

DESIGNS ON TAUTOLOGICAL BUNDLE

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1. INTRODUCTION

In the area of combinatorics, many researchers have studied point arrangements called “designs”. Roughly speaking, “designs” are “good” point arrangements with approximate given space. Here, we introduce a difinition of spherical t -design.

Definition 1.1. (Delsart, Gothals, Seidel(1977) [4]) Let

$$(1.1) \quad S^{n-1} = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 = 1\}.$$

A finite subset X of S^{n-1} is called a spherical t -design if

$$(1.2) \quad \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f d\mu = \frac{1}{|X|} \sum_{u \in X} f(u)$$

for any polynomial $f(x_1, \dots, x_n)$ of degree t or less.

For spherical designs, refer to [2]. Also designs on following spaces have been studied:

- Unitary groups [5]
- Grassmannian spaces [1]
- Compact symmetric spaces [3]

In this paper, we see a new definition of designs and its examples.

2. DEFINITION OF τ -DESIGN

Definition 2.1. Let

- Ω :a set,
- W, \mathcal{H} : \mathbb{C} -vector spaces,
- \mathcal{H}_0 :a subset of \mathcal{H} ,
- $\{V_p\}_{p \in \Omega}$:a family of vector spaces,
- $\{e_p : \mathcal{H} \rightarrow V_p\}_{p \in \Omega}$:a family of linear maps,
- $\tau : \mathcal{H}_0 \rightarrow W$:a linear map.

For a finite subset $X \subset \Omega$ and linear functions $\lambda_x : V_x \rightarrow W$, a pair $(X, \{\lambda_x\}_{x \in X})$ is a τ -design if for any $s \in \mathcal{H}_0$ the following equation holds:

$$(2.1) \quad \tau(s) = \sum_{x \in X} (\lambda_x \circ e_x)(s)$$

Example 2.2. Now, we rewrite the spherical design with the definition of τ -design. Let

- $\Omega = S^{n-1}$,
- $V = \mathbb{R}, \mathcal{H} = C^\infty(\mathbb{R}^n), \mathcal{H}_0 = \text{Pol}_{\leq t}(\mathbb{R}^n)|_{S^{n-1}}$,
- $V_p = \mathbb{R} (p \in \Omega), e_p : \mathcal{H} \rightarrow V_p, f \mapsto f(p)$,

•

$$(2.2) \quad \tau : \mathcal{H}_0 \rightarrow W, f \mapsto \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f d\mu$$

where μ is a radon measure of S^{n-1} ,

• For $x \in X \subset \Omega$

$$(2.3) \quad \lambda_x : V_x \rightarrow W, z \mapsto \frac{1}{|X|} z.$$

Then, it is easy to see that

$$X \text{ is a spherical } t\text{-design} \Leftrightarrow (X, \lambda) \text{ is a } \tau\text{-design.}$$

3. DESIGNS ON TAUTOLOGICAL BUNDLE

3.1. TAUTOLOGICAL BUNDLE

We define an action $SU(2) \curvearrowright \mathbb{C}P^1$ as follows:

$$(3.1) \quad g \cdot V = \{gv \mid v \in V\}$$

and let $v_0 = \{(x, 0) \mid x \in \mathbb{C}\}$.

Proposition 3.1.

$$(3.2) \quad \varpi : SU(2) \rightarrow \mathbb{C}P^1, g \mapsto g \cdot v_0$$

is a principal bundle. Moreover, an isotropy subgroup

$$(3.3) \quad \text{Iso}_{v_0}(SU(2)) := \{g \in SU(2) \mid g \cdot v_0 = v_0\} = S(U(1) \times U(1))$$

leads the following isomorphism:

$$(3.4) \quad \mathbb{C}P^1 \cong SU(2)/S(U(1) \times U(1)),$$

that is (3.2) is a principal $S(U(1) \times U(1))$ -bundle.

Now, we define the following action $S(U(1) \times U(1)) \curvearrowright \mathbb{C}$:

$$(3.5) \quad g \cdot x = g_1 x \quad \left(g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \in S(U(1) \times U(1)), x \in \mathbb{C} \right).$$

Then,

$$(3.6) \quad \pi : T_{2,1} \rightarrow \mathbb{C}P^1$$

denotes an associated bundle to (3.2) with fiber \mathbb{C} .

Lemma 3.2.

$$(3.7) \quad T_{2,1} = \{(V, v) \in \mathbb{C}P^1 \times \mathbb{C}^2 \mid v \in V\}.$$

The right-hand side of (3.7) is called a tautological bundle.

3.2. SECTION OF TAUTOLOGICAL BUNDLE

In this section, we see a decomposition of $\Gamma(T_{2,1})$, which is a set of smooth section of $T_{2,1}$.

Proposition 3.3. $\Gamma(T_{2,1})$ has the following irreducible decomposition of $SU(2)$ representation:

$$(3.8) \quad \Gamma(T_{2,1}) = \bigoplus_{k \geq 1} \Gamma^{2k}(T_{2,1}),$$

where $\dim_{\mathbb{C}} \Gamma^{2k}(T_{2,1}) = 2k$.

From the general theory of representations in $SU(2)$, we can see $\Gamma^{2k}(T_{2,1})$ as a symmetric tensor.

Lemma 3.4. Let $S^n(\mathbb{C}^2)$ be the set of all symmetric tensors of order n defined on \mathbb{C}^2 and we define

$$(3.9) \quad s : S^{2k-1}(\mathbb{C}^2) \rightarrow \Gamma^{2k}(T_{2,1})$$

as

$$(3.10) \quad s_{\alpha}(l) = \left(l, \alpha_k^l \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right) \quad (\alpha \in S^{2k-1}(\mathbb{C}^2), l \in \mathbb{C}P^1)$$

where α_k^l is a coefficient of $e_1^k e_2^{n-k}$ of α when the basis of \mathbb{C}^2 are $e_1 = (x, y), e_2 = (-\bar{y}, \bar{x})$ for $l = [x, y] ((x, y) \in S^3)$. Then, s gives a homeomorphism between $S^{2k-1}(\mathbb{C}^2)$ and $\Gamma^{2k}(T_{2,1})$.

Then, let $H = \bigoplus_{k=1}^5 \Gamma^{2k}(T_{2,1})$ and

$$(3.11) \quad \tau : H \rightarrow \Gamma^2(T_{2,1})$$

be the projection.

3.3. τ -DESIGN

Recalling that the associated bundle to (3.2) is

$$(3.12) \quad \pi : T_{2,1}^1 \rightarrow \mathbb{C}P^1,$$

$\Omega = \mathbb{C}P^1, \mathcal{H} = \Gamma(T_{2,1})$ and $\mathcal{H}_0 = H (= \bigoplus_{k=1}^5 \Gamma^{2k}(T_{2,1}))$. Also, let $\{V_l\}_{l \in \mathbb{C}P^1} = \{l \in \mathbb{C}P^1\}$ and

$$(3.13) \quad e_l : \Gamma(T_{2,1}^1) \rightarrow l, e_l(s) = s^{(2)}(l).$$

where $s^{(2)}(l)$ is the second component of $s(l)$.

Now we define a G -invariant polynomial ring.

Definition 3.5. For a subgroup $G \subset GL(n, \mathbb{C})$, we define

$$(3.14) \quad \mathbb{C}[x_1, \dots, x_n]^G = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(Ax_1, \dots, Ax_n) = f(x_1, \dots, x_n)\}$$

and $\mathbb{C}[x_1, \dots, x_n]_d^G$ denotes a set of all elements of $\mathbb{C}[x_1, \dots, x_n]^G$ whose degree is d . The following series is called Hilbert series:

$$(3.15) \quad P_n^G(t) = \sum_{d=0}^{\infty} (\dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]_d^G) t^d.$$

Then, let

$$(3.16) \quad S = \begin{pmatrix} \varepsilon^3 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}, T = \frac{1}{\varepsilon^2 - \varepsilon^3} \begin{pmatrix} \varepsilon + \varepsilon^4 & 1 \\ 1 & -\varepsilon - \varepsilon^4 \end{pmatrix}, U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where ε is an primitive 5th root of 1.

Proposition 3.6.

$$(3.17) \quad G_{icosa} = \langle S, T, U \rangle$$

is a subgroup of $SU(2)$ and $\#G_{icosa} = 120$.

$$(3.18) \quad P_2^{G_{icosa}}(t) = \frac{1+t^{30}}{(1-t^{12})(1-t^{20})} = 1+t^{12}+t^{20}+t^{24}+\dots$$

For a subset $X \subset \mathbb{C}P^1$, linear functions $\lambda_x : V_x \rightarrow \Gamma^2(T_{2,1})$ and $g \in G \subset SU(2)$, we define

$$(3.19) \quad \Psi_{(X,\lambda)} : H \rightarrow \Gamma^2(T_{2,1}), \Psi_{(X,\lambda)}(s) = \sum_{x \in X} \lambda_x \circ e_x(s),$$

and

$$(3.20) \quad \Psi_{g \cdot (X,\lambda)} : H \rightarrow \Gamma^2(T_{2,1}), \Psi_{g \cdot (X,\lambda)}(s) = \sum_{x \in X} (g \cdot \lambda_x) \circ e_{gx}(s)$$

where

$$(3.21) \quad g \cdot \lambda_x : V_{gx} \rightarrow \Gamma^2(T_{2,1}), (g \cdot \lambda_x)(z) = g \cdot \lambda_x(g^{-1}z).$$

For subsets $X, Y \subset \mathbb{C}P^1$, we define the sum of (X, λ_X) and (Y, λ_Y) as follows:

$$(3.22) \quad (X, \lambda_X) + (Y, \lambda_Y) = (X \cup Y, \lambda_{(X \cup Y)})$$

where

$$(3.23) \quad \lambda_{(X \cup Y)}(z) = \begin{cases} \lambda_X(z) & (z \in X \setminus Y) \\ \lambda_X(z) + \lambda_Y(z) & (z \in X \cap Y) \\ \lambda_Y(z) & (z \in Y \setminus X) \end{cases}.$$

Then, we define

$$(3.24) \quad G \cdot (X, \lambda) = \sum_{g \in G} (g \cdot X, g \cdot \lambda_x).$$

Lemma 3.7. Let G be a finite subset of $SU(2)$ and $(Y, \lambda_Y) = \frac{1}{\#G} G \cdot (X, \lambda_X)$. Then, $\Psi_{(Y,\lambda_Y)}$ is a G -intertwining operator.

Theorem 3.8. For a subset $X \subset \mathbb{C}P^1$, if $\text{tr}_{\mathbb{C}}(\Psi_{(X,\lambda)}) = 2$, $(Y, \lambda_Y) = \frac{1}{\#G_{icosa}} G_{icosa} \cdot (X, \lambda)$ is a τ -design.

This theorem insists that

$$(3.25) \quad \Psi_{(Y,\lambda)}|_{\Gamma^{2k}(T_{2,1})} = \begin{cases} \text{id} & (k=1) \\ 0 & (k=2,3,4,5) \end{cases}.$$

Finally, we compose a τ -design. Let $x_0 = \{(x, 0) \mid x \in \mathbb{C}\}$ and $\lambda_0 : x_0 \rightarrow \Gamma^2(T_{2,1}^1)$ be

$$(3.26) \quad \lambda_0(x, 0)(l) = (l, 2\text{pr}_l(x, 0)).$$

Then, $\text{tr}_{\mathbb{C}}(\Psi_{(x_0,\lambda_0)}) = 2$. Therefore, from Theorem 3.8,

$$(3.27) \quad \frac{1}{\#G_{icosa}} G_{icosa} \cdot (x_0, \lambda_0)$$

is a τ -design. That is

$$(3.28) \quad \tau = \frac{1}{120} \sum_{g \in G_{icosa}} (g \cdot \lambda_0) \circ e_{gx_0}$$

Furthermore, since the following lemma, we can calculate the equation (3.28) as follows:

$$(3.29) \quad \tau = \frac{1}{12} \sum_{i=1}^{12} (g_i \cdot \lambda_0) \circ e_{g_i x_0}$$

where $\{g_i\}$ are the representatives of $G_{icosa}/\text{Iso}_{x_0}(G_{icosa})$.

Lemma 3.9.

$$(3.30) \quad \#\text{Iso}_{x_0}(G_{icosa}) = 10$$

and

$$(3.31) \quad \forall k \in \#\text{Iso}_{x_0}(G_{icosa}), k \cdot \lambda_0 = \lambda_0,$$

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