde{R}$-gmod (Theorem 7.4) and mention that it is isomorphic to the cellular crystal $\mathbb{B}_{\mathbf{i}}$ (Theorem 7.5), where $\mathbf{i}$ is a reduced word for the longest Weyl group element w_0.

Lemma 7.1 ([4, Proposition 2.18])**.** For any $i \in I$, $\beta, \gamma \in Q_+$, any modules $M \in R(\beta)$-gmod and $N \in R(\gamma)$-gmod, one has the following exact sequence in $R(\beta + \gamma - \alpha_i)$-gmod:

$$0 \longrightarrow E_iM \circ N \longrightarrow E_i(M \circ N) \longrightarrow q^{-(\alpha,\beta)}M \circ E_iN \longrightarrow 0. \tag{7.1}$$

For $i \in I$, let $i^* \in I$ be a unique index satisfying $\Lambda_{i^*} = -w_0\Lambda_i$.

Lemma 7.2. (1) For $S \in R$-gmod and $i \in I$, if $E_iS = 0$, then the module $E_iC_{\Lambda_{i^*}} \circ S$ is a simple module.
(2) If $E_iS = 0$ for $S \in R$-gmod, then we get for $\Lambda \in P_+$ with $\langle h_{i^*}, \Lambda\rangle > 0$,

$$\mathrm{soc}(E_i(C_\Lambda \circ S)) \cong C_{\Lambda-\Lambda_{i^*}} \circ (E_iC_{i^*} \circ S), \tag{7.2}$$

up to grading shift.

We set

$$\mathbb{B}(\widetilde{R}\text{-gmod}) := \{L \,|\, L \text{ is a self-dual simple module in } \widetilde{R}\text{-gmod}\}.$$

Lemma 7.3 ([5])**.** For any simple $L \in \widetilde{R}$-gmod, there exists a unique $n \in \mathbb{Z}$ such that q^nL is self-dual simple. For a simple module $L \in \widetilde{R}$-gmod we define $\delta(L)$ to be this integer n.

Then by this lemma, we find that $\mathbb{B}(\widetilde{R}\text{-gmod})$ includes all simple modules in $\widetilde{R}$-gmod up to grading shift. For a simple object $\widetilde{C_\Lambda} \circ \Phi(S) \in \widetilde{R}$-gmod we write simply $C_\Lambda \circ S$ if there is no confusion.

Now let us define the Kashiwara operators $\widetilde{F}_i$ and $\widetilde{E}_i$ $(i \in I)$ on $\mathbb{B}(\widetilde{R}\text{-gmod})$ by

$$\widetilde{F}_i(C_\Lambda \circ S) = q^{\delta(C_\Lambda \circ \widetilde{F}_iS)}C_\Lambda \circ \widetilde{F}_iS, \tag{7.3}$$

$$\widetilde{E}_i(C_\Lambda \circ S) = \begin{cases} q^{\delta(C_\Lambda \circ \widetilde{E}_iS)}C_\Lambda \circ \widetilde{E}_iS & \text{if } E_iS \neq 0, \\ q^{\delta(C_{\Lambda-\Lambda_{i^*}} \circ (\widetilde{E}_iC_{\Lambda_{i^*}} \circ S))}C_{\Lambda-\Lambda_{i^*}} \circ (\widetilde{E}_iC_{\Lambda_{i^*}} \circ S) & \text{if } E_iS = 0, \end{cases} \tag{7.4}$$

where $C_\Lambda \circ S$ is a self-dual simple module in $\widetilde{R}$-gmod, the actions $\widetilde{E}_iS$ and $\widetilde{F}_iS$ are given in (4.2), which is defined on the family of all self-dual simple modules in R-gmod and in (7.4) the module $\widetilde{E}_iC_{\Lambda_{i^*}} \circ S$ is simple by Lemma 7.2. Note that for any $m > 0$, $\widetilde{E}_i^m(C_\Lambda \circ S) \neq 0$, $\widetilde{F}_i^m(C_\Lambda \circ S) \neq 0$.

Let $\Psi : \mathbb{B}(R\text{-gmod}) \xrightarrow{\sim} B(\infty)$ be as in Theorem 4.4. For $C_\Lambda \circ S \in \mathbb{B}(\widetilde{R}\text{-gmod})$, we also define

$$\begin{aligned} &\varepsilon_i(C_\Lambda \circ S) = \varepsilon_i(\Psi(S)) - \langle h_i, w_0\Lambda\rangle, \qquad \mathrm{wt}(C_\Lambda \circ S) = \mathrm{wt}(\Psi(S)) + w_0\Lambda - \Lambda, \\ &\varphi_i(C_\Lambda \circ S) = \varepsilon_i(\Psi(C_\Lambda \circ S)) + \langle h_i, \mathrm{wt}(C_\Lambda \circ S)\rangle. \end{aligned} \tag{7.5}$$

Theorem 7.4. The 6-tuple $(\mathbb{B}(\widetilde{R}\text{-gmod}), \mathrm{wt}, \{\varepsilon_i\}, \{\varphi_i\}, \{\widetilde{E}_i\}, \{\widetilde{F}_i\})_{i\in I}$ is a crystal.

Here, by Proposition 6.2 we observe that there seems to exist a certain correspondence:

$$\begin{array}{rcl} \{C_\Lambda \mid \Lambda \in P_+\} \subset R\text{-gmod} & \longleftrightarrow & \mathcal{H}_{\mathbf{i}} \\ C_\Lambda = \widetilde{F}_{i_1}^{m_1} \cdots \widetilde{F}_{i_N}^{m_N} \mathbf{1} & \longleftrightarrow & \mathbf{h}_\Lambda = \tilde{f}_{i_1}^{m_1} \tilde{f}_{i_2}^{m_2} \cdots \tilde{f}_{i_N}^{m_N}((0)_{i_1} \otimes (0)_{i_2} \otimes \cdots \otimes (0)_{i_N}) \end{array}$$

Together with the result of Proposition 3.12, we obtain the following:

Theorem 7.5. For any reduced longest word $\mathbf{i} = i_1 i_2 \cdots i_N$, there exists an isomorphism of crystals:

$$\begin{array}{rcl} \widetilde{\Psi} : \mathbb{B}(\widetilde{R}\text{-gmod}) & \stackrel{\sim}{\longrightarrow} & \mathbb{B}_{\mathbf{i}} = \bigcup_{h \in \mathcal{H}_{\mathbf{i}}} B^h(\infty) \\ C_\Lambda \circ S & \longmapsto & \mathbf{h}_\Lambda + \Psi(S) \in B^{\mathbf{h}_\Lambda}(\infty), \end{array}$$

where $\Psi : \mathbb{B}(R\text{-gmod}) \stackrel{\sim}{\longrightarrow} B(\infty)$ is the isomorphism of crystals given in Theorem 4.4, S is simple in $\mathbb{B}(R\text{-gmod})$ and for $\Lambda = \sum_i a_i \Lambda_i$ set $\mathbf{h}_\Lambda = \sum_i a_i \mathbf{h}_i$.

8. Application and further problems

8.1. **Operator $\tilde{\mathfrak{a}}$.** Define the $\mathbb{Q}(q)$-linear anti-automorphism $\star$ of $U_q(\mathfrak{g})$ by

$$(q^h)^\star = q^{-h}, \quad e_i^\star = e_i, \quad f_i^\star = f_i.$$

Theorem 8.1 ([2])**.** Set $L^\star(\infty) := \{u^\star \mid u \in L(\infty)\}$, $B^\star(\infty) := \{b^\star \mid b \in B(\infty)\}$. Then we have

$$L^\star(\infty) = L(\infty), \quad B^\star(\infty) = B(\infty).$$

From the proof of Theorem 5.13 in [5] we get

Proposition 8.2 ([5])**.** For $\nu = (\nu_1, \nu_2, \cdots, \nu_{m-1}, \nu_m) \in I^\beta$ ($m := |\beta|$) set $\overline{\nu} = (\nu_m, \nu_{m-1}, \cdots, \nu_2, \nu_1)$. Define the automorphism $\mathfrak{a}$ on $R(\beta)$ by

$$\mathfrak{a}(e(\nu)) = e(\overline{\nu}), \quad \mathfrak{a}(x_i e(\nu)) = x_{m-i+1} e(\overline{\nu}), \quad \mathfrak{a}(\tau_j e(\nu)) = -\tau_{m-j} e(\overline{\nu}).$$

Then, there exists the functor $\mathfrak{a} : R\text{-gmod} \to R\text{-gmod}$ such that $\mathfrak{a}(C_i) = C_{i^*}$ ($\forall i \in I$), $\mathfrak{a}^2 \cong \mathrm{id}$ and $\mathfrak{a}(X \circ Y) \cong \mathfrak{a}(Y) \circ \mathfrak{a}(X)$ for $X, Y \in R\text{-gmod}$. Furthermore, it is extended to the functor $\tilde{\mathfrak{a}} : \widetilde{R}\text{-gmod} \to \widetilde{R}\text{-gmod}$ which satisfies

$$\tilde{\mathfrak{a}}^2 \cong \mathrm{id}, \quad \text{and} \quad \tilde{\mathfrak{a}}(X \circ Y) \cong \tilde{\mathfrak{a}}(Y) \circ \tilde{\mathfrak{a}}(X) \quad \text{for } X, Y \in \widetilde{R}\text{-gmod}. \tag{8.1}$$

Note that $\mathfrak{a}$(resp. $\tilde{\mathfrak{a}}$) induces the operation $\star$ on $\mathcal{A}_q(\mathfrak{n})$ (resp. $\widetilde{\mathcal{A}_q}(\mathfrak{n})$) since $\mathfrak{a}(L(i)) = L(i)$ and then one has $\mathfrak{a}(f_i) = f_i$ (resp. $\tilde{\mathfrak{a}}(f_i) = f_i$) on $\mathcal{A}_q(\mathfrak{n})$ (resp. $\widetilde{\mathcal{A}_q}(\mathfrak{n})$). Now, we obtain the following:

Proposition 8.3. Let $\tilde{\mathfrak{a}} : \widetilde{R}\text{-gmod} \to \widetilde{R}\text{-gmod}$ be the functor as above. It yields

$$\tilde{\mathfrak{a}}(\mathbb{B}(\widetilde{R}\text{-gmod})) = \mathbb{B}(\widetilde{R}\text{-gmod}). \tag{8.2}$$

Here note that Proposition 8.3 can be seen as a generalization of Theorem 8.1.

Since as crystals $\mathbb{B}(\widetilde{R}\text{-gmod}) \cong \mathbb{B}_{\mathbf{i}}$ for any reduced longest word $\mathbf{i}$, the proposition above gives rise to the following problem.

Problem 1. Can we describe $\tilde{\mathfrak{a}}$-operation on $\mathbb{B}_{\mathbf{i}} = B_{i_1} \otimes \cdots \otimes B_{i_N}$ explicitly?

Of course, this problem is non-trivial since even for the case $B(\infty)$ the explicit description has not yet been done before in $\mathbb{B}_{\mathbf{i}}$.

8.2. **Category $\widetilde{\mathscr{C}_w}$.** In [5], it has been shown that for an arbitrary symmetrizable Kac-Moody Lie algebra and any Weyl group element $w \in W$, there exists a subcategory $\mathscr{C}_w \subset R$-gmod and it admits a localization

$$\widetilde{\mathscr{C}_w} = \mathscr{C}_w[C_i^{\circ -1} \mid i \in I], \quad (C_i = M(w\Lambda_i, \Lambda_i))$$

Indeed, note that for finite type Lie algebra setting, $\mathscr{C}_{w_0} = R$-gmod.

Problem 2. We conjecture that the localization $\widetilde{\mathscr{C}_w}$ possess a crystal $\mathbb{B}(\widetilde{\mathscr{C}_w})$. If so, we also conjecture that there is an isomorphism of crystals

$$\mathbb{B}(\widetilde{\mathscr{C}_w}) \xrightarrow{\sim} B_{i_1} \otimes \cdots \otimes B_{i_m},$$

where $i_1 \cdots i_m$ is a reduced word of w.

8.3. **Rigidity.**

Definition 8.4. Let X, Y be objects in a monoidal category $\mathcal{T}$, and $\varepsilon : X \otimes Y \to 1$ and $\eta : 1 \to Y \otimes X$ morphisms in $\mathcal{T}$. We say that a pair (X, Y) is *dual pair* or X is a *left dual* to Y or Y is a *right dual* to X if the following compositions are identities:

$$X \simeq X \otimes 1 \xrightarrow{\mathrm{id}\otimes\eta} X \otimes Y \otimes X \xrightarrow{\varepsilon\otimes\mathrm{id}} 1 \otimes X \simeq X, \;\; Y \simeq 1 \otimes Y \xrightarrow{\eta\otimes\mathrm{id}} Y \otimes X \otimes Y \xrightarrow{\mathrm{id}\otimes\varepsilon} Y \otimes 1 \simeq Y$$

We denote a right dual to X by $\mathcal{D}(X)$ and a left dual to X by $\mathcal{D}^{-1}(X)$.

Theorem 8.5 ([5])**.** For any finite type R, $\widetilde{R}$-gmod is rigid, i.e., every object in $\widetilde{R}$-gmod has left and right duals.

Note that in [6], it is shown that for any symmetrizable Kac-Moody setting the localized category $\widetilde{\mathscr{C}_w}$ is rigid.

Problem 3. For a simple object $C_\Lambda \circ S \in \mathbb{B}(\widetilde{R}\text{-gmod})$, describe the right and left duals explicitly:

$$\widetilde{\Psi}(\mathcal{D}(C_\Lambda \circ S)), \quad \widetilde{\Psi}(\mathcal{D}^{-1}(C_\Lambda \circ S)) \in \mathbb{B}_{\mathbf{i}}.$$

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