

Computation of weighted Bergman inner products on bounded symmetric domains and Parseval–Plancherel-type formulas for $(Sp(r, \mathbb{R}), Sp(r', \mathbb{R}) \times Sp(r'', \mathbb{R}))$

Ryosuke Nakahama^{*†}

NTT Institute for Fundamental Mathematics

Abstract

Let $(G, G') = (G, (G^\sigma)_0)$ be a symmetric pair of holomorphic type, and we consider a pair of Hermitian symmetric spaces $D' = G'/K' \subset D = G/K$, realized as bounded symmetric domains in complex vector spaces $\mathfrak{p}_1^+ := (\mathfrak{p}^+)^\sigma \subset \mathfrak{p}^+$ respectively. Then the universal covering group \tilde{G} of G acts unitarily on the weighted Bergman space $\mathcal{H}_\lambda(D) \subset \mathcal{O}(D) = \mathcal{O}_\lambda(D)$ on D for sufficiently large λ . Its restriction to the subgroup \tilde{G}' decomposes discretely and multiplicity-freely, and its branching law is given explicitly by Hua–Kostant–Schmid–Kobayashi’s formula in terms of the \tilde{K}' -decomposition of the space $\mathcal{P}(\mathfrak{p}_2^+)$ of polynomials on $\mathfrak{p}_2^+ := (\mathfrak{p}^+)^{-\sigma} \subset \mathfrak{p}^+$. Our goal is to understand the decomposition of the restriction $\mathcal{H}_\lambda(D)|_{\tilde{G}'}$ by studying the weighted Bergman inner product on each \tilde{K}' -type in $\mathcal{P}(\mathfrak{p}_2^+) \subset \mathcal{H}_\lambda(D)$. In this article we mainly deal with the symmetric pair $(G, G') = (Sp(r, \mathbb{R}), Sp(r', \mathbb{R}) \times Sp(r'', \mathbb{R}))$.

1 Setting

First we review a family of representations, called *holomorphic discrete series representations*, of a Hermitian Lie group G , in the case $G = Sp(r, \mathbb{R})$. We realize the real symplectic group $G = Sp(r, \mathbb{R})$ as

$$G = Sp(r, \mathbb{R}) := \left\{ g \in GL(2r, \mathbb{C}) \mid g \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} t g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, g \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \bar{g} \right\}.$$

Then this is isomorphic to the usual $Sp(r, \mathbb{R})$ via the Cayley transform. Under this realization, G acts transitively on

$$D_r := \{x \in \text{Sym}(r, \mathbb{C}) \mid I - x\bar{x} \text{ is positive definite}\}$$

by the linear fractional transform

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .x := (ax + b)(cx + d)^{-1},$$

and D_r gives the *bounded symmetric domain realization (Harish-Chandra realization)* of the Hermitian symmetric space $Sp(r, \mathbb{R})/U(r)$. Next let $\lambda \in \mathbb{C}$, and let (τ, V) be a finite dimensional representation of $GL(r, \mathbb{C})$, with the $K := U(r)$ -invariant inner product $(\cdot, \cdot)_V$. Then

^{*}This work was supported by Grant-in-Aid for JSPS Fellows Grant Number JP20J00114.

[†]This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

the universal covering group \tilde{G} of G acts on the space of V -valued holomorphic functions $\mathcal{O}(D_r, V) = \mathcal{O}_\lambda(D_r, V)$ by

$$\tau_\lambda \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \right) f(x) := \det(cx + d)^{-\lambda} \tau({}^t(cx + d)) f((ax + b)(cx + d)^{-1}).$$

We note that $\det(cx + d)^{-\lambda}$ is not well-defined on $G \times D$ unless $\lambda \in \mathbb{Z}$, but is well-defined on the universal covering space $\tilde{G} \times D$. Let $\mathcal{H}_\lambda(D_r, V) \subset \mathcal{O}_\lambda(D_r, V)$ be the non-zero unitary subrepresentation of \tilde{G} if it exists. We note that such subrepresentation is unique, since the corresponding reproducing kernel is proportional to $\tau(I - x\bar{y}) \det(I - x\bar{y})^{-\lambda}$ by the transitivity of the action of G on D_r . Especially, if $\lambda \in \mathbb{R}$ is sufficiently large, then such unitary subrepresentation exists, and its inner product is given by the explicit converging integral

$$\langle f, g \rangle_{\lambda, V} := C_{\lambda, V} \int_{D_r} (\tau(I - x\bar{x})^{-1} f(x), g(x))_V \det(I - x\bar{x})^{\lambda - (r+1)} dx.$$

This is called a *weighted Bergman inner product*, and the unitary representation $(\tau_\lambda, \mathcal{H}_\lambda(D_r, V))$ is called a *holomorphic discrete series representation*. Especially when $(\tau, V) = \mathbb{C}$ is trivial, then we write $\mathcal{H}_\lambda(D_r, \mathbb{C}) = \mathcal{H}_\lambda(D_r)$, and call it of *scalar type*. In this case $\mathcal{H}_\lambda(D)$ becomes a holomorphic discrete series representation if $\lambda > r$, with the inner product

$$\langle f, g \rangle_\lambda := C_\lambda \int_{D_r} f(x) \overline{g(x)} \det(I - x\bar{x})^{\lambda - (r+1)} dx. \quad (1.1)$$

Here we determine the constant C_λ such that $\|1\|_\lambda = 1$ holds.

Next suppose (G, G') is a symmetric pair of *holomorphic type*, that is, both G/K and G'/K' are Hermitian symmetric spaces and the natural embedding $G'/K' \hookrightarrow G/K$ is holomorphic, and let $\mathcal{H}_\lambda(D)$ be a holomorphic discrete series representation of \tilde{G} of scalar type. Then it is known that the restriction $\mathcal{H}_\lambda(D)|_{\tilde{G}'}$ decomposes discretely and multiplicity-freely, and its branching law is explicitly determined (see Kobayashi [13]). In the following, we consider the case $(G, G') := (Sp(r, \mathbb{R}), Sp(r', \mathbb{R}) \times Sp(r'', \mathbb{R}))$ with $r = r' + r''$, $r' \leq r''$, and give the description of the branching law of $\mathcal{H}_\lambda(D_r)|_{\tilde{G}'}$. To do this, let

$$\mathfrak{p}^+ := \text{Sym}(r, \mathbb{C}), \quad \mathfrak{p}_{11}^+ := \text{Sym}(r', \mathbb{C}), \quad \mathfrak{p}_{12}^+ := M(r', r''; \mathbb{C}), \quad \mathfrak{p}_{22}^+ := \text{Sym}(r'', \mathbb{C}),$$

and write the elements $x \in \mathfrak{p}^+$ as

$$\mathfrak{p}^+ = \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{12}^+ \oplus \mathfrak{p}_{22}^+ \ni x = \begin{pmatrix} x_{11} & x_{12} \\ {}^t x_{12} & x_{22} \end{pmatrix}.$$

Also, let

$$\mathbb{Z}_{++}^r := \{\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r \mid k_1 \geq \dots \geq k_r \geq 0\}.$$

Then the space of polynomials $\mathcal{P}(\mathfrak{p}_{12}^+)$ on \mathfrak{p}_{12}^+ is decomposed under $K' := U(r') \times U(r'')$ as

$$\mathcal{P}(\mathfrak{p}_{12}^+) = \mathcal{P}(M(r', r''; \mathbb{C})) = \bigoplus_{\mathbf{k} \in \mathbb{Z}_{++}^{r'}} \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+) \simeq \bigoplus_{\mathbf{k} \in \mathbb{Z}_{++}^{r'}} V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{k}}^{(r'')\vee},$$

where $V_{\mathbf{k}}^{(r')\vee}$ is the irreducible representation of $U(r')$ with the lowest weight $-\mathbf{k}$ under a suitable identification of the weight lattice for $U(r')$ and $\mathbb{Z}^{r'}$, and similar for $V_{\mathbf{k}}^{(r'')\vee}$, where we identify \mathbf{k} and $(\mathbf{k}, 0, \dots, 0)$. According to this decomposition, for $\lambda > r$, $\mathcal{H}_\lambda(D_r)|_{\tilde{G}'}$ is decomposed as

$$\mathcal{H}_\lambda(D_r)|_{\tilde{G}'} \simeq \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r'}}^{\oplus} \mathcal{H}_\lambda(D_{r'}, V_{\mathbf{k}}^{(r')\vee}) \boxtimes \mathcal{H}_\lambda(D_{r''}, V_{\mathbf{k}}^{(r'')\vee}) \quad (1.2)$$

(see Kobayashi [13, Theorem 8.3]). Now we want to understand this decomposition concretely by considering the inner product

$$\left\langle f(x_{12}), e^{\operatorname{tr}(x\bar{z})} \right\rangle_{\lambda, x} \quad \left(f(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+), x = \begin{pmatrix} x_{11} & x_{12} \\ t x_{12} & x_{22} \end{pmatrix}, z \in \mathfrak{p}^+ \right), \quad (1.3)$$

where the subscript x stands for the variable of integration.

For example, suppose $r' = r''$, and we consider the case $\mathbf{k} = (k, \dots, k)$. Then we have $\mathcal{P}_{(k, \dots, k)}(\mathfrak{p}_{12}^+) = \mathcal{P}_{(k, \dots, k)}(M(r', \mathbb{C})) = \mathbb{C} \det(x_{12})^k$. In this setting, the above inner product is explicitly computable.

Theorem 1.1 ([21, Theorem 6.8 (2)]). *Suppose $r' = r''$. Then for $k \in \mathbb{Z}_{\geq 0}$, $\operatorname{Re} \lambda > 2r'$, $z = \begin{pmatrix} z_{11} & z_{12} \\ t z_{12} & z_{22} \end{pmatrix} \in \mathfrak{p}^+$, we have*

$$\begin{aligned} & \left\langle \det(x_{12})^k, e^{\operatorname{tr}(x\bar{z})} \right\rangle_{\lambda, x} \\ &= \frac{\prod_{i=r'+1}^{2r'} (\lambda + \lceil \frac{k}{2} \rceil - \frac{i}{2})_{\lfloor k/2 \rfloor}}{\prod_{i=1}^{r'} (\lambda - \frac{i-1}{2})_k \prod_{i=r'+1}^{2r'} (\lambda - \frac{i-1}{2})_{\lfloor k/2 \rfloor}} \det(z_{12})^k {}_2F_1 \left(\begin{matrix} -\frac{k}{2}, -\frac{k-1}{2} \\ -\lambda - k + r' + 1 \end{matrix}; z_{11} t z_{12}^{-1} z_{22} z_{12}^{-1} \right). \end{aligned}$$

Here, $(\lambda)_m := \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+m-1)$. We omit the definition of ${}_2F_1$, but this coincides with a special case of Heckman–Opdam’s multivariate hypergeometric function of type $BC_{r'}$ under a suitable change of variables. By using this, we can construct explicitly the \tilde{G}' -intertwining operator (symmetry breaking operator) from $\mathcal{H}_\lambda(D_r)|_{\tilde{G}'}$ to $\mathcal{H}_{\lambda+k}(D_{r'}) \hat{\boxtimes} \mathcal{H}_{\lambda+k}(D_{r'})$ (see [21, Theorem 8.6]). Also, by the theorem we can immediately determine the top term (i.e., the value at $z_{11} = 0, z_{22} = 0$) of (1.3) as

$$\left\langle \det(x_{12})^k, e^{\operatorname{tr}(x\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_{11}=0, z_{22}=0} = \frac{\prod_{i=r'+1}^{2r'} (\lambda + \lceil \frac{k}{2} \rceil - \frac{i}{2})_{\lfloor k/2 \rfloor}}{\prod_{i=1}^{r'} (\lambda - \frac{i-1}{2})_k \prod_{i=r'+1}^{2r'} (\lambda - \frac{i-1}{2})_{\lfloor k/2 \rfloor}} \det(z_{12})^k,$$

and determine the poles of (1.3) with respect to $\lambda \in \mathbb{C}$, that is,

$$\prod_{i=1}^{r'} \left(\lambda - \frac{i-1}{2} \right)_k \prod_{i=r'+1}^{2r'} \left(\lambda - \frac{i-1}{2} \right)_{\lfloor k/2 \rfloor} \left\langle \det(x_{12})^k, e^{\operatorname{tr}(x\bar{z})} \right\rangle_{\lambda, x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$. In the following, we consider general partitions $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$. Then we cannot compute explicitly (1.3) so far, but can compute the top term and the poles. This is applied for the determination of the Parseval–Plancherel-type formula for the decomposition of $\mathcal{H}_\lambda(D_r)|_{\tilde{G}'}$.

2 Main theorems and applications

As before, let $\mathfrak{p}^+ := \operatorname{Sym}(r, \mathbb{C}) \supset \mathfrak{p}_{12}^+ := M(r', r''; \mathbb{C})$ with $r = r' + r''$, $r' \leq r''$, and write $x = \begin{pmatrix} x_{11} & x_{12} \\ t x_{12} & x_{22} \end{pmatrix} \in \mathfrak{p}^+$. First we give a result on top terms of (1.3).

Theorem 2.1. *Let $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$ and put $k_{r'+1} := 0$. Then for $\operatorname{Re} \lambda > r$, $f(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$, we have*

$$\left\langle f(x_{12}), e^{\operatorname{tr}(x\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_{11}=0, z_{22}=0} = C(\lambda, \mathbf{k}) f(z_{12}),$$

where

$$\begin{aligned}
C(\lambda, \mathbf{k}) &= \frac{2^{|\mathbf{k}|} \prod_{1 \leq i < j \leq r'} (2\lambda - (i+j))_{k_i+k_j} \prod_{i=1}^{r'} (\lambda - i)_{k_i}}{\prod_{1 \leq i < j \leq r'+1} (2\lambda - (i+j-1))_{k_i+k_j} \prod_{i=1}^{r'} (\lambda - (i-1))_{k_i}} \\
&= \frac{\prod_{1 \leq i < j \leq r'} \left(\lambda - \frac{i+j-1}{2} \right)_{\lfloor \frac{k_i+k_j}{2} \rfloor} \prod_{1 \leq i \leq j \leq r'} \left(\lambda - \frac{i+j}{2} \right)_{\lceil \frac{k_i+k_j}{2} \rceil}}{\prod_{1 \leq i \leq j \leq r'+1} \left(\lambda - \frac{i+j-2}{2} \right)_{\lfloor \frac{k_i+k_j}{2} \rfloor} \prod_{1 \leq i < j \leq r'+1} \left(\lambda - \frac{i+j-1}{2} \right)_{\lceil \frac{k_i+k_j}{2} \rceil}} \\
&= \frac{\prod_{a=3}^{2r'-1} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor - 1} \left(\lambda - \frac{a-1}{2} \right)_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor} \prod_{a=2}^{2r'} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor} \left(\lambda - \frac{a}{2} \right)_{\lceil \frac{k_i+k_{a-i}}{2} \rceil}}{\prod_{a=1}^{2r'} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor} \left(\lambda - \frac{a-1}{2} \right)_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor} \prod_{a=2}^{2r'} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor} \left(\lambda - \frac{a}{2} \right)_{\lceil \frac{k_i+k_{a+1-i}}{2} \rceil}}.
\end{aligned} \tag{2.1}$$

Next we give a result on poles of (1.3).

Theorem 2.2. For $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, define $\phi(\mathbf{k}) \in \mathbb{Z}_{++}^{2r'}$ by

$$\phi(\mathbf{k})_a := \min \left\{ \left\lfloor \frac{k_i + k_j}{2} \right\rfloor \mid 1 \leq i \leq j \leq r' + 1, i + j = a + 1 \right\} \quad (1 \leq a \leq 2r'), \tag{2.2}$$

where $k_{r'+1} := 0$. Then for $f(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$,

$$\prod_{a=1}^{2r'} \left(\lambda - \frac{a-1}{2} \right)_{\phi(\mathbf{k})_a} \left\langle f(x_{12}), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$.

Remark 2.3. We can easily verify the restriction of Theorem 2.2 to $z_{11} = 0$, $z_{22} = 0$ by using Theorem 2.1, that is,

$$\begin{aligned}
\prod_{a=1}^{2r'} \left(\lambda - \frac{a-1}{2} \right)_{\phi(\mathbf{k})_a} C(\lambda, \mathbf{k}) &= \prod_{a=1}^{2r'} \left(\lambda - \frac{a-1}{2} \right)_{\min_i \lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor} \\
&\times \frac{\prod_{a=3}^{2r'-1} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor - 1} \left(\lambda - \frac{a-1}{2} \right)_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor} \prod_{a=2}^{2r'} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor} \left(\lambda - \frac{a}{2} \right)_{\lceil \frac{k_i+k_{a-i}}{2} \rceil}}{\prod_{a=1}^{2r'} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor} \left(\lambda - \frac{a-1}{2} \right)_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor} \prod_{a=2}^{2r'} \prod_{i=\max\{1, a-r'\}}^{\lfloor a/2 \rfloor} \left(\lambda - \frac{a}{2} \right)_{\lceil \frac{k_i+k_{a+1-i}}{2} \rceil}}
\end{aligned}$$

is holomorphic for all $\lambda \in \mathbb{C}$.

Next we consider some applications of the theorems. Let $(G, G') = (Sp(r, \mathbb{R}), Sp(r', \mathbb{R}) \times Sp(r'', \mathbb{R}))$ as before. Then since $\mathcal{H}_\lambda(D_r)|_{\tilde{G}'}$ decomposes as in (1.2) for $\lambda > r$, for each $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$ there exists uniquely (up to scalar) a \tilde{G}' -intertwining operator (symmetry breaking operator)

$$\mathcal{F}_{\lambda, \mathbf{k}}: \mathcal{H}_\lambda(D_r)|_{\tilde{G}'} \longrightarrow \mathcal{H}_\lambda(D_{r'}, V_{\mathbf{k}}^{(r')\vee}) \hat{\boxtimes} \mathcal{H}_\lambda(D_{r''}, V_{\mathbf{k}}^{(r'')\vee}).$$

We fix the normalization of $\mathcal{F}_{\lambda, \mathbf{k}}$ such that

$$\|\mathcal{F}_{\lambda, \mathbf{k}}(f(x_{12}))\|_{\mathcal{H}_\lambda(D_{r'}, V_{\mathbf{k}}^{(r')\vee}) \hat{\boxtimes} \mathcal{H}_\lambda(D_{r''}, V_{\mathbf{k}}^{(r'')\vee})}^2 = \bar{f} \left(\frac{1}{2} \frac{\partial}{\partial z_{12}} \right) f(z_{12}) \Big|_{z_{12}=0} \quad (f(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+))$$

holds, independent of λ . Then we can easily prove the following.

Corollary 2.4. For $\lambda > r$, for $f \in \mathcal{H}_\lambda(D_r)$, we have

$$\|f\|_{\mathcal{H}_\lambda(D_r)}^2 = \sum_{\mathbf{k} \in \mathbb{Z}_{++}^{r'}} C(\lambda, \mathbf{k}) \|\mathcal{F}_{\lambda, \mathbf{k}} f\|_{\mathcal{H}_\lambda(D_{r'}, V_{\mathbf{k}}^{(r')\vee}) \hat{\boxtimes} \mathcal{H}_\lambda(D_{r''}, V_{\mathbf{k}}^{(r'')\vee})}^2,$$

where $C(\lambda, \mathbf{k})$ is as in (2.1).

We omit the proof of Corollary 2.4. Next we consider the decomposition of $\mathcal{H}_\lambda(D_r)|_{\tilde{G}'}$ for smaller λ . $\tilde{G} = \tilde{Sp}(r, \mathbb{R})$ acts on $\mathcal{O}_\lambda(D_r)$, and it is known that there exists a non-zero unitary subrepresentation $\mathcal{H}_\lambda(D_r) \subset \mathcal{O}_\lambda(D_r)$ if and only if

$$\lambda \in \left\{0, \frac{1}{2}, 1, \dots, \frac{r-1}{2}\right\} \cup \left(\frac{r-1}{2}, \infty\right)$$

(see, e.g., [5, Theorem XIII.2.7]). This set is called the *Wallach set*. Especially, $\mathcal{H}_\lambda(D_r)$ is a holomorphic discrete series representation (i.e., the integral (1.1) converges) for $\lambda > r$. If $\lambda > \frac{r-1}{2}$, then the decomposition of $\mathcal{H}_\lambda(D_r)|_{\tilde{G}'}$ is again given by (1.2). On the other hand, for smaller λ the following holds. Here, $\phi(\mathbf{k})_a$ is defined as in (2.2) for $1 \leq a \leq 2r'$, and we set $\phi(\mathbf{k})_a := 0$ for $2r' < a \leq r$.

Corollary 2.5. For $a = 0, 1, 2, \dots, r-1$, we have

$$\begin{aligned} \mathcal{H}_{\frac{a}{2}}(D_r)|_{\tilde{G}'} &\simeq \sum_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{r'} \\ \phi(\mathbf{k})_{a+1}=0}}^{\oplus} \mathcal{H}_\lambda(D_{r'}, V_{\mathbf{k}}^{(r')\vee}) \hat{\boxtimes} \mathcal{H}_\lambda(D_{r''}, V_{\mathbf{k}}^{(r'')\vee}) \\ &= \sum_{\substack{0 \leq b \leq c \leq r' \\ \bar{b}+c \leq \bar{a} \\ k_b \geq 2}}^{\oplus} \sum_{\mathbf{k} \in \mathbb{Z}_{++}^b}^{\oplus} \mathcal{H}_{\frac{a}{2}}(D_{r'}, V_{(\mathbf{k}, \underbrace{1, \dots, 1}_{c-b}, \underbrace{0, \dots, 0}_{r'-c})}^{(r')\vee}) \hat{\boxtimes} \mathcal{H}_{\frac{a}{2}}(D_{r''}, V_{(\mathbf{k}, \underbrace{1, \dots, 1}_{c-b}, \underbrace{0, \dots, 0}_{r''-c})}^{(r'')\vee}). \end{aligned}$$

Remark 2.6. (1) *Parseval–Plancherel-type formulas for Hermitian symmetric pairs (G, K) , i.e., cases such that $K \subset G$ is a maximal compact subgroup, are studied by, e.g., Ørsted [23], Faraut–Korányi [4, 5], Ørsted–Zhang [24, 25], Hwang–Liu–Zhang [10] and the author [20].*

(2) *Parseval–Plancherel-type formulas for general symmetric pairs of holomorphic type (G, G') are studied by, e.g., Hilgert–Krötz [7, 8], Ben Saïd [1, 2] and Kobayashi–Pevzner [18], under different realization of holomorphic discrete series representations.*

(3) *In this article, we treat the explicit forms of symmetry breaking operators as black boxes. Construction of differential symmetry breaking operators are studied by, e.g., Rankin [27], Cohen [3], Peng–Zhang [26], Juhl [12], Ibukiyama–Kuzumaki–Ochiai [11], Kobayashi–Ørsted–Somberg–Souček [15], Kobayashi–Pevzner [16, 17], Kobayashi–Kubo–Pevzner [14] and the author [21].*

(4) *Branching laws of unitary highest weight modules for discrete Wallach sets are studied by, e.g., Sekiguchi [28] and Möllers–Oshima [19]. We can also study branching laws of unitary highest weight modules by using the seesaw dual pair theory (see, e.g., [9, Section 3]) as in [19] when (G, G') is classical.*

3 Proof of Theorem 2.2 and Corollary 2.5

In this section we give proofs of Theorem 2.2 and Corollary 2.5. To do this, we observe the $\tilde{K} = \tilde{U}(r)$ -type decomposition of $\mathcal{H}_\lambda(D_r)$. The \tilde{K} -finite part of $\mathcal{H}_\lambda(D_r)$ is given by

$$\mathcal{O}_\lambda(D_r)_{\tilde{K}} = \det^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}^+) = \det^{-\lambda} \otimes \mathcal{P}(\text{Sym}(r, \mathbb{C})),$$

and the space of polynomials $\mathcal{P}(\mathfrak{p}^+) = \mathcal{P}(\text{Sym}(r, \mathbb{C}))$ is decomposed under $K = U(r)$ as

$$\mathcal{P}(\mathfrak{p}^+) = \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq \bigoplus_{\mathbf{m} \in \mathbb{Z}_{++}^r} V_{2\mathbf{m}}^{(r)\vee}.$$

According to this decomposition, the following holds.

Theorem 3.1 (Faraut–Korányi [5, Corollary XIII.2.3]). *Let $\mathbf{m} \in \mathbb{Z}_{++}^r$. Then for $\text{Re } \lambda > r$, $f(x) \in \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+)$, we have*

$$\left\langle f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} = \frac{1}{\prod_{a=1}^r \left(\lambda - \frac{a-1}{2} \right)_{m_a}} f(z).$$

Especially, for $f(x) \in \mathcal{P}(\mathfrak{p}^+)$ and for $\mathbf{l} \in \mathbb{Z}_{++}^r$,

$$\prod_{a=1}^r \left(\lambda - \frac{a-1}{2} \right)_{l_a} \left\langle f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x}$$

is holomorphically continued for all $\lambda \in \mathbb{C}$ if and only if

$$f(x) \in \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_a \leq l_a}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

Proof of Theorem 2.2. Under Theorem 3.1, Theorem 2.2 is equivalent to

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_a \leq \phi(\mathbf{k})_a}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+), \quad (3.1)$$

and hence it is enough to prove this inclusion. Since $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+) \simeq V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{k}}^{(r'')\vee}$ as a $K' = U(r') \times U(r'')$ -module and $\mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \simeq V_{2\mathbf{m}}^{(r)\vee}$ as a $K = U(r)$ -module, it is enough to show that

$$\text{Hom}_{U(r') \times U(r'')}(V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{k}}^{(r'')\vee}, V_{2\mathbf{m}}^{(r)\vee}) \neq \{0\} \quad \text{implies} \quad m_a \leq \phi(\mathbf{k})_a \quad (1 \leq a \leq r),$$

or equivalently by the definition of $\phi(\mathbf{k})_a$,

$$\text{Hom}_{U(r') \times U(r'')}(V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{k}}^{(r'')\vee}, V_{2\mathbf{m}}^{(r)\vee}) \neq \{0\} \quad \text{implies} \quad 2m_{i+j-1} \leq k_i + k_j \quad (1 \leq i, j \leq r' + 1),$$

with $k_{r'+1} := 0$. On the other hand, for $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$, $\mathbf{m} \in \mathbb{Z}_{++}^r$ with $r' + r'' = r$,

$$\dim \text{Hom}_{U(r') \times U(r'')}(V_{\mathbf{k}}^{(r')\vee} \boxtimes V_{\mathbf{l}}^{(r'')\vee}, V_{\mathbf{m}}^{(r)\vee}) = \dim \text{Hom}_{U(r)}(V_{\mathbf{m}}^{(r)\vee}, V_{\mathbf{k}}^{(r)\vee} \otimes V_{\mathbf{l}}^{(r)\vee})$$

holds in general by [6, Theorem 9.2.3], and by the Littlewood–Richardson rule, we can show that for $\mathbf{k}, \mathbf{l}, \mathbf{m} \in \mathbb{Z}_{++}^r$,

$$\text{Hom}_{U(r)}(V_{\mathbf{m}}^{(r)\vee}, V_{\mathbf{k}}^{(r)\vee} \otimes V_{\mathbf{l}}^{(r)\vee}) \neq \{0\} \quad \text{implies} \quad m_{i+j-1} \leq k_i + l_j \quad (1 \leq i, j, i+j \leq r+1)$$

(see [22, Lemma 3.6]). Hence the theorem follows. \square

We note that this proof for $(G, G') = (Sp(r, \mathbb{R}), Sp(r', \mathbb{R}) \times Sp(r'', \mathbb{R}))$ is not available for other symmetric pairs in general.

Next, to prove Corollary 2.5, we observe the \tilde{K} -type formula for the discrete Wallach set. By Theorem 3.1, $\langle \cdot, \cdot \rangle_\lambda$ is meromorphically continued for all $\lambda \in \mathbb{C}$, and is positive definite on $\mathcal{P}(\mathfrak{p}^+)$ for $\lambda > \frac{r-1}{2}$. That is, $\mathcal{H}_\lambda(D_r)_{\tilde{K}} = \det^{-\lambda} \otimes \mathcal{P}(\mathfrak{p}^+)$ holds for $\lambda > \frac{r-1}{2}$. On the other hand, $\langle \cdot, \cdot \rangle_\lambda$ has poles at $\lambda \in \frac{r-1}{2} - \frac{1}{2}\mathbb{Z}_{\geq 0}$, and $\mathcal{O}_\lambda(D_r)_{\tilde{K}}$ becomes reducible for such λ . If the restriction of $\langle \cdot, \cdot \rangle_\lambda$ to the irreducible $(\mathfrak{g}, \tilde{K})$ -submodule of $\mathcal{O}_\lambda(D_r)_{\tilde{K}}$ is positive definite, then this becomes an infinitesimally unitary submodule. This occurs when $\lambda = 0, \frac{1}{2}, 1, \dots, \frac{r-1}{2}$. That is, the following holds.

Corollary 3.2 (Faraut–Korányi [5, Theorem XIII.2.7]). *For $a = 0, 1, 2, \dots, r-1$, we have*

$$\mathcal{H}_{\frac{a}{2}}(D_r)_{\tilde{K}} = \det^{-\lambda} \otimes \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_{a+1}=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

Especially, $f(x) \in \mathcal{H}_{\frac{a}{2}}(D_r)_{\tilde{K}}$ holds if and only if $\langle f(x), e^{\text{tr}(x\bar{z})} \rangle_{\lambda, x}$ is holomorphic for $\lambda > \frac{a-1}{2}$.

Proof of Corollary 2.5. We embed $\mathfrak{p}^+ = \text{Sym}(r, \mathbb{C})$ into $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(r, \mathbb{C})$, the complexified Lie algebra of $G = Sp(r, \mathbb{R})$, by $x \mapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, and similarly embed $\mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$ into $\mathfrak{g}'^{\mathbb{C}} = \mathfrak{sp}(r', \mathbb{C}) \oplus \mathfrak{sp}(r'', \mathbb{C})$ compatibly, so that we have

$$\begin{array}{ccc} \mathfrak{g}^{\mathbb{C}} & \supset & \mathfrak{p}^+ & := \text{Sym}(r, \mathbb{C}) \\ \cup & & \cup & \\ \mathfrak{g}'^{\mathbb{C}} & \supset & \mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+ & := \text{Sym}(r', \mathbb{C}) \oplus \text{Sym}(r'', \mathbb{C}). \end{array}$$

Then since $\mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$ acts on $\mathcal{O}_\lambda(D_r)_{\tilde{K}} = \mathcal{P}(\mathfrak{p}^+)$ by constant coefficient differential operators along $\mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+ \subset \mathfrak{p}^+$, the $\mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+$ -null part of $\mathcal{H}_{\frac{a}{2}}(D_r)_{\tilde{K}}$ is given by

$$\mathcal{H}_{\frac{a}{2}}(D_r)_{\tilde{K}}^{\mathfrak{p}_{11}^+ \oplus \mathfrak{p}_{22}^+} = \mathcal{P}(\mathfrak{p}_{12}^+) \cap \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_{a+1}=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+),$$

with $\mathfrak{p}_{12}^+ := M(r', r''; \mathbb{C})$. Since every $(\mathfrak{g}', \tilde{K}')$ -submodule in $\mathcal{H}_{\frac{a}{2}}(D_r)$ intersects the above space, it is enough to show that

$$\mathcal{P}(\mathfrak{p}_{12}^+) \cap \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_{a+1}=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) = \bigoplus_{\substack{\mathbf{k} \in \mathbb{Z}_{++}^{r'} \\ \phi(\mathbf{k})_{a+1}=0}} \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$$

holds. To prove the inclusion from right to left, suppose $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$ satisfies $\phi(\mathbf{k})_{a+1} = 0$. Then by (3.1), we have

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_j \leq \phi(\mathbf{k})_j}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+) \subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_{a+1}=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

To prove the opposite inclusion, suppose $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$ satisfies $\phi(\mathbf{k})_{a+1} \neq 0$, and take the smallest $a' > a$ such that $\phi(\mathbf{k})_{a'+1} = 0$. Then for $f(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$, by Theorem 2.1, we can show that

$$\left\langle f(x_{12}), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_{11}=0, z_{22}=0} = C(\lambda, \mathbf{k})f(z_{12})$$

has a pole at $\lambda = \frac{a'-1}{2}$. Especially, $\langle f(x_{12}), e^{\text{tr}(x\bar{z})} \rangle_{\lambda, x}$ is not holomorphic on $\lambda > \frac{a-1}{2}$, and hence by Theorem 3.1 we have

$$\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+) \not\subset \bigoplus_{\substack{\mathbf{m} \in \mathbb{Z}_{++}^r \\ m_{a+1}=0}} \mathcal{P}_{\mathbf{m}}(\mathfrak{p}^+).$$

This completes the proof of Corollary 2.5. □

4 Proof of Theorem 2.1

In this section we give a proof of Theorem 2.1. First, for $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, we define a polynomial $\Delta_{\mathbf{k}}(x_{12})$ on $\mathfrak{p}_{12}^+ = M(r', r''; \mathbb{C})$ by

$$\Delta_{\mathbf{k}}(x_{12}) := \prod_{l=1}^{r'} \det(((x_{12})_{ij})_{1 \leq i, j \leq l})^{k_l - k_{l+1}},$$

where $k_{r'+1} := 0$. Then $\mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$ is generated by $\Delta_{\mathbf{k}}(x_{12})$ as a $K' = U(r') \times U(r'')$ -module. Since the inner product (1.1) is K' -equivariant, it is enough to prove the theorem when $f(x_{12}) = \Delta_{\mathbf{k}}(x_{12}) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_{12}^+)$.

To prove the theorem, we prepare some lemmas. First, for $s < r$ we fix an inclusion $\text{Sym}(s, \mathbb{C}) \hookrightarrow \text{Sym}(r, \mathbb{C})$ suitably, and for $x \in \text{Sym}(r, \mathbb{C})$, let $x' \in \text{Sym}(s, \mathbb{C})$ denote the orthogonal projection of x . Then the following holds.

Lemma 4.1. *For $\text{Re } \lambda > r$, for $f(x') \in \mathcal{P}(\text{Sym}(s, \mathbb{C})) \subset \mathcal{P}(\text{Sym}(r, \mathbb{C}))$, we have*

$$\left\langle f(x'), e^{\text{tr}(x\bar{z})} \right\rangle_{\mathcal{H}_{\lambda}(D_r), x} = \left\langle f(x'), e^{\text{tr}(x'\bar{z}')} \right\rangle_{\mathcal{H}_{\lambda}(D_s), x'}.$$

Proof. Let $\mathbf{m} \in \mathbb{Z}_{++}^s$. Then we have $\mathcal{P}_{\mathbf{m}}(\text{Sym}(s, \mathbb{C})) \subset \mathcal{P}_{\mathbf{m}}(\text{Sym}(r, \mathbb{C})) = \mathcal{P}_{(\mathbf{m}, \overbrace{0, \dots, 0}^{r-s})}(\text{Sym}(r, \mathbb{C}))$, and by Theorem 3.1, for $f(x') \in \mathcal{P}_{\mathbf{m}}(\text{Sym}(s, \mathbb{C})) \subset \mathcal{P}_{\mathbf{m}}(\text{Sym}(r, \mathbb{C}))$ we have

$$\left\langle f(x'), e^{\text{tr}(x\bar{z})} \right\rangle_{\mathcal{H}_{\lambda}(D_r), x} = \frac{1}{\prod_{a=1}^s (\lambda - \frac{a-1}{2})_{m_a}} f(z') = \left\langle f(x'), e^{\text{tr}(x'\bar{z}')} \right\rangle_{\mathcal{H}_{\lambda}(D_s), x'}.$$

Since this holds for every $\mathbf{m} \in \mathbb{Z}_{++}^s$, we get the lemma. \square

Suppose $r = r' + r''$, $r' \leq r''$, and let $s = 2r'$. Then by applying the above lemma for the inclusion

$$\begin{array}{ccc} \text{Sym}(2r', \mathbb{C}) & \subset & \text{Sym}(r, \mathbb{C}) \\ \cup & & \cup \\ M(r', \mathbb{C}) & \subset & M(r', r''; \mathbb{C}), \end{array}$$

for every $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$ we get

$$\left\langle \Delta_{\mathbf{k}}(x_{12}), e^{\text{tr}(x\bar{z})} \right\rangle_{\mathcal{H}_{\lambda}(D_r), x} = \left\langle \Delta_{\mathbf{k}}(x'_{12}), e^{\text{tr}(x'\bar{z}')} \right\rangle_{\mathcal{H}_{\lambda}(D_{2r'}), x'}.$$

Hence it is enough to prove Theorem 2.1 when $r' = r''$.

In the following suppose $r' = r''$, and let $\mathfrak{p}_{12}^+ := M(r', \mathbb{C})$. For $x \in \mathfrak{p}^+ = \text{Sym}(r, \mathbb{C})$, let

$$\det'(x) := \det \left(x \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right),$$

so that $\det' \begin{pmatrix} 0 & x_{12} \\ x_{12} & 0 \end{pmatrix} = \det(x_{12})^2$ holds. Then the following holds.

Proposition 4.2. *For $\text{Re } \lambda > r$, $k \in \mathbb{Z}_{\geq 0}$, $f(x) \in \mathcal{P}(\mathfrak{p}^+)$, we have*

$$\begin{aligned} & \left\langle \det(x_{12})^k f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} \\ &= \frac{1}{\prod_{i=1}^r (\lambda - \frac{i-1}{2})_k} \det'(z)^{-\lambda + \frac{r+1}{2}} \det \left(\frac{1}{2} \frac{\partial}{\partial z_{12}} \right)^k \det'(z)^{\lambda + k - \frac{r+1}{2}} \left\langle f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda + k, x}. \end{aligned}$$

The proof of Proposition 4.2 is given later. We also need the following.

Lemma 4.3 ([5, Proposition VII.1.6]). *For $\mu \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r'}$, $z_{12} \in \mathfrak{p}_{12}^+ = M(r', \mathbb{C})$, we have*

$$\det \left(\frac{\partial}{\partial z_{12}} \right)^k \det(z_{12})^\mu \Delta_{\mathbf{l}}(z_{12}) = \prod_{i=1}^{r'} (\mu + l_i - k + r' - i + 1)_k \det(z_{12})^{\mu-k} \Delta_{\mathbf{l}}(z_{12}).$$

Proof of Theorem 2.1. The 2nd equality of (2.1) follows from $(2\mu)_k = 2^k (\mu)_{\lceil k/2 \rceil} (\mu + \frac{1}{2})_{\lfloor k/2 \rfloor}$, and the 3rd equality is easy. For the 1st equality, it is enough to prove when $r' = r''$ and $f(x_{12}) = \Delta_{\mathbf{k}}(x_{12})$, as explained before. We prove this by induction on $r' = r/2$. First, when $\mathbf{k} = (0, \dots, 0)$ (“ $r' = 0$ case”), this is clear. Next we assume the theorem for $r' - 1$, and prove it for r' . We write $\underline{k}_{r'} := \underbrace{(k_{r'}, \dots, k_{r'})}_{r'}$. Then by Proposition 4.2 we have

$$\begin{aligned} & \left\langle \Delta_{\mathbf{k}}(x_{12}), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_{11}=0, z_{22}=0} = \left\langle \det(x_{12})^{k_{r'}} \Delta_{\mathbf{k}-\underline{k}_{r'}}(x_{12}), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_{11}=0, z_{22}=0} \\ &= \frac{1}{\prod_{i=1}^{r'} (\lambda - \frac{i-1}{2})_{k_{r'}}} \det' \begin{pmatrix} 0 & z_{12} \\ t_{z_{12}} & 0 \end{pmatrix}^{-\lambda + \frac{r+1}{2}} \det \left(\frac{1}{2} \frac{\partial}{\partial z_{12}} \right)^{k_{r'}} \det' \begin{pmatrix} 0 & z_{12} \\ t_{z_{12}} & 0 \end{pmatrix}^{\lambda + k_{r'} - \frac{r+1}{2}} \\ & \quad \times \left\langle \Delta_{\mathbf{k}-\underline{k}_{r'}}(x_{12}), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda + k_{r'}, x} \Big|_{z_{11}=0, z_{22}=0} \\ &= \frac{C(\lambda + k_{r'}, \mathbf{k} - \underline{k}_{r'})}{\prod_{i=1}^{2r'} (\lambda - \frac{i-1}{2})_{k_{r'}}} \det(z_{12})^{2(-\lambda + r' + \frac{1}{2})} \det \left(\frac{1}{2} \frac{\partial}{\partial z_{12}} \right)^{k_{r'}} \det(z_{12})^{2(\lambda + k_{r'} - r' - \frac{1}{2})} \Delta_{\mathbf{k}-\underline{k}_{r'}}(z_{12}) \\ &= \frac{C(\lambda + k_{r'}, \mathbf{k} - \underline{k}_{r'})}{2^{k_{r'} r'} \prod_{i=1}^{2r'} (\lambda - \frac{i-1}{2})_{k_{r'}}} \prod_{i=1}^{r'} (2\lambda + k_i - i - r')_{k_{r'}} \det(z_{12})^{k_{r'}} \Delta_{\mathbf{k}-\underline{k}_{r'}}(z_{12}) \\ &= \frac{1}{2^{k_{r'} r'} \prod_{i=1}^{2r'} (\lambda - \frac{i-1}{2})_{k_{r'}}} \frac{2^{|\mathbf{k}| - k_{r'} r'} \prod_{1 \leq i < j \leq r'-1} (2(\lambda + k_{r'}) - (i + j))_{(k_i - k_{r'}) + (k_j - k_{r'})}}{\prod_{1 \leq i < j \leq r'} (2(\lambda + k_{r'}) - (i + j - 1))_{(k_i - k_{r'}) + (k_j - k_{r'})}} \\ & \quad \times \frac{\prod_{i=1}^{r'-1} (\lambda + k_{r'} - i)_{k_i - k_{r'}}}{\prod_{i=1}^{r'-1} (\lambda + k_{r'} - (i - 1))_{k_i - k_{r'}}} \prod_{i=1}^{r'} (2\lambda + k_i - i - r')_{k_{r'}} \Delta_{\mathbf{k}}(z_{12}) \\ &= \frac{2^{|\mathbf{k}| - 2k_{r'} r'}}{\prod_{i=1}^{2r'} (\lambda - \frac{i-1}{2})_{k_{r'}}} \frac{\prod_{1 \leq i < j \leq r'-1} (2\lambda - (i + j))_{k_i + k_j}}{\prod_{1 \leq i < j \leq r'} (2\lambda - (i + j - 1))_{k_i + k_j}} \frac{\prod_{1 \leq i < j \leq r'} (2\lambda - (i + j - 1))_{2k_{r'}}}{\prod_{1 \leq i < j \leq r'-1} (2\lambda - (i + j))_{2k_{r'}}} \\ & \quad \times \frac{\prod_{i=1}^{r'} (\lambda - i)_{k_i}}{\prod_{i=1}^{r'} (\lambda - (i - 1))_{k_i}} \frac{\prod_{i=1}^{r'} (\lambda - (i - 1))_{k_{r'}}}{\prod_{i=1}^{r'} (\lambda - i)_{k_{r'}}} \frac{\prod_{i=1}^{r'} (2\lambda - (i + r'))_{k_i + k_{r'}}}{\prod_{i=1}^{r'} (2\lambda - (i + (r' + 1) - 1))_{k_i}} \Delta_{\mathbf{k}}(z_{12}) \\ &= \frac{2^{|\mathbf{k}| - 2k_{r'} r'}}{\prod_{i=1}^{2r'} (\lambda - \frac{i-1}{2})_{k_{r'}}} \frac{\prod_{1 \leq i < j \leq r'} (2\lambda - (i + j))_{k_i + k_j}}{\prod_{1 \leq i < j \leq r'+1} (2\lambda - (i + j - 1))_{k_i + k_j}} \frac{\prod_{1 \leq i < j \leq r'-1} (2\lambda - (i + j))_{2k_{r'}}}{\prod_{1 \leq i < j \leq r'-1} (2\lambda - (i + j))_{2k_{r'}}} \\ & \quad \times \frac{\prod_{i=1}^{r'} (\lambda - i)_{k_i}}{\prod_{i=1}^{r'} (\lambda - (i - 1))_{k_i}} \frac{\prod_{i=1}^{r'} (\lambda - (i - 1))_{k_{r'}}}{\prod_{i=1}^{r'} (\lambda - i)_{k_{r'}}} (2\lambda - 2r')_{2k_{r'}} \Delta_{\mathbf{k}}(z_{12}) \\ &= \frac{2^{|\mathbf{k}|} \prod_{1 \leq i < j \leq r'} (2\lambda - (i + j))_{k_i + k_j}}{\prod_{1 \leq i < j \leq r'+1} (2\lambda - (i + j - 1))_{k_i + k_j}} \frac{\prod_{i=1}^{r'} (\lambda - i)_{k_i}}{\prod_{i=1}^{r'} (\lambda - (i - 1))_{k_i}} \\ & \quad \times \frac{2^{-2k_{r'} r'} \prod_{i=1}^{r'} (2\lambda - 2i)_{2k_{r'}}}{\prod_{i=1}^{r'} (\lambda - (i - 1))_{k_{r'}} \prod_{i=1}^{r'} (\lambda - (i - \frac{1}{2}))_{k_{r'}}} \frac{\prod_{i=1}^{r'} (\lambda - (i - 1))_{k_{r'}}}{\prod_{i=1}^{r'} (\lambda - i)_{k_{r'}}} \Delta_{\mathbf{k}}(z_{12}) \\ &= C(\lambda, \mathbf{k}) \Delta_{\mathbf{k}}(z_{12}), \end{aligned}$$

where we have used Lemma 4.1 and the induction hypothesis at the 3rd equality, and Lemma 4.3 at the 4th equality. Hence the theorem holds for all r' . \square

Now the proof of Proposition 4.2 is remaining. To prove this, for $\mathfrak{p}^+ = \text{Sym}(r, \mathbb{C})$ with $r = 2r'$, let $\mathfrak{n}^+ \subset \mathfrak{p}^+$ be the real form and $\Omega \subset \mathfrak{n}^+$ be the open cone given by

$$\begin{aligned}\mathfrak{n}^+ &:= \mathfrak{p}^+ \cap \text{Herm}(r, \mathbb{C}) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (\simeq \text{Sym}(r, \mathbb{R})), \\ \Omega &:= \mathfrak{p}^+ \cap \text{Herm}_+(r, \mathbb{C}) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad (\simeq \text{Sym}_+(r, \mathbb{R})),\end{aligned}$$

where $\text{Herm}_+(r, \mathbb{C})$ is the set of $r \times r$ positive definite Hermitian matrices. Also, for $\lambda \in \mathbb{C}$ let $\Gamma_r(\lambda) := (2\pi)^{r(r-1)/4} \prod_{i=1}^r \Gamma(\lambda - \frac{i-1}{2})$, and let $n := \dim \mathfrak{p}^+ = r(r+1)/2$. Then the following holds.

Lemma 4.4. *For $\text{Re } \lambda > r$, $f \in \mathcal{P}(\mathfrak{p}^+)$, $z, a \in \Omega$, we have*

$$\left\langle f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} = \det'(z)^{-\lambda + \frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\text{tr}(zw)} f(w^{-1}) \det'(w)^{-\lambda} dw.$$

Proof. Let $\mathfrak{m} \in \mathbb{Z}_{++}^r$. Then for $f(x) \in \mathcal{P}_{\mathfrak{m}}(\mathfrak{p}^+)$, by Theorem 3.1 we have

$$\left\langle f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} = \frac{1}{\prod_{i=1}^r (\lambda - \frac{i-1}{2})_{m_i}} f(z),$$

and by the inverse Laplace transform (Gindikin, see [5, Lemma XI.2.3, Section IX.3]), we have

$$\det'(z)^{-\lambda + \frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\text{tr}(zw)} f(w^{-1}) \det'(w)^{-\lambda} dw = \frac{1}{\prod_{i=1}^r (\lambda - \frac{i-1}{2})_{m_i}} f(z).$$

Hence the both sides coincide. Since this holds for every $\mathfrak{m} \in \mathbb{Z}_{++}^r$, the both sides coincide for all $f(x) \in \mathcal{P}_{\mathfrak{m}}(\mathfrak{p}^+)$. \square

Proof of Proposition 4.2. First, let $\text{Proj}_{12}: \mathfrak{p}^+ \rightarrow \mathfrak{p}_{12}^+$ be the orthogonal projection. Then for $w = \begin{pmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{pmatrix} \in \mathfrak{p}^+$, we have

$$\det(\text{Proj}_{12}(w^{-1})) = \det(({}^t w_{12} - w_{22} w_{12}^{-1} w_{11})^{-1}) = \det'(w)^{-1} \det(w_{12}).$$

Now let $z = \begin{pmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{pmatrix} \in \Omega \subset \mathfrak{p}^+$ and $f(x) \in \mathcal{P}(\mathfrak{p}^+)$. Then by using lemma 4.4 twice we have

$$\begin{aligned}\left\langle \det(x_{12})^k f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} &= \left\langle \det(\text{Proj}_{12}(x))^k f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda, x} \\ &= \det'(z)^{-\lambda + \frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\text{tr}(zw)} \det(\text{Proj}_{12}(w^{-1}))^k f(w^{-1}) \det'(w)^{-\lambda} dw \\ &= \det'(z)^{-\lambda + \frac{r+1}{2}} \frac{\Gamma_r(\lambda)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\text{tr}(zw)} \det(w_{12})^k f(w^{-1}) \det'(w)^{-\lambda-k} dw \\ &= \frac{\det'(z)^{-\lambda + \frac{r+1}{2}}}{\prod_{i=1}^r (\lambda - \frac{i-1}{2})_k} \det\left(\frac{1}{2} \frac{\partial}{\partial z_{12}}\right)^k \frac{\Gamma_r(\lambda+k)}{(2\pi\sqrt{-1})^n} \int_{a+\sqrt{-1}\mathfrak{n}^+} e^{\text{tr}(zw)} f(w^{-1}) \det'(w)^{-\lambda-k} dw \\ &= \frac{\det'(z)^{-\lambda + \frac{r+1}{2}}}{\prod_{i=1}^r (\lambda - \frac{i-1}{2})_k} \det\left(\frac{1}{2} \frac{\partial}{\partial z_{12}}\right)^k \det'(z)^{\lambda+k - \frac{r+1}{2}} \left\langle f(x), e^{\text{tr}(x\bar{z})} \right\rangle_{\lambda+k, x}.\end{aligned}$$

Since both sides are single-valued holomorphic with respect to $z \in \mathfrak{p}^+$, both sides coincide for all $z \in \mathfrak{p}^+$. \square

5 Results for other symmetric pairs of holomorphic type

In this section we state the theorems in the preprint [22] on top terms and poles of weighted Bergman inner products $\langle \cdot, \cdot \rangle_\lambda$ on bounded symmetric domains $D \simeq G/K$ for other symmetric pairs $(G, (G^\sigma)_0)$ of holomorphic type. Let G be a connected simple Hermitian Lie group with a Cartan involution θ , that is, the maximal compact subgroup $K = G^\theta$ has a 1-dimensional center $Z(K)$. Let σ be an involution of G . Without loss of generality we may assume σ commutes with θ . Then $(G, (G^\sigma)_0)$ is called a symmetric pair of *holomorphic type* if $Z(K) \subset G^\sigma$ (see [13, Section 3.4]). Then the complexified Lie algebras $\mathfrak{g}^\mathbb{C}$, $(\mathfrak{g}^\mathbb{C})^\sigma$, $(\mathfrak{g}^\mathbb{C})^{\sigma\theta}$ are decomposed into the $Ad(Z(K))$ -eigenspaces as

$$\begin{aligned} \mathfrak{g}^\mathbb{C} &= \mathfrak{p}^+ \oplus \mathfrak{k}^\mathbb{C} \oplus \mathfrak{p}^-, \\ \cup &\quad \cup \quad \cup \quad \cup \\ (\mathfrak{g}^\mathbb{C})^\sigma &= (\mathfrak{p}^+)^\sigma \oplus (\mathfrak{k}^\mathbb{C})^\sigma \oplus (\mathfrak{p}^-)^\sigma, \\ (\mathfrak{g}^\mathbb{C})^{\sigma\theta} &= (\mathfrak{p}^+)^{-\sigma} \oplus (\mathfrak{k}^\mathbb{C})^\sigma \oplus (\mathfrak{p}^-)^{-\sigma}. \end{aligned}$$

Let $(\mathfrak{p}^+)^\sigma =: \mathfrak{p}_1^+$, $(\mathfrak{p}^+)^{-\sigma} =: \mathfrak{p}_2^+$ so that $\mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$, and write $x = (x_1, x_2) \in \mathfrak{p}^+ = \mathfrak{p}_1^+ \oplus \mathfrak{p}_2^+$. In the following, for simplicity we assume that both \mathfrak{g} and the non-compact ideals of $\mathfrak{g}^{\sigma\theta}$ are of tube type, that is, there exists $e \in \mathfrak{p}_2^+ \subset \mathfrak{p}^+$ such that $ad([e, \bar{e}])|_{\mathfrak{p}^+} = 2I_{\mathfrak{p}^+}$ holds, where $x \mapsto \bar{x}$ is the complex conjugate with respect to the real form $\mathfrak{g} \subset \mathfrak{g}^\mathbb{C}$, although this assumption is not essential. Then both \mathfrak{p}^+ and \mathfrak{p}_2^+ have Jordan algebra structures with the unit element e , and we have $\text{rank } \mathfrak{p}^+ = \text{rank}_\mathbb{R} G =: r$, $\text{rank } \mathfrak{p}_2^+ = \text{rank}_\mathbb{R} (G^\sigma)_0 =: r_2$. Let $n := \dim \mathfrak{p}^+$, and when $r \geq 2$ let $d := \frac{2(n-r)}{r(r-1)}$. Also, if \mathfrak{p}_2^+ is simple, then we define d_2 similarly for \mathfrak{p}_2^+ . If $r_2 = 1$, then we cannot define d_2 by this way, and we set $d_2 := 2d$. Then one of the following holds.

- (1) $\mathfrak{p}_2^+ = \mathfrak{p}^{+'} \oplus \mathfrak{p}^{+''}$ and $r = \text{rank } \mathfrak{p}^{+'} + \text{rank } \mathfrak{p}^{+''} =: r' + r''$,
- (2) \mathfrak{p}_2^+ is simple, $r = 2r_2$ and $d = d_2/2$,
- (3) \mathfrak{p}_2^+ is simple, $r = r_2$ and $d = 2d_2$,
- (4) \mathfrak{p}_2^+ is simple, $r = r_2 = 2$ and $d > d_2$.

First we consider Case (1). Suppose $(G, (G^\sigma)_0, (G^{\sigma\theta})_0)$ is one of the following.

$$\begin{aligned} & (SO_0(2, d+2), \quad SO_0(2, d) \times SO(2), \quad SO_0(2, 2) \times SO(d)) \quad (d = d), \\ & (Sp(r, \mathbb{R}), \quad U(r', r''), \quad Sp(r', \mathbb{R}) \times Sp(r'', \mathbb{R})) \quad (d = 1), \\ & (U(r, r), \quad U(r', r'') \times U(r'', r'), \quad U(r', r') \times U(r'', r'')) \quad (d = 2), \\ & (SO^*(4r), \quad U(2r', 2r''), \quad SO^*(4r') \times SO^*(4r'')) \quad (d = 4), \\ & (E_{7(-25)}, \quad U(1) \times E_{6(-14)}, \quad SL(2, \mathbb{R}) \times Spin_0(2, 10)) \quad (d = 8). \end{aligned}$$

Let $(r, r', r'') = (2, 1, 1)$ for the 1st case, $(r, r', r'') = (3, 1, 2)$ for the 5th case. Then we have the following.

Theorem 5.1. *Let $\mathbf{k} \in \mathbb{Z}_{++}^{r'}$, $\mathbf{l} \in \mathbb{Z}_{++}^{r''}$ and put $k_{r'+1} = l_{r''+1} := 0$. Let $f(x_2) \in \mathcal{P}_{(\mathbf{k}, \mathbf{l})}(\mathfrak{p}_2^+)$.*

- (1) *For $\text{Re } \lambda > \frac{2n}{r} - 1$, we have*

$$\begin{aligned} \left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_1=0} &= \frac{\prod_{i=1}^{r'} \prod_{j=1}^{r''} (\lambda - \frac{d}{2}(i+j-1))_{k_i+l_j}}{\prod_{i=1}^{r'+1} \prod_{j=1}^{r''+1} (\lambda - \frac{d}{2}(i+j-2))_{k_i+l_j}} f(z_2) \\ &= \frac{\prod_{a=2}^r \prod_{i=\max\{1, a-r'\}}^{\min\{a-1, r'\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a-i}}}{\prod_{a=1}^r \prod_{i=\max\{1, a-r''\}}^{\min\{a, r'+1\}} (\lambda - \frac{d}{2}(a-1))_{k_i+l_{a+1-i}}} f(z_2). \end{aligned}$$

(2) The following is holomorphically continued for all $\lambda \in \mathbb{C}$.

$$\prod_{a=1}^r \left(\lambda - \frac{d}{2}(a-1) \right)_{\min\{k_i+l_j \mid 1 \leq i \leq r'+1, 1 \leq j \leq r''+1, i+j=a+1\}} \left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x}.$$

Next we consider Case (2). Suppose $(G, (G^\sigma)_0, (G^{\sigma\theta})_0)$ is one of the following.

$$\begin{aligned} & \left(SO_0(2, n), \quad SO_0(2, n-1), \quad SO_0(2, 1) \times SO(n-1) \right) \quad (d = n-2), \\ & \left(Sp(2r_2, \mathbb{R}), \quad Sp(r_2, \mathbb{R}) \times Sp(r_2, \mathbb{R}), \quad U(r_2, r_2) \right) \quad (d = 1), \\ & \left(SU(2r_2, 2r_2), \quad Sp(2r_2, \mathbb{R}), \quad SO^*(4r_2) \right) \quad (d = 2). \end{aligned}$$

Let $r_2 = 1$ for the 1st case. Then we have the following.

Theorem 5.2. Let $\mathbf{k} \in \mathbb{Z}_{++}^{r_2}$ and put $k_{r_2+1} := 0$. Let $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$.

(1) For $\operatorname{Re} \lambda > \frac{2n}{r} - 1$, we have

$$\begin{aligned} & \left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_1=0} \\ &= \frac{\prod_{1 \leq i < j \leq r_2} \left(\lambda - \frac{d}{2}(i+j-1) \right)_{\lfloor \frac{k_i+k_j}{2} \rfloor} \prod_{1 \leq i \leq j \leq r_2} \left(\lambda - \frac{1}{2} - \frac{d}{2}(i+j-1) \right)_{\lceil \frac{k_i+k_j}{2} \rceil}}{\prod_{1 \leq i \leq j \leq r_2+1} \left(\lambda - \frac{d}{2}(i+j-2) \right)_{\lfloor \frac{k_i+k_j}{2} \rfloor} \prod_{1 \leq i < j \leq r_2+1} \left(\lambda - \frac{1}{2} - \frac{d}{2}(i+j-2) \right)_{\lceil \frac{k_i+k_j}{2} \rceil}} f(z_2) \\ &= \frac{\prod_{a=3}^{2r_2-1} \prod_{i=\max\{1, a-r_2\}}^{\lceil a/2 \rceil - 1} \left(\lambda - \frac{d}{2}(a-1) \right)_{\lfloor \frac{k_i+k_{a-i}}{2} \rfloor}}{\prod_{a=1}^{2r_2} \prod_{i=\max\{1, a-r_2\}}^{\lceil a/2 \rceil} \left(\lambda - \frac{d}{2}(a-1) \right)_{\lfloor \frac{k_i+k_{a+1-i}}{2} \rfloor}} \\ & \quad \times \frac{\prod_{a=2}^{2r_2} \prod_{i=\max\{1, a-r_2\}}^{\lceil a/2 \rceil} \left(\lambda - \frac{1}{2} - \frac{d}{2}(a-1) \right)_{\lceil \frac{k_i+k_{a-i}}{2} \rceil}}{\prod_{a=2}^{2r_2} \prod_{i=\max\{1, a-r_2\}}^{\lceil a/2 \rceil} \left(\lambda - \frac{1}{2} - \frac{d}{2}(a-1) \right)_{\lceil \frac{k_i+k_{a+1-i}}{2} \rceil}} f(z_2). \end{aligned}$$

(2) The following is holomorphically continued for all $\lambda \in \mathbb{C}$.

$$\prod_{a=1}^{2r_2} \left(\lambda - \frac{d}{2}(a-1) \right)_{\min\{\lfloor \frac{k_i+k_j}{2} \rfloor \mid 1 \leq i \leq j \leq r+1, i+j=a+1\}} \left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x}.$$

Next we consider Case (3). Suppose $(G, (G^\sigma)_0, (G^{\sigma\theta})_0)$ is one of the following.

$$\begin{aligned} & \left(SO^*(4r), \quad SO^*(2r) \times SO^*(2r), \quad U(r, r) \right) \quad (d = 4), \\ & \left(SU(r, r), \quad SO^*(2r), \quad Sp(r, \mathbb{R}) \right) \quad (d = 2), \\ & \left(E_{7(-25)}, \quad SU(2, 6), \quad SO^*(12) \right) \quad (d = 8). \end{aligned}$$

Let $r = 3$ for the 3rd case. Then we have the following.

Theorem 5.3. Let $\mathbf{k} \in \mathbb{Z}_{++}^r$ and put $k_{r+1} := 0$. Let $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$.

(1) For $\operatorname{Re} \lambda > \frac{2n}{r} - 1$, we have

$$\begin{aligned} & \left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_1=0} = \frac{\prod_{1 \leq i < j \leq r} \left(\lambda - \frac{d}{4}(i+j-2) \right)_{k_i+k_j}}{\prod_{1 \leq i < j \leq r+1} \left(\lambda - \frac{d}{4}(i+j-3) \right)_{k_i+k_j}} f(z_2) \\ &= \frac{\prod_{a=2}^{2r-2} \prod_{i=\max\{1, a+1-r\}}^{\lceil a/2 \rceil} \left(\lambda - \frac{d}{4}(a-1) \right)_{k_i+k_{a+1-i}}}{\prod_{a=1}^{2r-1} \prod_{i=\max\{1, a+1-r\}}^{\lceil a/2 \rceil} \left(\lambda - \frac{d}{4}(a-1) \right)_{k_i+k_{a+2-i}}} f(z_2). \end{aligned}$$

(2) The following is holomorphically continued for all $\lambda \in \mathbb{C}$.

$$\prod_{a=1}^r \left(\lambda - \frac{d}{2}(a-1) \right)_{\min\{k_i+k_j \mid 1 \leq i < j \leq r+1, i+j=2a+1\}} \left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x}.$$

Finally we consider Case (4). Suppose

$$(G, (G^\sigma)_0, (G^{\sigma\theta})_0) = (SO_0(2, n), SO_0(2, n') \times SO(n''), SO_0(2, n'') \times SO(n')),$$

with $n = n' + n''$, $n'' \geq 3$. Then we have the following.

Theorem 5.4. *Let $\mathbf{k} \in \mathbb{Z}_{++}^2$, and let $f(x_2) \in \mathcal{P}_{\mathbf{k}}(\mathfrak{p}_2^+)$.*

(1) For $\operatorname{Re} \lambda > n - 1$, we have

$$\left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x} \Big|_{z_1=0} = \frac{(\lambda + k_1 - \frac{n'}{2})_{k_2}}{(\lambda)_{k_1+k_2} (\lambda - \frac{n-2}{2})_{k_2}} f(z_2).$$

(2) The following is holomorphically continued for all $\lambda \in \mathbb{C}$.

$$(\lambda)_{k_1+k_2} \left(\lambda - \frac{n-2}{2} \right)_{k_2} \left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x}.$$

In fact, for this case we have

$$\left\langle f(x_2), e^{(x|\bar{z})} \right\rangle_{\lambda, x} = \frac{(\lambda + k_1 - \frac{n'}{2})_{k_2}}{(\lambda)_{k_1+k_2} (\lambda - \frac{n-2}{2})_{k_2}} {}_2F_1 \left(\begin{matrix} -k_2, -k_1 - \frac{n''}{2} + 1 \\ -\lambda - k_1 - k_2 + \frac{n'}{2} + 1 \end{matrix}; -\frac{q(z_1)}{q(z_2)} \right) f(z_2),$$

where $q(z_1)$, $q(z_2)$ are suitable quadratic forms on $\mathfrak{p}_1^+ \simeq \mathbb{C}^{n'}$, $\mathfrak{p}_2^+ \simeq \mathbb{C}^{n''}$. By using these results, we can determine the Parseval–Plancherel-type formulas for the decomposition of $\mathcal{H}_\lambda(D)|_{(\tilde{G}^\sigma)_0}$, and can determine the branching laws for the discrete Wallach sets. For more detail see the preprint [22].

References

- [1] S. Ben Saïd, *Espaces de Bergman pondérés et série discrète holomorphe de $\widetilde{U}(p, q)$* . J. Funct. Anal. **173** (2000), no. 1, 154–181.
- [2] S. Ben Saïd, *Weighted Bergman spaces on bounded symmetric domains*. Pacific J. Math. **206** (2002), no. 1, 39–68.
- [3] H. Cohen, *Sums involving the values at negative integers of L-functions of quadratic characters*. Math. Ann. **217** (1975), no. 3, 271–285.
- [4] J. Faraut and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*. J. Funct. Anal. **88** (1990), no. 1, 64–89.
- [5] J. Faraut and A. Korányi, *Analysis on symmetric cones*. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994.

- [6] R. Goodman and N.R. Wallach, *Symmetry, representations, and invariants*. Graduate Texts in Mathematics, 255. Springer, Dordrecht, 2009. xx+716 pp.
- [7] J. Hilgert and B. Krötz, *Weighted Bergman spaces associated with causal symmetric spaces*. Manuscripta Math. **99** (1999), no. 2, 151–180.
- [8] J. Hilgert and B. Krötz, *The Plancherel theorem for invariant Hilbert spaces*. Math. Z. **237** (2001), no. 1, 61–83.
- [9] R. Howe, E.C. Tan and J.F. Willenbring, *Stable branching rules for classical symmetric pairs*. Trans. Amer. Math. Soc. **357** (2005), no. 4, 1601–1626.
- [10] S. Hwang, Y. Liu and G. Zhang, *Hilbert spaces of tensor-valued holomorphic functions on the unit ball of \mathbb{C}^n* . Pacific J. Math. **214** (2004), no. 2, 303–322.
- [11] T. Ibukiyama, T. Kuzumaki and H. Ochiai, *Holonomic systems of Gegenbauer type polynomials of matrix arguments related with Siegel modular forms*. J. Math. Soc. Japan **64** (2012), no. 1, 273–316.
- [12] A. Juhl, *Families of conformally covariant differential operators, Q -curvature and holography*. Progress in Mathematics, 275. Birkhäuser Verlag, Basel, 2009.
- [13] T. Kobayashi, *Multiplicity-free theorems of the restrictions of unitary highest weight modules with respect to reductive symmetric pairs*. Representation theory and automorphic forms, 45–109, Progr. Math., 255, Birkhäuser Boston, Boston, MA, 2008.
- [14] T. Kobayashi, T. Kubo and M. Pevzner, *Conformal symmetry breaking operators for differential forms on spheres*. Lecture Notes in Math. 2170, Springer, Singapore, 2016, ix+192 pp.
- [15] T. Kobayashi, B. Ørsted, P. Somberg and V. Souček, *Branching laws for Verma modules and applications in parabolic geometry. I*. Adv. Math. **285** (2015), 1796–1852.
- [16] T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators: I. Genreal theory and F -method*. Selecta Math. (N.S.) **22** (2016), no. 2, 801–845.
- [17] T. Kobayashi and M. Pevzner, *Differential symmetry breaking operators: II. Rankin–Cohen Operators for Symmetric Pairs*. Selecta Math. (N.S.) **22** (2016), no. 2, 847–911.
- [18] T. Kobayashi and M. Pevzner, *Inversion of Rankin–Cohen operators via Holographic Transform*. Ann. Inst. Fourier (Grenoble) **70** (2020), no. 5, 2131–2190.
- [19] J. Möllers and Y. Oshima, *Discrete branching laws for minimal holomorphic representations*. J. Lie Theory **25** (2015), no. 4, 949–983.
- [20] R. Nakahama, *Norm computation and analytic continuation of vector valued holomorphic discrete series representations*. J. Lie Theory **26** (2016), no. 4, 927–990.
- [21] R. Nakahama, *Computation of weighted Bergman inner products on bounded symmetric domains and restriction to subgroups*. SIGMA Symmetry Integrability Geom. Methods Appl. **18** (2022), 033, 105 pages.
- [22] R. Nakahama, *Computation of weighted Bergman inner products on bounded symmetric domains and Parseval–Plancherel-type formulas under subgroups*. preprint (2022), arXiv:2207.11663.

- [23] B. Ørsted, *Composition series for analytic continuations of holomorphic discrete series representations of $SU(n, n)$* . Trans. Amer. Math. Soc. **260** (1980), no. 2, 563–573.
- [24] B. Ørsted and G. Zhang, *Reproducing kernels and composition series for spaces of vector-valued holomorphic functions on tube domains*. J. Funct. Anal. **124** (1994), no. 1, 181–204.
- [25] B. Ørsted and G. Zhang, *Reproducing kernels and composition series for spaces of vector-valued holomorphic functions*. Pacific J. Math. **171** (1995), no. 2, 493–510.
- [26] L. Peng and G. Zhang, *Tensor products of holomorphic representations and bilinear differential operators*. J. Funct. Anal. **210** (2004), no. 1, 171–192.
- [27] R.A. Rankin, *The construction of automorphic forms from the derivatives of a given form*. J. Indian Math. Soc. (N.S.) **20** (1956), 103–116.
- [28] H. Sekiguchi, *Branching rules of singular unitary representations with respect to symmetric pairs (A_{2n-1}, D_n)* . Internat. J. Math. **24** (2013), no. 4, 1350011, 25 pp.

NTT Institute for Fundamental Mathematics
 3-9-11, Midori-cho Musashino-shi, Tokyo, 180-8585, Japan
 Email: ryosuke.nakahama@ntt.com