TRIANGULATIONS OF CYCLIC POLYTOPES AND THE HIGHER AUSLANDER ALGEBRAS OF TYPE A

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Abstract. We outline the relationship between triangulations of cyclic polytopes and the representation theory of the higher Auslander algebras of type A. This relationship includes algebraic interpretations of the two partial orders on the set of triangulations of a cyclic polytope known as the higher Stasheff–Tamari orders. These orders were subsequently shown to be equal by the author, thereby giving new information about the representation theory of the higher Auslander algebras of type A.

1. Introduction

One of the earliest examples of a cluster algebra produced by Fomin and Zelevinsky was the type A cluster algebra [FZ02]. This cluster algebra has a particularly simple combinatorial model. Namely, the cluster variables of this cluster algebra are in bijection with the arcs in a convex polygon in such a way that clusters correspond to triangulations of the polygon.

Categorical versions of this relationship were later discovered in the representation theory of algebras. For instance, one can categorify the type A cluster algebra by taking the cluster category of a type A path algebra [Bua+06]. In this category, indecomposable objects correspond to arcs in the polygon in such a way that cluster-tilting objects correspond to triangulations.

It is natural to wonder whether there are versions of this phenomenon that use polytopes of higher dimension than two-dimensional ones. This was discovered by Oppermann and Thomas [OT12], who showed that triangulations of $2d$-dimensional cyclic polytopes were in bijection with cluster-tilting objects for the higher Auslander algebras of type A, introduced by Iyama [Iya11]. An alternative framework for essentially the same result gives that triangulations of $2d$-dimensional cyclic polytopes are in bijection with certain silting complexes over the higher Auslander algebra of type A.

One of the many fascinating aspects of this discovery is that the Calabi–Yau dimension of the cluster category is equal to the dimension of the corresponding polytope. In the classical case, the type A cluster categories are 2-Calabi–Yau, corresponding to a two-dimensional convex polygon; in the general case, the cluster categories of the higher Auslander algebras of type A are $2d$-Calabi–Yau, corresponding to a $2d$-dimensional cyclic polytope. It is also intriguing that this picture only involves even-dimensional triangulations, especially noting the relationship discovered between the higher Auslander algebras of type A and symplectic geometry [DJL21].

The relationship between triangulations of cyclic polytopes and the higher Auslander algebras of type A has much more structure than simply being a bijection. Indeed, there are two natural orders on the set of triangulations of a cyclic polytope known as the 'higher Stasheff–Tamari orders' [ER96]. For even dimensions, these partial orders correspond, via the bijection, to partial orders on silting complexes already known in representation theory from [AI12], following earlier work in [RS91].

The interpretation of the higher Stasheff–Tamari orders on the representation-theoretic side can be used to bring triangulations of odd-dimensional cyclic polytopes into the picture. Indeed, it is shown in [Ram97] that triangulations of a $(\delta+1)$-dimensional cyclic polytope are given by equivalence-classes of maximal chains in the δ-dimensional higher Stasheff–Tamari order. Hence, triangulations of a $(2d+1)$-dimensional cyclic polytope are given my maximal chains of $2d$-dimensional triangulations, which in turn correspond to maximal chains of silting complexes. Such maximal chains of silting complexes were already known in representation theory for $d = 1$ as 'maximal green sequences'. For $d > 1$, we call them 'd-maximal green sequences'. The difference between the representation-theoretic interpretations of even- and odd-dimensional triangulations is therefore quite striking.

One can then interpret the odd-dimensional higher Stasheff–Tamari orders in terms of equivalence classes of d-maximal green sequences, giving natural orders not previously appreciated in representation theory. Indeed, we obtain an altogether new perspective on maximal green sequences: that they should be considered subject to an equivalence relation, and that when one does this more structure becomes visible—namely, the partial orders on the equivalence classes.

The two higher Stasheff–Tamari orders were conjectured to be equal in [ER96]. This conjecture remained open for some time, despite various papers on the subject [Ram97; ERR00; Tho02; Tho03]. It was eventually shown to be true in [Wil21b]. With the algebraic interpretations of the higher Stasheff–Tamari orders from [Wil22], we thereby obtain new information about the representation theory of the higher Auslander algebras of type A, in particular, that the two partial orders on silting complexes coincide with each other, as do the two partial orders on equivalence classes of d-maximal green sequences.

The structure of the paper is as follows. We begin in Section 2 by giving background on cyclic polytopes and the higher Auslander algebras of type A. In Section 3, we describe how triangulations of even-dimensional cyclic polytopes may be interpreted in terms of the representation theory of the higher Auslander algebras of type A. We also describe the even-dimensional higher Stasheff–Tamari orders in these terms. We do the same for odd dimensions in Section 4. In the final section, Section 5, we describe the consequences of the equality of the higher Stasheff–Tamari orders on the representation theory of the higher Auslander algebras of type A.

2. Background

2.1. **Cyclic polytopes and their triangulations.**

2.1.1. *Cyclic polytopes.* Cyclic polytopes are very special polytopes in combinatorics. They satisfy the Upper Bound Theorem [McM70; Sta75], meaning that they have the largest number of faces possible in every given dimension. Furthermore, every sufficiently large generic collection of points in $\mathbb{R}^\delta$ contains the vertex set of a polytope combinatorially equivalent to a cyclic polytope [CD00].

The *moment curve* $p\colon \mathbb{R} \to \mathbb{R}^\delta$ is defined by $p(t) := (t, t^2, \dots, t^\delta) \subset \mathbb{R}^\delta$, where $\delta \in \mathbb{N}_{\geqslant 1}$. Choose $t_1, t_2, \dots, t_m \in \mathbb{R}$ such that $t_1 < t_2 < \cdots < t_m$. The convex hull $\operatorname{conv}\{p(t_1), \dots, p(t_m)\}$ is a *cyclic polytope* $C(m, \delta)$. The combinatorial properties of $C(m, \delta)$ are independent of the initial choices of $t_1, t_2, \dots, t_m$, so, for ease, we set $t_i = i$. We label the vertices of $C(m, \delta)$ by $[m] = \{1, 2, \dots, m\}$, in the natural way. There is a natural projection map from $C(m, \delta)$ to $C(m, \delta - 1)$ given by forgetting the last coordinate.

2.1.2. *Triangulations.* The set of triangulations of a cyclic polytope has a rich combinatorial structure which appears in many different areas of mathematics [DK19; DM12; Wil21a; AT14].

A *triangulation* of a cyclic polytope $C(m, \delta)$ is a subdivision of $C(m, \delta)$ into δ-dimensional simplices whose vertices are elements of $[m]$. We identify a triangulation of $C(m, \delta)$ with the corresponding set of δ-simplices.

We specify a k-simplex in $C(m, \delta)$ using its vertex set in $\binom{[m]}{k+1}$, the set of subsets of $[m]$ of size $k+1$. Given $A \in \binom{[m]}{k+1}$, we write $|A|_\delta$ for the corresponding geometric simplex in dimension δ. When the dimension is clear, we will drop the subscript. We will always label the elements of $A \in \binom{[m]}{k+1}$ as $a_0 < a_1 < \cdots < a_d$, and use the analogous labelling for different letter of the alphabet.

2.1.3. *The higher Stasheff–Tamari orders.* The first higher Stasheff–Tamari orders were introduced by Kapranov and Voevodsky to give examples in higher category theory [KV91]. The second higher Stasheff–Tamari order was later introduced by [ER96]. Both orders are higher-dimensional generalisations of the Tamari lattice [Tam51; Tam62], a ubiquitous partial order in mathematics [MPS12].

The cyclic polytope $C(\delta + 2, \delta)$ has two triangulations, one of which is known as the 'lower triangulation', and the other which is known as the 'upper triangulation'. Suppose that a triangulation $\mathcal{T}$ of $C(m, \delta)$ restricts to a triangulation of a copy of $C(\delta + 2, \delta)$ on a subset of the vertices, such that this $C(\delta + 2, \delta)$ subpolytope is given the lower triangulation by $\mathcal{T}$. Let $\mathcal{T}'$ be the triangulation obtained from $\mathcal{T}$ by replacing the part of $\mathcal{T}$ within the $C(\delta + 2, \delta)$ subpolytope by the upper triangulation. Then $\mathcal{T}'$ is called an *increasing bistellar flip* of $\mathcal{T}$. The *first higher Stasheff–Tamari order* is defined such that $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if $\mathcal{T}'$ is an increasing bistellar flip of $\mathcal{T}$. Here $\lessdot_1$ denotes a covering relation in the first higher Stasheff–Tamari order.

Every triangulation $\mathcal{T}$ of $C(m, \delta)$ determines a unique piecewise-linear section

$$s_{\mathcal{T}}\colon C(m, \delta) \to C(m, \delta + 1)$$

of $C(m, \delta + 1)$ by sending each δ-simplex $|A|_\delta$ of $\mathcal{T}$ to $|A|_{\delta+1}$ in $C(m, \delta + 1)$, in the natural way. This is a section—that is, a right inverse—of the projection map from

FIGURE 1. Examples of the quivers $Q^{(d,n)}$

$Q^{(1,3)}$

$$\begin{tikzcd}[column sep=small]
 & & 3 \\
 & 2 \arrow[ur] & \\
1 \arrow[ur] & &
\end{tikzcd}$$

$Q^{(2,3)}$

$$\begin{tikzcd}[column sep=small]
 & & 15 \arrow[dr] & & \\
 & 14 \arrow[ur] \arrow[dr] & & 25 \arrow[dr] & \\
13 \arrow[ur] & & 24 \arrow[ur] & & 35
\end{tikzcd}$$

$Q^{(3,3)}$

$$\begin{tikzcd}[column sep=small]
 & & 137 \arrow[rr] & & 147 \arrow[rr] \arrow[dr] & & 157 \arrow[dr] & & \\
 & 136 \arrow[ur] \arrow[rr] & & 146 \arrow[ur] \arrow[dr] & & 247 \arrow[rr] & & 257 \arrow[dr] & \\
135 \arrow[ur] & & & & 246 \arrow[ur] & & & & 357
\end{tikzcd}$$

$C(m,\delta+1)$ to $C(m,\delta)$. The *second higher Stasheff–Tamari order* on triangulations of $C(m,\delta)$ is defined such that

$$\mathcal{T} \leqslant_2 \mathcal{T}' \iff s_{\mathcal{T}}(x)_{\delta+1} \leqslant s_{\mathcal{T}'}(x)_{\delta+1} \; \forall x \in C(m,\delta),$$

where $s_{\mathcal{T}}(x)_{\delta+1}$ denotes the $(\delta+1)$-th coordinate of the point $s_{\mathcal{T}}(x)$. We write $\mathcal{S}_2(m,\delta)$ for the poset this gives.

2.2. **Representation theory of finite-dimensional algebras.** We let K be a field. By 'modules' we mean right modules.

2.2.1. *The higher Auslander algebras of type A.* The higher Auslander algebras of type A are the canonical examples of algebras appearing in the higher homological algebra of Iyama [Iya07; Iya11].

In order to define these algebras, we first need to define the set of subsets

$$\mathbf{I}^d_m := \{A \in \tbinom{[m]}{d+1} : a_{i-1} \leqslant a_i - 2 \; \forall i \in [d]\}.$$

Then, let $Q^{(d,n)}$ be the quiver with vertices

$$Q_0^{(d,n)} := \mathbf{I}^{d-1}_{n+2d-2}$$

and arrows

$$Q_1^{(d,n)} := \{\, A \to \sigma_i(A) : A, \sigma_i(A) \in Q_0^{(d,n)} \,\},$$

where

$$\sigma_i(A) := \{a_0, a_1, \ldots, a_{i-1}, a_i + 1, a_{i+1}, \ldots, a_d\}.$$

We multiply arrows as if we were composing functions, so that $\xrightarrow{\alpha}\xrightarrow{\beta} = \beta\alpha$.

Let A^d_n be the quotient of the path algebra $KQ^{(d,n)}$ by the relations:

$$A \to \sigma_i(A) \to \sigma_j(\sigma_i(A)) = \begin{cases} A \to \sigma_j(A) \to \sigma_j(\sigma_i(A)) & \text{if } \sigma_j(A) \in Q_0^{(d,n)} \\ 0 & \text{otherwise.} \end{cases}$$

The algebra A^d_n has a distinguished basic module $M^{(d,n)}$ which has the property of being *d-cluster-tilting*. This means that

$$\begin{aligned} \operatorname{add} M^{(d,n)} &= \{X \in \operatorname{mod}\Lambda \mid \forall i \in [d-1],\ \operatorname{Ext}^i_\Lambda(X, M^{(d,n)}) = 0\} \\ &= \{X \in \operatorname{mod}\Lambda \mid \forall i \in [d-1],\ \operatorname{Ext}^i_\Lambda(M^{(d,n)}, X) = 0\}. \end{aligned}$$

The module $M^{(d,n)}$ is the unique basic A_n^d-module which satisfies these properties, up to isomorphism. The beautiful result due to [Iya11] is that we in fact have

$$A_n^{d+1} \cong \operatorname{End}_{A_n^d} M^{(d,n)}.$$

The idea of higher homological algebra is that there are no non-trivial extensions of degree less than d in $\operatorname{add} M^{(d,n)}$, so that the shortest non-trivial exact sequences are those with $d+2$ non-zero terms, rather than short exact sequences. Indeed, $\operatorname{add} M^{(d,n)}$ is a "d-abelian" category [Jas16], meaning that it satisfies axioms analogous to those of abelian categories, only with short exact sequences replaced by longer exact sequences. Few algebras Λ posses d-cluster-tilting subcategories in $\operatorname{mod} \Lambda$, so the algebras A_n^d are quite special.

2.2.2. *Silting theory.* Silting complexes were introduced in [KV88] to classify aisles in derived categories. We write $D^b(A_n^d)$ for the bounded derived category of A_n^d.

A complex T of $D^b(A_n^d)$ is *pre-silting* if $\operatorname{Hom}_{D^b(A_n^d)}(T, T[i]) = 0$ for all $i > 0$. A pre-silting complex T is *silting* if, additionally, $\operatorname{thick} T = D^b(A_n^d)$. Here $\operatorname{thick} T$ denotes the smallest full subcategory of $D^b(A_n^d)$ which contains T and is closed under cones, $[\pm 1]$, direct summands, and isomorphisms. We consider the subcategory $\mathcal{E}(A_n^d) := \operatorname{add}(M^{(d,n)} \oplus A_n^d[d])$ of $D^b(A_n^d)$ and call a silting complex T of $D^b(A_n^d)$ *d-silting* if, additionally, it lies in $\mathcal{E}(A_n^d)$.

Remark 2.1. Note that for objects T, T' of $\mathcal{E}(A_n^d)$ we have $\operatorname{Hom}_{D^b(A_n^d)}(T, T'[i]) = 0$ if $i \notin \{-d, 0, d\}$, since $\operatorname{add} M^{(d,n)}$ is a d-cluster-tilting subcategory of $\operatorname{mod} A_n^d$ and $\operatorname{gl.dim} \Lambda \leqslant d$. Hence, for an object T of $\mathcal{E}(A_n^d)$ to be d-silting, it suffices that $\operatorname{Hom}_{D^b(A_n^d)}(T, T[d]) = 0$ and $\operatorname{thick} T = D^b(A_n^d)$.

3. Triangulations of even-dimensional cyclic polytopes

We now describe the relation between d-silting complexes over A_n^d and triangulations of the even-dimensional cyclic polytope $C(n+2d+1, 2d)$.

3.1. Describing $\mathcal{E}(A_n^d)$.

We begin by explaining how one can describe the category $\mathcal{E}(A_n^d)$ combinatorially, which is the first step in describing the relationship with cyclic polytopes.

Since $\operatorname{End}_{A_n^d} M^{(d,n)} = A_n^d$, we have that the vertices of $Q^{(n,d+1)}$ correspond to the indecomposable modules in $\operatorname{add} M^{(d,n)}$. Hence, we have that the indecomposable modules in $\operatorname{add} M^{(d,n)}$ are labelled by $\mathbf{I}_{n+2d}^d$. Given a subset $B \in \mathbf{I}_{n+2d}^d$, we write U_B for the corresponding indecomposable A_n^d-module in $\operatorname{add} M^{(d,n)}$. This labelling of the modules allows us to identify the projectives and injectives.

Proposition 3.1 ([OT12]). *There is a bijection from $\mathbf{I}_{n+2d}^d$ to the isomorphism classes of indecomposable modules in* $\operatorname{add} M^{(d,n)}$ *via*

$$A \mapsto U_A$$

such that

(1) *U_A is a projective A_n^d-module if and only if $a_0 = 1$; and*
(2) *U_A is an injective A_n^d-module if and only if $a_d = n + 2d$.*

One can extend this labelling of the indecomposables of $\operatorname{add} M^{(d,n)}$ by $\mathbf{I}^d_{n+2d}$ to a labelling of the indecomposables of $\mathcal{E}(A^d_n)$ in a natural way. For this, we need the set of subsets

$$^{\circlearrowleft}\mathbf{I}^d_m := \{\, A \in \tbinom{[m]}{d+1} : a_{i-1} \leqslant a_i - 2 \,\forall i \in [d], \text{ and } a_d \leqslant a_0 + m - 2 \,\}.$$

Then, if U_A is a projective A^d_n-module, we write

$$U_{\{a_1-1, a_2-1, \ldots, a_d-1, a_0+n+2d\}} := U_A[d].$$

Definition 3.2 ([OT12])**.** If $A, B \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ are such that

$$a_0 < b_0 < a_1 < b_1 < \cdots < a_d < b_d,$$

then we say that A *intertwines* B, and write $A \wr B$.

Theorem 3.3 ([OT12])**.** *There is a bijection from* ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ *to the isomorphism classes of indecomposable complexes in* $\mathcal{E}(A^d_n)$

$$A \mapsto U_A$$

such that given $U_A, U_B \in \mathcal{E}(A^d_n)$*, we have that* $\operatorname{Hom}_{D^b(A^d_n)}(U_B, U_A[d]) \neq 0$ *if and only if* $A \wr B$ *and in this case the* Hom*-space is one-dimensional.*

To summarise, we have a labelling of $\mathcal{E}(A^d_n)$ by ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ which encodes certain homological properties.

3.2. **Describing even-dimensional triangulations.** The easiest way to describe a triangulation of a convex polygon is as a set of non-crossing arcs, rather than as a set of triangles. Oppermann and Thomas show that, remarkably, this observation extends to all even-dimensional cyclic polytopes. That is, a triangulation of a $2d$-dimensional cyclic polytope can be described as a set of non-crossing internal d-simplices, rather than as a set of $2d$-simplices. Here, a simplex is *internal* in $C(n+2d+1, 2d)$ if it does not lie in the boundary.

Theorem 3.4 ([OT12])**.** *There is a bijection between triangulations of* $C(n+2d+1, 2d)$ *and sets of internal* d*-simplices of size* $\binom{n+d-1}{d}$ *whose interiors do not intersect each other, given*

$$\mathcal{T} \mapsto \{A \in \tbinom{n+2d+1}{d+1} : |A| \text{ is an internal } d\text{-simplex of } \mathcal{T}\}.$$

This theorem can be made combinatorial using the following proposition, which gives combinatorial criteria for a d-simplex to be internal and for two d-simplices to intersect in their interiors.

Proposition 3.5 ([OT12; Bre73; Gal63])**.**
(1) *A* d*-simplex* $|A|$ *is internal in* $C(n+2d+1, 2d)$ *if and only if* $A \in {}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$.
(2) *Two* d*-simplices* $|A|$ *and* $|B|$ *in* $C(n+2d+1, 2d)$ *intersect in their interiors if and only if either* $A \wr B$ *or* $B \wr A$.

Hence, we obtain the following combinatorial version of Theorem 3.4.

Theorem 3.6 ([OT12])**.** *There is a bijection between triangulations of* $C(n+2d+1, 2d)$ *and non-intertwining subsets of* ${}^{\circlearrowleft}\mathbf{I}^d_{n+2d+1}$ *of size* $\binom{n+d-1}{d}$*, given by sending a triangulation to the set of vertex sets of its set of internal* d*-simplices.*

3.3. **Even-dimensional triangulations and representation theory.** This description of triangulations of even-dimensional cyclic polytopes can be used to draw a connection with representation theory. Indeed, putting together the description of the category $\mathcal{E}(A_n^d)$ from Section 3.1 and the triangulations of $C(n+2d+1, 2d)$ from Section 3.2, along with some additiona work, we obtain the following theorem.

Theorem 3.7 ([OT12; Wil]). *There is a bijection*

$$|A| \mapsto U_A$$

between internal d-simplices in $C(n+2d+1, 2d)$ and indecomposable objects of $\mathcal{E}(A_n^d)$ which induces a bijection between triangulations of $C(n+2d+1, 2d)$ and basic d-silting complexes in $\mathcal{E}(A_n^d)$.

Having interpreted triangulations of even-dimensional cyclic polytopes in terms of representation theory, we can now consider whether the higher Stasheff–Tamari orders can be described algebraically too. The interpretation of both orders is very natural, with the first order being interpreted as follows.

Theorem 3.8 ([Wil22; Wil], [BK04, $d=1$]). *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(n+2d+1, 2d)$ with corresponding basic d-silting complexes T and T' in $\mathcal{E}(A_n^d)$. Then $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if T' is a left mutation of T.*

Here T' is a *left mutation* of T if and only if $T = E \oplus X$, $T' = E \oplus Y$ where X and Y are indecomposable and such that $\operatorname{Hom}(X, Y[d]) = 0$. This theorem is explained by the fact that an increasing bistellar flip of a triangulation of a $2d$-dimensional cyclic polytope is given by replacing a d-simplex by one which it intertwines, which corresponds to a left mutation by Proposition 3.3.

The second higher Stasheff–Tamari order is then interpreted as follows.

Theorem 3.9 ([Wil22; Wil]). *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(n+2d+1, 2d)$ with corresponding basic d-silting complexes $T, T' \in \mathcal{E}(A_n^d)$. Then $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if ${}^{\perp}T \subseteq {}^{\perp}T'$.*

Here

$$\begin{aligned} {}^{\perp}T &:= \{X \in \mathcal{E}(A_n^d) : \operatorname{Hom}_{D^b(A_n^d)}(X, T[i]) = 0 \, \forall i > 0\} \\ &= \{X \in \mathcal{E}(A_n^d) : \operatorname{Hom}_{D^b(A_n^d)}(X, T[d]) = 0\}. \end{aligned}$$

The explanation of this theorem is that for an internal simplex $|A|_{2d}$ in $C(n+2d+1, 2d)$, we have that $|A|_{2d+1}$ lies below the section of a triangulation $\mathcal{T}$ if and only if we have $U_A \in {}^{\perp}T$ for the corresponding d-silting complex T.

Remark 3.10. The remarkable thing about Theorem 3.8 and Theorem 3.9 is that the orders obtained on silting complexes are higher-dimensional versions of well-known orders introduced in [AI12], based on analogous orders on tilting modules defined in [RS91]. It is beautiful that orders defined independently in representation theory and combinatorics should turn out to be the same.

4. Triangulations of odd-dimensional cyclic polytopes

What is especially interesting about Theorem 3.8 is that it allows us to interpret odd-dimensional triangulations in the representation theory of A_n^d. This completes the picture from [OT12], as it were.

The key is a result of [Ram97], which states that triangulations of $C(m, \delta + 1)$ are in bijection with equivalence classes of maximal chains in $\mathcal{S}_1(m, \delta)$. By Theorem 3.8, maximal chains in $\mathcal{S}_1(n+2d+1, 2d)$ correspond to maximal sequences of left mutations of d-silting complexes over A_n^d. For $d = 1$, such sequences are known as 'maximal green sequences' [Kel11; DIJ19; BST19]. Hence, we make the following definition.

Definition 4.1 ([Wil22; Wil])**.** A *d-maximal green sequence* of A_n^d is a sequence $(T_0, T_1, \dots, T_r)$ of d-silting complexes in $\mathcal{E}(A_n^d)$ such that $T_0 = A_n^d$, $T_r = A_n^d[d]$, and, for $i \in [r]$, T_i is a left mutation of T_{i-1}.

Since we have that triangulations of $C(n + 2d + 1, 2d + 1)$ are in bijection with equivalence classes of maximal chains in $\mathcal{S}_1(n + 2d + 1, 2d)$, we need an equivalence relation on d-maximal green sequences of A_n^d. Given a d-maximal green sequence G, we denote the set of indecomposable summands of d-silting complexes occurring in G by $\Sigma(G)$. We write $G \sim G'$ if and only if $\Sigma(G) = \Sigma(G')$. We then obtain the following theorem by applying [Ram97, Theorem 1.1] to Theorem 3.8.

Theorem 4.2 ([Wil22; Wil])**.** *There is a bijection between triangulations of $C(n+2d+1, 2d+1)$ and equivalence classes of d-maximal green sequences of A_n^d.*

Having interpreted triangulations of odd-dimensional cyclic polytopes in the representation theory of A_n^d, we can now ask the same question we asked before, namely, how the higher Stasheff–Tamari orders may be interpreted. The first order has the following description in terms of d-maximal green sequences.

Theorem 4.3 ([Wil22; Wil])**.** *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulations of $C(n+2d+1, 2d+1)$ corresponding to equivalence classes of d-maximal green sequences $[G], [G']$ of A_n^d. Then $\mathcal{T} \lessdot_1 \mathcal{T}'$ if and only if there are equivalence class representatives $\widehat{G} \in [G]$ and $\widehat{G'} \in [G']$ such that $\widehat{G'}$ is an increasing elementary polygonal deformation of $\widehat{G}$.*

Here, an increasing elementary polygonal deformation is defined as follows. An *oriented polygon* is a sub-poset of $\mathcal{S}_1(m, 2d)$ formed of the union of a chain of $d+2$ covering relations with a chain of $d + 1$ covering relations, such that these chains intersect only at the top and bottom. If two d-maximal green sequences G and G' differ only in that G contains the longer side of an oriented polygon and G' contains the shorter side, then we say that G' is *an increasing elementary polygonal deformation* of G. Note that an increasing elementary polygonal deformation decreases the length of the chain.

The explanation of Theorem 4.3 is that the chain of length $d + 2$ forming part of the polygon gives the lower triangulation of a copy of $C(2d + 3, 2d + 1)$, whilst the chain of length $d + 1$ gives the upper triangulation of the same copy of $C(2d +$

$3, 2d+1)$. Hence, the increasing elementary polygonal deformation corresponds to an increasing bistellar flip.

The second order admits the following elegant description in terms of d-maximal green sequences.

Theorem 4.4 ([Wil22; Wil]). *Given two triangulations $\mathcal{T}$ and $\mathcal{T}'$ of $C(n+2d+1, 2d+1)$ corresponding to equivalence classes of d-maximal green sequences $[G]$ and $[G']$ of A_n^d, we have that $\mathcal{T} \leqslant_2 \mathcal{T}'$ if and only if $\Sigma(G) \supseteq \Sigma(G')$.*

The explanation here is that the elements of $\Sigma(G)$ which are neither projectives nor shifted projectives correspond to the internal d-simplices of $\mathcal{T}$, in a similar manner to Theorem 3.7. One can then show that in dimension $2d+1$, the second higher Stasheff–Tamari order corresponds to reverse inclusion of internal d-simplices, which yields the result.

5. Equality of the orders

In [ER96], Edelman and Reiner conjectured that the two higher Stasheff–Tamari orders were actually the same. This conjecture was proven in [Wil21b].

Theorem 5.1 ([Wil21b]). *Let $\mathcal{T}, \mathcal{T}'$ be triangulations of $C(m, \delta)$. Then $\mathcal{T} \leqslant_1 \mathcal{T}'$ if and only if $\mathcal{T} \leqslant_2 \mathcal{T}'$.*

The difficult direction in this theorem is to show that if $\mathcal{T} \leqslant_2 \mathcal{T}'$, then $\mathcal{T} \leqslant_1 \mathcal{T}'$. It is straightforward to show that if $\mathcal{T} \leqslant_1 \mathcal{T}'$, then $\mathcal{T} \leqslant_2 \mathcal{T}'$: it suffices to note that increasing bistellar flips always move the section upwards. However, to show that $\mathcal{T} \leqslant_1 \mathcal{T}'$ whenever $\mathcal{T} \leqslant_2 \mathcal{T}'$ requires one to show that one can always find a sequence of increasing bistellar flips from $\mathcal{T}$ to $\mathcal{T}'$ whenever the section of $\mathcal{T}'$ lies above the section of $\mathcal{T}$. However, whilst every arc of a polygon triangulation can be flipped, the analogous statement is not true in higher-dimensional triangulations. In fact, as the dimension of a cyclic polytope increases, bistellar flips become increasingly scarce. Hence, finding such a sequence of increasing bistellar flips is in general very hard.

By the results of the preceding sections, Theorem 5.1 also gives new results about the representation theory of A_n^d. Indeed, by applying Theorem 5.1 to Theorem 3.8 and Theorem 3.9, we obtain the following.

Corollary 5.2 ([Wil]). *Let T, T' be d-silting complexes in $\mathcal{E}(A_n^d)$. Then there is a sequence of left mutations from T to T' if and only if ${}^{\perp}T \subseteq {}^{\perp}T'$.*

This is essentially the analogue in higher Auslander–Reiten theory of [AI12, Proposition 2.36] for the higher Auslander algebras of type A. Conjecturally, the same result should hold for other algebras in higher homological algebra, but the proof techniques from [AI12] break down for $d > 1$.

Theorem 5.1 can also be applied to Theorem 4.3 and Theorem 4.4 to obtain results on the two orders on d-maximal green sequences.

Corollary 5.3 ([Wil]). *Let $[G], [G']$ be two equivalence classes of d-maximal green sequences for A_n^d. Then there is a sequence of increasing elementary polygonal deformations from $[G]$ to $[G']$ if and only if $\Sigma(G) \supseteq \Sigma(G')$.*

This can be seen as a "no-gap" result for d-maximal green sequences. The "no-gap" conjecture of Brüstle, Dupont, and Perotin states that the set of lengths of maximal green sequences of a finite-dimensional algebra should not contain any gaps [BDP14]. Cases of this conjecture were proven in [GM19; HI19]. Corollary 5.3 implies that if there is a sequence of increasing elementary polygonal deformations from $[G]$ to $[G']$, then is a d-maximal green sequence of every length between the lengths of G and G'. There exists a d-maximal green sequence G' of A_n^d such that $\Sigma(G')$ consists only of the projectives and shifted projectives; there also exists G such that $\Sigma(G)$ consists of all of the indecomposable objects of $\mathcal{E}(A_n^d)$. By Corollary 5.3, we get a sequence of increasing elementary polygonal deformations from $[G]$ to $[G']$, since $\Sigma(G) \supseteq \Sigma(G')$. Since G is clearly the longest possible d-maximal green sequence, whilst G' is the shortest, the "no-gap" conjecture therefore holds for A_n^d. Corollary 5.3 is stronger than simply saying that there are no gaps, of course, since it also takes account of how the d-maximal green sequences are ordered.

For the case $d = 1$, we get a stronger result on the poset of maximal green sequences, namely that it is a lattice.

Corollary 5.4 ([Wil22])**.** *The set of equivalence classes of (1-)maximal green sequences of A_1 forms a lattice under the order given by reverse inclusion of summands, or, equivalently, the order whose covering relations are given by increasing elementary polygonal deformations.*

The fact that the set of equivalence classes of maximal green sequences has such a nice structure shows the virtues of considering maximal green sequences subject to an equivalence relation. Equivalence of maximal green sequences will be considered further in [GW].

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