THE CLOSURE PROPERTY OF THE FOKKER–PLANCK EQUATION, GAUSSIAN HYPERCONTRACTIVTY, AND LOGARITHMIC SOBOLEV INEQUALITIES

SHOHEI NAKAMURA

Abstract. An importance of functional inequalities can be usually seen by being applied to analysis of differential equations. In this report, we explain an idea reversing such understanding, namely applying properties of differential equations to analyze functional inequalities. This idea is motivated from the work on the theory of Brascamp–Lieb inequality due to Bennett–Carbery–Christ–Tao [5] and Carlen–Lieb–Loss [9]. More precisely, we report that one can improve the best constant of Nelson's hypercontractivity inequality and Gross's logarithmic Sobolev inequality via the regularizing property of the Fokker–Planck equation, which is the main result in the work with Bez and Tsuji [7].

1. Introduction

This report is based on the work with Bez and Tsuji [7].

1.1. **Closure property of the Fokker–Planck equation.** Once one finds out a functional which is monotone along inputs flow of a given evolution equation, one would obtain certain inequality which leads us to a deep understanding about the functional. Such a framework known as *flow monotonicity* or *semigroup interpolation* has been developed and employed over the years as a powerful tool for proving geometric and functional inequalities as stated in the abstract in Ledoux's survey paper [14]. We simply refer the textbook due to Bakry–Gentil–Ledoux [2] for the comprehensive treatment of this framework. Despite of such a fruitful consequences of the flow monotonicity framework, it is yet unclear, as far as we are aware, what is a theory sitting behind the flow monotonicity phenomena. Toward this investigation, the closure property of diffusion equations recently have been paid attention, see [1, 3, 4]. Let us demonstrate the idea of the closure property with the simplest

Date: March 30, 2022.
2010 *Mathematics Subject Classification.* 26D10, 47D07, 52A40 (primary); 35K05, 60J60 (secondary).
Key words and phrases. Hypercontractivity, logarithmic Sobolev inequality, Fokker–Planck equation.
This work was supported by Grant-in-Aid for JSPS Research Fellow no. 17J01766 and JSPS Kakenhi grant numbers 19K03546, 19H01796 and 21K13806. Author appreciates to RIMS and organizers of this conference.

example which yields the Cauchy–Schwarz inequality. Take an sufficiently regular non-negative functions $f_1, f_2 \in L^1(\mathbb{R}^n)$ and aim to prove

$$\int_{\mathbb{R}^n} f_1(x)^{\frac12} f_2(x)^{\frac12}\,dx \le \Big(\int f_1\Big)^{\frac12}\Big(\int f_2\Big)^{\frac12}. \tag{1.1}$$

Let u_1, u_2 be positive and suitably regular solutions to the heat equation on $\mathbb{R}^n$:

$$\partial_t u_j = \Delta u_j, \quad u_j(0,x) = f_j(x), \quad j = 1,2.$$

Then from a direct computation, one can see that $\widetilde{u} := u_1^{\frac12} u_2^{\frac12}$ also becomes a supersolution, namely $\widetilde{u}$ satisfies[1]

$$\partial_t \widetilde{u} \ge \Delta \widetilde{u}.$$

This property shows the monotonicity of the functional

$$\Lambda(t) := \int_{\mathbb{R}^n} u_1(t,x)^{\frac12} u_2(t,x)^{\frac12}\,dx = \int_{\mathbb{R}^n} \widetilde{u}(t,x)\,dx$$

in the sense that

$$\frac{d}{dt}\Lambda(t) = \int_{\mathbb{R}^n} \partial_t \widetilde{u}(t,x)\,dx \ge \int_{\mathbb{R}^n} \Delta\widetilde{u}(t,x)\,dx = 0$$

thanks to the integration by parts. Once this monotonicity is ensured, one can easily prove the Cauchy–Schwarz inequality as follows. In fact, the monotonicity shows that

$$\Lambda(0) \le \lim_{t\to\infty} \Lambda(t).$$

On the one hand, we know that

$$\Lambda(0) = \int_{\mathbb{R}^n} f_1(x)^{\frac12} f_2(x)^{\frac12}\,dx.$$

On the other hand, if we do the rescaling $x = \sqrt{t}y$ for each $t > 0$, then we obtain that

$$\Lambda(t) = \int_{\mathbb{R}^n} \big(\sqrt{t}u_1(t,\sqrt{t}y)\big)^{\frac12}\big(\sqrt{t}u_2(t,\sqrt{t}y)\big)^{\frac12}\,dy.$$

It is straightforward to see that $\lim_{t\to\infty}\sqrt{t}u_1(t,\sqrt{t}y) = \big(\int f_j\big)\gamma_1(y)$, where $\gamma_1(y) := (2\pi)^{\frac{n}{2}} e^{-\frac12|y|^2}$, from the explicit form of the heat-solution and hence we obtain that

$$\lim_{t\to\infty}\Lambda(t) = \Big(\int f_1\Big)\Big(\int f_2\Big)\int_{\mathbb{R}^n} \gamma_1(y)^{\frac12}\gamma_1(y)^{\frac12}\,dy = \Big(\int f_1\Big)\Big(\int f_2\Big).$$

Putting altogether, we conclude (1.1).

It is worth to remark that this idea is robust enough to generate the monotonicity of functionals which relate to the Brascamp–Lieb inequalities, see [4] and functionals which relate to the sharp Young's convolution inequalities, see [3]. More recently in [1], authors observed the closure property of the Gaussian heat equation or backward Kolmogorov equation. In order to state their result, let us introduce notations. For a diffusion parameter $\beta > 0$, let $\mathcal{L}_\beta$ be a generator of the Ornstein–Uhlenbeck semigroup with a diffusion parameter β given by $\mathcal{L}_\beta\phi(x) := \beta\Delta\phi(x) -$

---

[1] Indeed, one can show slightly stronger statement

$$\partial_t u_j \ge \Delta u_g \quad \Rightarrow \quad \partial_t \widetilde{u} \ge \Delta\widetilde{u}.$$

This statement is more appropriate for the name of *closure* property.

$x\cdot\nabla\phi(x)$ and $P_s=e^{s\mathcal{L}_1}$ be the Ornstein–Uhlenbeck semigroup according to $\mathcal{L}_1$. Explicitly, P_s is an integral operator defined by

$$P_s\phi(x):=\int_{\mathbb{R}^n}\phi(e^{-s}x+\sqrt{1-e^{-2s}}y)\,d\gamma_1(y),\quad x\in\mathbb{R}^n,$$

for a test function ϕ. For a fixed time parameter $s>0$, let $1<p<q<\infty$ satisfy

$$\frac{q-1}{p-1}=e^{2s}. \tag{1.2}$$

Then it is observed in [1] that if $u=u_t(x)$ is a positive and sufficiently regular supersolution on $\mathbb{R}^n$:

$$\partial_t u\ge\mathcal{L}_1u,\quad (t,x)\in(0,\infty)\times\mathbb{R}^n,$$

then

$$\partial_t\widetilde{u}\ge\mathcal{L}_1\widetilde{u},\quad (t,x)\in(0,\infty)\times\mathbb{R}^n,\quad \widetilde{u}(t,x)^{\frac{1}{q}}:=P_s\big[u_t^{\frac{1}{p}}\big](x). \tag{1.3}$$

Again this closure property ensures $\frac{d}{dt}\Lambda(t)\ge0$, where the functional is given by

$$\Lambda(t):=\int_{\mathbb{R}^n}P_s\big[u_t^{\frac{1}{p}}\big](x)^q\,d\gamma_1(x),$$

and $d\gamma_a(x):=(2\pi a)^{-\frac{n}{2}}e^{-\frac{|x|^2}{2a}}$ is a normalised Gaussian with variance $a>0$. From the Ergodicity $\lim_{t\to\infty}u_t=\int u_0\,d\gamma_1$, one can in particular derives Nelson's celebrated hypercontractivity inequality

$$\big\|P_s\big[u_0^{\frac{1}{p}}\big]\big\|_{L^q(\gamma_1)}\le\Big(\int_{\mathbb{R}^n}u_0\,d\gamma_1\Big)^{\frac{1}{p}} \tag{1.4}$$

for all sufficiently regular initial data $u_0:\mathbb{R}^n\to(0,\infty)$. Note that authors in [1] worked on the framework of the Markov semigoup which is more general setting than the Ornstein–Uhlenbeck semigroup.

Our first aim is to obtain new closure property of the Fokker–Planck equation with a diffusion parameter $\beta\ge1$:

$$\partial_t v=\mathcal{L}_\beta^*v=\beta\Delta v+x\cdot\nabla v+v,\quad (t,x)\in(0,\infty)\times\mathbb{R}^n, \tag{1.5}$$

where $\mathcal{L}_\beta^*$ is a dual of $\mathcal{L}_\beta$ with respect to $L^2(dx)$. It is worth to mention that the Fokker–Planck equation (1.5) with $\beta=1$ is closely related to the Gaussian heat equation

$$\partial_t u=\mathcal{L}_1u$$

via a transformation $u=\frac{v}{\gamma_1}$. Moreover, from this relation, the closure property of the Gaussian heat equation (1.3) yields that if $v=v_t$ is sufficiently regular and positive supersolution of the Fokker–Planck equation:

$$\partial_t v\ge\mathcal{L}_1^*v,$$

then it holds that

$$\partial_t\widetilde{v}\ge\mathcal{L}_1^*\widetilde{v},\quad (t,x)\in(0,\infty)\times\mathbb{R}^n,\quad \Big(\frac{\widetilde{v}(t,x)}{\gamma_1}\Big)^{\frac{1}{q}}:=P_s\Big[\Big(\frac{v_t}{\gamma_1}\Big)^{\frac{1}{p}}\Big](x),$$

regardless of the initial condition on v_t. Hence the closure property of the Fokker–Planck equation is a direct consequence from the one of the Gaussian heat equation due to [1] when $\beta=1$. The first result in this report is about the validity of this closure property when $\beta>1$. Usually modifying the diffusion parameter may be regarded as a minor modification. However, regarding the closure property, this is not the case. For example, it is clear that the above argument reducing the matters to the closure property of the Gaussian heat equation does not work well when $\beta > 1$. This is because $u := \frac{v}{\gamma_1}$ is not always the solution to the Guassian heat equation even if v is a solution to the Fokker–Planck equation (1.5)[2] unless $\beta = 1$. More importantly, we find that the closure property of the Fokker–Planck equation does not always hold when $\beta > 1$ and an appropriate initial condition should be involved. The condition we introduce is related to the semi-log-convexity, namely

$$\nabla^2 \log v_0 \geq -\frac{1}{\beta}. \tag{1.6}$$

Typical examples satisfying (1.6) are centered gaussians whose variance is greater than β, namely γ_a with $a \geq \beta$. More generally, if v_0 is give by

$$v_0(x) = \frac{1}{(2\pi\beta)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{1}{2\beta}|x-y|^2}\, d\mu(y), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n \tag{1.7}$$

for some positive finite measure $d\mu$, then the assumption (1.6) can be ensured as the log-convexity is preserved under the superposition. We first exhibit the closure property of the Fokker–Planck equation for $\beta > 1$ under the semi-log-convexity assumption (1.6).

Theorem 1.1 (Theorem 3.5 in [7])**.** *Let $s > 0$, $\beta > 1$, and $1 < p < q < \infty$ satisfy $\frac{q-1}{p-1} = e^{2s}$. Suppose that the twice differentiable $v : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$ is in $L^2(\gamma_\beta^{-1})$ and satisfies (1.6). Then*

$$\partial_t v_t \geq \mathcal{L}_\beta^* v_t \quad \Rightarrow \quad \partial_t \widetilde{v} \geq \mathcal{L}_{\beta_{s,p}}^* \widetilde{v}, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \tag{1.8}$$

where $\widetilde{v}_t$ is defined by

$$\big(\frac{\widetilde{v}(t,x)}{\gamma_1}\big)^{\frac{1}{q}} := P_s\big[\big(\frac{v_t}{\gamma_1}\big)^{\frac{1}{p}}\big](x),$$

and $\beta_{s,p} > 1$ is given by

$$\beta_{s,p} := 1 + (\beta - 1)\frac{q}{p} e^{-2s}. \tag{1.9}$$

We give few remarks.

(1) The $\beta_{s,p}$ introduced here is natural. In fact, when $v_t = \gamma_\beta$ which is the stationary solution to (1.5), one can see that

$$\widetilde{v}_t(x) := \gamma_1(x) P_s\big[\big(\frac{\gamma_\beta}{\gamma_1}\big)^{\frac{1}{p}}\big](x)^q = \beta^{\frac{qn}{2p'}} \beta_{s,p}^{-\frac{qn}{2q'}} \gamma_{\beta_{s,p}}(x).$$

Hence, we see that

$$v_t = \gamma_\beta \quad \Rightarrow \quad \partial_t \widetilde{v}_t = \mathcal{L}_{\beta_{s,p}}^* \widetilde{v}_t,$$

which strongly suggests the validity of (1.8) with $\beta_{s,p}$. Hence the moral of Theorem 1.1 is that if one has some property which can be established for gaussian, then it should hold true on more general setting. Such idea can

[2] One may wonder another option to transform by $u = \frac{v}{\gamma_\beta}$. However, this does not suit for our later purpose and moreover one needs to modify the semigroup P_s by $e^{s\mathcal{L}_\beta}$.

be found in several places over sciences. We here simply mention the work of Lieb [15].

(2) If one considers the shrunk gaussian γ_a for $a \leq \beta$, then the assumption (1.6) fails true. For such example, one can also see that the closure property (1.8) cannot be true if $\beta > 1$.

(3) It is remarkable that even if one combines Theorem 1.1 and the transformation $u = \frac{v}{\gamma_1}$, one cannot expect any closure property of Ornsten–Uhlenbeck semigroup unless $\beta = 1$. This distinguishes the situation of the Ornsten–Uhlenbeck flow and the Fokker–Planck flow and this is one of the reason of the preference of the Fokker–Planck equation.

As a consequence of Theorem 1.1, we next show that Nelson's hypercontractivity (1.4) can be improved in terms of the Fokker–Planck equation.

1.2. **Reguralised hypercontractivity and logarithmic Sobolev inequality.** Nelson's hypercontractivity inequality, which manifests the smoothing property of the Ornstein–Uhlenbeck semigroup P_s, states that the inequality (1.4) holds for all positive $u_0 \in L^1(d\gamma_1)$. It is worth to note that the equality in (1.4) is established for $u_0(x) = e^{a\cdot x+b}$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and these are the only case (among smooth initial data), see Ledoux's work [13]. Therefore, $u_0 = \frac{\gamma_a}{\gamma_1}$ does not attain the equality in (1.4) unless $a = 1$. In fact, one can see from the direct calculations that

$$\big\| P_s\big[\big(\frac{\gamma_\beta}{\gamma_1}\big)^{\frac{1}{p}}\big]\big\|_{L^q(\gamma_1)} = \beta^{\frac{n}{2p'}} \beta_{s,p}^{-\frac{n}{2q'}} < 1, \tag{1.10}$$

where $\beta_{s,p} > 1$ is given by (1.9). Our next result on the hypercontractivity claims that this simple observation works for more general initial data described by Fokker–Planck equation, or the assumption (1.6).

Theorem 1.2 (Theorem 1.4 in [7]). *Let $\beta > 1$, $s > 0$, and $1 < p < q < \infty$ satisfy $\frac{q-1}{p-1} = e^{2s}$. Then for any twice differentiable $v_0 \in L^2(\gamma_\beta^{-1})$ satisfying* (1.6)*, we have that*

$$\big\| P_s\big[\big(\frac{v_0}{\gamma_1}\big)^{\frac{1}{p}}\big]\big\|_{L^q(\gamma_1)} \leq \beta^{\frac{n}{2p'}} \beta_{s,p}^{-\frac{n}{2q'}} \big(\int_{\mathbb{R}^n} \frac{v_0}{\gamma_1}\, d\gamma_1\big)^{\frac{1}{p}}, \tag{1.11}$$

where $\beta_{s,p}$ is given in (1.9)*. Moreover the equality is established for $v_0 = \gamma_\beta$.*

In fact, Theorem 1.2 is an immediate consequence from our closure property Theorem 1.1 as follows. Let v_0 be an $L^1(dx)$-normalized function satisfying assumptions in Theorem 1.2 and v_t be a β-Fokker–Planck solution to (1.5) with initial data v_0. Then we see that $\widetilde{v}_t$ satisfies

$$\partial_t \widetilde{v}_t \geq \mathcal{L}^*_{\beta_{s,p}} \widetilde{v}_t$$

from Theorem 1.1. If we introduce the quantity $Q(t)$ by

$$Q(t) := \int_{\mathbb{R}^n} \widetilde{v}_t\, dx,$$

then the closure property ensures the monotonicity of $Q(t)$:

$$\frac{d}{dt} Q(t) = \int_{\mathbb{R}^n} \partial_t \widetilde{v}_t\, dx \geq \int_{\mathbb{R}^n} \mathcal{L}^*_{\beta_{s,p}} \widetilde{v}_t\, dx = 0$$

thanks to the integration by parts. Note that the definition of $\widetilde{v}_t$ reveals that

$$Q(0)=\int_{\mathbb{R}^n} P_s\big[\big(\frac{v_0}{\gamma_1}\big)^{\frac{1}{p}}\big]^q\,d\gamma_1=\big\|P_s\big[\big(\frac{v_0}{\gamma_1}\big)^{\frac{1}{p}}\big]\big\|_{L^q(\gamma_1)}^q$$

which is the left-hand side of (1.11). On the other hand, β-Fokker–Plank solution converges to its stationary solution γ_β as $t\to\infty$: $\lim_{t\to\infty} v_t=\gamma_\beta$ and hence we obtain that

$$\lim_{t\to\infty} Q(t)=\big\|P_s\big[\big(\frac{\gamma_\beta}{\gamma_1}\big)^{\frac{1}{p}}\big]\big\|_{L^q(\gamma_1)}^q.$$

Putting altogether, as well as (1.10), we conclude

$$\big\|P_s\big[\big(\frac{v_0}{\gamma_1}\big)^{\frac{1}{p}}\big]\big\|_{L^q(\gamma_1)}^q=Q(0)\le\lim_{t\to\infty}Q(t)=\beta^{\frac{qn}{2p'}}\beta_{s,p}^{-\frac{qn}{2q'}}.$$

Since we assumed $\int \frac{v_0}{\gamma_1}\,d\gamma_1=1$, (1.11) follows.

As a special case of Theorem 1.2, we may observe the phenomena that we mentioned in the abstract, namely the Fokker–Planck equation improves the hypercontractivity. In order to provide a precise statement, we introduce further notations. First we introduce a class of initial data *reguralised* by the Fokker–Planck equation. Considering general setting, for $\beta>0$ and initial data any non-negative finite measure $d\mu$, let v_* be the corresponding 2β-Fokker–Planck solution:

$$\partial_t v_*=\mathcal{L}_{2\beta}^* v_*,\quad (t,x)\in(0,\infty)\times\mathbb{R}^n,\quad v_*(0,x)=d\mu(x). \tag{1.12}$$

Then for fixed time $t_*=\frac{1}{2}\log 2>0$, we introduce our class

$$\mathrm{FP}(\beta):=\big\{v=v_*(t_*,\cdot):\ v_*\,\mathrm{is\,non-negative\,solution\,to\,(1.12)}\big\}$$

Here we chose somewhat artificial parameters 2β and $t_*=\frac{1}{2}\log 2$ but this should not be worried. These choice are taken in order to ensure the fact that γ_β, which is our base of investigation, is in $\mathrm{FP}(\beta)$. The moral is that $\mathrm{FP}(\beta)$ is a class of functions regularized for certain time by the Fokker–Planck equation. Let us provide basic facts about the class $\mathrm{FP}(\beta)$. First, it is easy to check the monotonicity $\mathrm{FP}(\beta_1)\subset\mathrm{FP}(\beta_2)$ for $\beta_1\le\beta_2$. Secondly, that $\frac{\gamma_a}{\gamma_1}$ is in $\mathrm{FP}(\beta)$ is equivalent to $a\ge\beta$. In particular, if $\beta>1$, then γ_1, which is the extremiser of the hypercontractivity inequality (1.4), is not in the class $\mathrm{FP}(\beta)$ and in some sense the difference between γ_1 and $\mathrm{FP}(\beta)$ are measured by $\beta-1$. Based on this observation, together with the reguralising property of the Fokker–Planck equation, one might expect an improvement of the hypercontractivity inequality by restricting input functions to the class $\mathrm{FP}(\beta)$. This is in fact true and follows from Theorem 1.2.

Theorem 1.3 (Theorem 1.2 in [7])**.** *Let $\beta>1$, $s>0$, and $1<p<q<\infty$ satisfy $\frac{q-1}{p-1}=e^{2s}$. Then* (1.11) *holds true for all $v_0\in\mathrm{FP}(\beta)$.*

Remark. Our reguralised class $\mathrm{FP}(\beta)$ is motivated from so-called *Type-G* functions introduced in the work on the reguralised Brascamp–Lieb inequality due to Bennett–Carbery–Christ–Tao [5], see also [6].

Among several applications of Nelson's hypercontractivity, perhapps one of the most important one is the equivalence to the Gaussian logarithmic Sobolev inequality (LSI for short) which is found by L. Gross [12]. The LSI states that

$$\mathrm{Ent}_{\gamma_1}(f)\le\frac{1}{2}\mathrm{I}_{\gamma_1}(f), \tag{1.13}$$

where the entropy $\mathrm{Ent}_{\gamma_1}(f)$ and Fisher information $\mathrm{I}_{\gamma_1}(f)$ are defined by

$$\begin{aligned}\mathrm{Ent}_{\gamma_1}(f) &:= \int_{\mathbb{R}^n} f\log f\,d\gamma_1 - \Big(\int_{\mathbb{R}^n} f\,d\gamma_1\Big)\log\Big(\int_{\mathbb{R}^n} f\,d\gamma_1\Big),\\ \mathrm{I}_{\gamma_1}(f) &:= \int_{\mathbb{R}^n} |\nabla \log f|^2 f\,d\gamma_1\end{aligned}\tag{1.14}$$

We refer to [10] for the history of the hypercontractivity, how it emerges from a motivation of the quantum field theory, and how important its link to LSI is. It is also figured out by Carlen [8] that the equality in (1.13) is attained if and only if $f(x) = e^{a\cdot x+b}$, $a \in \mathbb{R}^n, b \in \mathbb{R}$. Therefore, one may expect to reguralise or improve LSI by restricting input functions to $\mathrm{FP}(\beta)$ for $\beta > 1$. This is indeed possible thanks to Theorem 1.3.

Theorem 1.4 (Theorem 1.3, 1.4 in [7])**.** *Let $\beta > 1$. Then*

$$\mathrm{Ent}_{\gamma_1}\big(\frac{v_0}{\gamma_1}\big) \le \frac{1}{2}\mathrm{I}_{\gamma_1}\big(\frac{v_0}{\gamma_1}\big) - D_n(\beta), \quad D_n(\beta) := \frac{n}{2}(\log\beta - 1 + \frac{1}{\beta})\tag{1.15}$$

holds for all $v_0 \in L^2(\gamma_\beta^{-1})$ satisfying (1.6)*. In particular,* (1.15) *holds for all $v_0 \in$* $\mathrm{FP}(\beta)$*. Moreover, the equality is established for $v_0 = \gamma_\beta$.*

Remark. One can rewrite the inequality (1.15) in more symmetric and dimension free way as follows. Introduce relative entropy and relative Fisher information for probability distribution ρ which is absolutely continuous w.r.t. γ_1 by

$$\mathrm{H}(\rho|\gamma_1) := \mathrm{Ent}_{\gamma_1}\big(\frac{d\rho}{d\gamma_1}\big), \quad \mathrm{I}(\rho|\gamma_1) := \mathrm{I}_{\gamma_1}\big(\frac{d\rho}{d\gamma_1}\big).\tag{1.16}$$

Also, one can check that

$$D_n(\beta) = \frac{1}{2}\mathrm{I}_{\gamma_1}\big(\frac{\gamma_\beta}{\gamma_1}\big) - \mathrm{Ent}_{\gamma_1}\big(\frac{\gamma_\beta}{\gamma_1}\big) \ge 0.$$

With these two in minds, the inequality (1.15) can be stated as

$$\mathrm{H}(v_0|\gamma_1) - \mathrm{H}(\gamma_\beta|\gamma_1) \le \frac{1}{2}\big(\mathrm{I}(v_0|\gamma_1) - \mathrm{I}(\gamma_\beta|\gamma_1)\big)\tag{1.17}$$

for any v_0 satisfying the assumption in Theorem 1.4.

It is interesting to compare Theorem 1.4 to the result on the stability of LSI recently obtained by Eldan–Lehec–Shenfeld [11]. Let us recall their result (Theorem 3) by restricting special cases. For a probability measure ρ, denote its covariance matrix by

$$\mathrm{cov}\,(\rho) := \Big(\int_{\mathbb{R}^n} x_i x_j\,d\rho - \big(\int_{\mathbb{R}^n} x_i\,d\rho\big)\big(\int_{\mathbb{R}^n} x_j\,d\rho\big)\Big)_{1\le i,j\le n}.$$

Theorem 1.5 (Theorem 3 in [11])**.** *Let $\beta \le 1$. Then for any probability measure ρ on $\mathbb{R}^n$ such that* $\mathrm{cov}\,(\rho) \le \beta\mathrm{id}$*,*

$$\mathrm{H}(\rho|\gamma_1) - \mathrm{H}(\gamma_\beta|\gamma_1) \le \frac{1}{2}\big(\mathrm{I}(\rho|\gamma_1) - \mathrm{I}(\gamma_\beta|\gamma_1)\big).\tag{1.18}$$

On the other hand, (1.18) *is completely wrong if the condition* $\mathrm{cov}\,(\rho) \le \beta\mathrm{id}$ *is removed.*

It is immediate to find that their inequality (1.18) is same as our (1.17). On the other hand, the condition of input function is somehow complement. In fact, our assumption (1.6) or $v_0 \in \mathrm{FP}(\beta)$ is stronger statement than that the covariance of v_0 is large:

$$v_0 \in \mathrm{FP}(\beta) \quad \Rightarrow \quad \nabla^2 \log v_0 \le -\frac{1}{\beta} \quad \Rightarrow \quad \mathrm{cov}\,(v_0) \ge \beta \mathrm{id},$$

see Lemma 1.5 in [7] for the proof. This means $\mathrm{cov}\,(v_1) \ge \mathrm{id}_{\mathbb{R}^n}$ as we impose $\beta > 1$, and our improvement (1.15) or (1.17) is under the case where the covariance is large which is the complement situation of Theorem 1.5. Also Eldan–Lehec–Shenfeld provide a *counterexample* $\rho_k := (1-\frac{1}{k})\gamma_1 + \frac{1}{k}\gamma_1(\cdot - k^2)$, $k \in \mathbb{N}$, on $\mathbb{R}$, which satisfies

$$\mathrm{cov}\,(\rho_k) \to \infty, \quad \frac{1}{2}\mathrm{I}(\rho_k|\gamma_1) - \mathrm{H}(\rho_k|\gamma_1) \to 0 \quad \text{as} \quad k \to \infty.$$

From this example, one cannot expect any improvement of LSI under the large covariance assumption in general. However, our result tells us that one can still improve LSI if one makes covariance large in a nice way according to the Fokker–Planck flow. In fact, their counterexample ρ_k is excluded from the class $\mathrm{FP}(\beta)$.

2. Proof of Theorem 1.1 for gaussian input

In this section, we provide a proof of Theorem 1.1 for specific input $v_0 = \gamma_a$, $a > 0$ in order to explains an idea of the proof. For the complete proof for general input, we refer [7]. Note that for $v_0 = \gamma_a$, the assumption (1.6) is equivalent to $a \ge \beta$.

Proposition 2.1. *Let $s > 0$, $\beta > 1$, and $1 < p < q < \infty$ satisfy $\frac{q-1}{p-1} = e^{2s}$. If $v_0 = \gamma_a$, $a \ge \beta$, then (1.8) holds true.*

2.1. Preliminaries.

In order to prove this proposition, we give several formulae regarding to the Ornstein–Uhlenbeck flow.

Lemma 2.2. *Let $\alpha \in \mathbb{R}$ satisfy*

$$\frac{\alpha + 1 - e^{-2t}}{\alpha} > 0. \tag{2.1}$$

Then for

$$\alpha_t := \frac{\alpha + 1 - e^{-2t}}{e^{-2t}} \in \mathbb{R}$$

we have $\frac{\alpha_t}{\alpha} > 0$ and

$$P_t\big[e^{-\frac{1}{2\alpha}x^2}\big](x) = \big(\frac{\alpha}{\alpha_t e^{-2t}}\big)^{\frac{1}{2}} e^{-\frac{1}{2\alpha_t}x^2}.$$

Remark. Practically speaking, we will always consider the case $\alpha \in (-\infty, -1) \cup (0, \infty)$. In this case, the assumption (2.1) automatically holds and

$$\begin{cases} \alpha_t \in (0,\infty), & \text{if} \quad \alpha \in (0,\infty), \\ \alpha_t \in (-\infty,-1), & \text{if} \quad \alpha \in (-\infty,-1). \end{cases}$$

Proof. This is a consequence from the definition of P_s and gaussian integral. Hence we omit the details. □

Using Lemma 2.2, we may compute $P_s\big[xe^{-\frac{1}{2\beta}x^2}\big]$.

Lemma 2.3. *Suppose $\beta \in \mathbb{R}$ satisfy*

$$\frac{\beta + 1 - e^{-2s}}{\beta} > 0. \tag{2.2}$$

Then for

$$\beta_s := \frac{\beta + 1 - e^{-2s}}{e^{-2s}} \in \mathbb{R},$$

we have $\frac{\beta}{\beta_s} > 0$ and

$$P_s\big[xe^{-\frac{1}{2\beta}x^2}\big](x) = e^{-s}\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac{3}{2}} xe^{-\frac{1}{2\beta_s}x^2}.$$

Remark. As before, we will always consider the case $\beta \in (-\infty, -1) \cup (0, \infty)$. In this case, the assumption (2.2) automatically holds and we have

$$\begin{cases} \beta_s \in (0, \infty), & \text{if} \quad \beta \in (0, \infty), \\ \beta_s \in (-\infty, -1), & \text{if} \quad \beta \in (-\infty, -1). \end{cases}$$

Proof. We first note that, under the assumption (2.2), we know $xe^{-\frac{1}{2\beta}x^2} \in L^1(\gamma_1)$ and hence $P_s\big[xe^{-\frac{1}{2\beta}x^2}\big]$ is well-defined. From the definition of P_s and Claim 2.2, we have that

$$\begin{aligned}
&P_s\big[xe^{-\frac{1}{2\beta}x^2}\big](x) \\
=&\int \big(e^{-s}x + (1 - e^{-2s})^{\frac{1}{2}}y\big)e^{-\frac{1}{2\beta}|e^{-s}x + (1-e^{-2s})^{\frac{1}{2}}y|^2} \frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac{1}{2}}}\,dy \\
=&e^{-s}xP_s\big[e^{-\frac{1}{2\beta}x^2}\big](x) + (1 - e^{-2s})^{\frac{1}{2}} \int ye^{-\frac{1}{2\beta}|e^{-s}x + (1-e^{-2s})^{\frac{1}{2}}y|^2} \frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac{1}{2}}}\,dy \\
=&e^{-s}x\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac{1}{2}} e^{-\frac{1}{2\beta_s}x^2} + (1 - e^{-2s})^{\frac{1}{2}} \int ye^{-\frac{1}{2\beta}|e^{-s}x + (1-e^{-2s})^{\frac{1}{2}}y|^2} \frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac{1}{2}}}\,dy.
\end{aligned}$$

Regarding the second term, we do the integration by parts to see that

$$
\begin{aligned}
I(x) :=& \int y e^{-\frac{1}{2\beta}|e^{-s}x+(1-e^{-2s})^{\frac12}y|^2} \frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac12}}\, dy \\
=& \int e^{-\frac{1}{2\beta}|e^{-s}x+(1-e^{-2s})^{\frac12}y|^2} \partial_y\big(-\frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac12}}\big)\, dy \\
=& \int \partial_y\big(e^{-\frac{1}{2\beta}|e^{-s}x+(1-e^{-2s})^{\frac12}y|^2}\big) \frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac12}}\, dy \\
=& \int \Big[-\frac{1}{\beta}\big(e^{-s}x+(1-e^{-2s})^{\frac12}y\big)\times(1-e^{-2s})^{\frac12}\Big] \\
&\times e^{-\frac{1}{2\beta}|e^{-s}x+(1-e^{-2s})^{\frac12}y|^2} \frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac12}}\, dy \\
=& -\frac{1}{\beta}e^{-s}(1-e^{-2s})^{\frac12} x P_s\big[e^{-\frac{1}{2\beta}x^2}\big](x) \\
&- \frac{1-e^{-2s}}{\beta}\int y e^{-\frac{1}{2\beta}|e^{-s}x+(1-e^{-2s})^{\frac12}y|^2} \frac{e^{-\frac{y^2}{2}}}{(2\pi)^{\frac12}}\, dy \\
=& -\frac{1}{\beta}e^{-s}(1-e^{-2s})^{\frac12} x P_s\big[e^{-\frac{1}{2\beta}x^2}\big](x) - \frac{1-e^{-2s}}{\beta} I(x).
\end{aligned}
$$

From this,

$$
\big(1+\frac{1-e^{-2s}}{\beta}\big) I(x) = -\frac{1}{\beta}e^{-s}(1-e^{-2s})^{\frac12} x P_s\big[e^{-\frac{1}{2\beta}x^2}\big](x)
$$

and then applying Claim 2.2 again, we arrive at

$$
I(x) = -e^{-s}\frac{(1-e^{-2s})^{\frac12}}{\beta}\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32} x e^{-\frac{1}{2\beta_s}x^2}.
$$

Hence,

$$
\begin{aligned}
&P_s\big[x e^{-\frac{1}{2\beta}x^2}\big](x) \\
=&\Big[\frac{\beta_s e^{-2s}}{\beta} - \frac{1-e^{-2s}}{\beta}\Big] e^{-s}\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32} x e^{-\frac{1}{2\beta_s}x^2} \\
=& e^{-s}\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32} x e^{-\frac{1}{2\beta_s}x^2}.
\end{aligned}
$$

□

We then next compute $P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big]$.

Lemma 2.4. *Let $\beta\in\mathbb{R}$ satisfy* (2.2). *For*

$$
\beta_s := \frac{\beta+1-e^{-2s}}{e^{-2s}} \in \mathbb{R},
$$

we have $\frac{\beta}{\beta_s} > 0$ and

$$
P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x) = \big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac52} e^{-2s} x^2 e^{-\frac{1}{2\beta_s}x^2} + (1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32} e^{-\frac{1}{2\beta_s}x^2}.
$$

Proof. Note that if β satisfies (2.2), then we have that $x^2 e^{-\frac{1}{2\beta}x^2} \in L^1(\gamma)$ and hence $P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x)$ is well-defined. One may directly compute $P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x)$ but here we argue in a different way. First it is reasonable to expect that the form of $P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x)$ would be

$$P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x) = A_\beta(s)x^2 e^{-\frac{1}{2\beta_s}x^2} + B_\beta(s)e^{-\frac{1}{2\beta_s}x^2}. \tag{2.3}$$

Hence we will identify $A_\beta(s), B_\beta(s)$ and then check if it satisfies the equation. On the one hand, we have that

$$P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](0) = B_\beta(s)$$

while the definition of P_s shows that

$$\begin{aligned}
&P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](0)\\
=&\int \big|(1-e^{-2s})^{\frac12}y\big|^2 e^{-\frac{1}{2\beta}|(1-e^{-2s})^{\frac12}y|^2}\,d\gamma(y)\\
=&(1-e^{-2s})\int y^2 e^{-\left(\frac{1-e^{-2s}}{2\beta}+\frac12\right)y^2}\,\frac{dy}{(2\pi)^{\frac12}}\\
=&(1-e^{-2s})\int y^2 e^{-\frac{1}{2\frac{\beta}{\beta_s e^{-2s}}}y^2}\,\frac{dy}{(2\pi)^{\frac12}}\\
=&(1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac12}\int y^2\,d\gamma_{\frac{\beta}{\beta_s e^{-2s}}}(y)\\
=&(1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32}.
\end{aligned}$$

Hence we see that

$$B_\beta(s) = (1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32}.$$

So we obtain that

$$P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x) = A_\beta(s)x^2 e^{-\frac{1}{2\beta_s}x^2} + (1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32} e^{-\frac{1}{2\beta_s}x^2}. \tag{2.4}$$

Next let us identify $A_\beta(s)$. To this end, we appeal to the fact that P_s preserves the $L^1(\gamma)$-mass, namely

$$\int P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x)\,d\gamma(x) = \int x^2 e^{-\frac{1}{2\beta}x^2}\,d\gamma(x).$$

For the right-hand side,

$$\int x^2 e^{-\frac{1}{2\beta}x^2}\,d\gamma(x) = \big(\frac{\beta}{\beta+1}\big)^{\frac12}\int x^2\,d\gamma_{\frac{\beta}{\beta+1}}(x) = \big(\frac{\beta}{\beta+1}\big)^{\frac32}.$$

Combining this with (2.4), we have that

$$A_\beta(s)\int x^2 e^{-\frac{1}{2\beta_s}x^2}\,d\gamma(x) + (1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32}\int e^{-\frac{1}{2\beta_s}x^2}\,d\gamma(x) = \big(\frac{\beta}{\beta+1}\big)^{\frac32}.$$

For the left-hand side, we argue in a similar way as before to see that

$$
\begin{aligned}
&A_\beta(s)\int x^2 e^{-\frac{1}{2\beta_s}x^2}\,d\gamma(x) + (1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32}\int e^{-\frac{1}{2\beta_s}x^2}\,d\gamma(x)\\
=&A_\beta(s)\big(\frac{\beta_s}{\beta_s+1}\big)^{\frac32} + (1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32}\big(\frac{\beta_s}{\beta_s+1}\big)^{\frac12}.
\end{aligned}
$$

Therefore, we conclude

$$
\begin{aligned}
A_\beta(s) =&\Big[\big(\frac{\beta}{\beta+1}\big)^{\frac32} - (1-e^{-2s})\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac32}\big(\frac{\beta_s}{\beta_s+1}\big)^{\frac12}\Big]\big(\frac{\beta_s}{\beta_s+1}\big)^{-\frac32}\\
=&e^{-2s}\big(\frac{\beta}{\beta_s e^{-2s}}\big)^{\frac52}.
\end{aligned}
$$

What we did in the above argument is that we identify the necessary condition on $A_\beta(s), B_\beta(s)$ by assuming $P_s\big[x^2 e^{-\frac{1}{2\beta}x^2}\big](x)$ has a form of (2.3). However, we may directly check that it is indeed the solution of the equation. □

2.2. **Proof of Proposition 2.1.** We recall that the formula

$$
\begin{aligned}
&\frac{pp'}{q}\frac{\partial_t\widetilde{v}_t - \mathcal{L}^*_{\beta_{s,p}}\widetilde{v}_t}{\widetilde{v}_t^{1-\frac{q}{2}}}\gamma_1^{\frac{2}{q}}\\
=&p(\beta_{s,p}-\beta)P_s\big[\big(\frac{v_t}{\gamma_1}\big)^{\frac1p}\big]P_s\big[\big(\frac{v_t}{\gamma_1}\big)^{\frac1p}\Delta\log\frac{v_t}{\gamma_1}\big]\\
&+\beta_{s,p}\bigg(P_s\big[\big(\frac{v_t}{\gamma_1}\big)^{\frac1p}\big]P_s\big[\big(\frac{v_t}{\gamma_1}\big)^{\frac1p}\big|\nabla\log\frac{v_t}{\gamma_1}\big|^2\big] - \big|P_s\big[\big(\frac{v_t}{\gamma_1}\big)^{\frac1p}\nabla\log\frac{v_t}{\gamma_1}\big]\big|^2\bigg).
\end{aligned}
\tag{2.5}
$$

holds true as long as $\frac{q-1}{p-1} = e^{2s}$, see Lemma 3.2 in [7]. In general, the β-Fokker–Planck solution v_t is explicitly given by

$$
v_t(x) = \frac{1}{(2\pi\beta(1-e^{-2t}))^{\frac n2}}\int_{\mathbb{R}^n} e^{-\frac{|x-e^{-t}y|^2}{2\beta(1-e^{-2t})}}v_0(y)\,dy,
$$

and hence in our case $v_0=\gamma_a$, we know that

$$
v_t = \gamma_{\beta^t(a)},\quad \beta^t(a) := e^{-2t}a + (1-e^{-2t})\beta.
$$

Hence (2.5) implies that

$$
\begin{aligned}
&\frac{pp'}{q}\frac{\partial_t\widetilde{v}_t - \mathcal{L}^*_{\beta_{s,p}}\widetilde{v}_t}{\widetilde{v}_t^{1-\frac{q}{2}}}\gamma_1^{\frac{2}{q}}\\
=&p(\beta_{s,p}-\beta)\big(1-\frac{1}{\beta^t(a)}\big)\big|P_s\big[\big(\frac{\gamma_{\beta^t(a)}}{\gamma_1}\big)^{\frac1p}\big]\big|^2\\
&+\beta_{s,p}\big(1-\frac{1}{\beta^t(a)}\big)^2\bigg(P_s\big[x^2\big(\frac{\gamma_{\beta^t(a)}}{\gamma_1}\big)^{\frac1p}\big]P_s\big[\big(\frac{\gamma_{\beta^t(a)}}{\gamma_1}\big)^{\frac1p}x^2\big] - \big|P_s\big[\big(\frac{\gamma_{\beta^t(a)}}{\gamma_1}\big)^{\frac1p}x\big]\big|^2\bigg).
\end{aligned}
$$

Now we are in the position to apply Lemmas 2.2–2.4 to compute the right-hand side. After that, it is just a coefficient computation and one can obtain that the right-hand side is non-negative in the end.

3. Example

In this section, we give an example showing that the closure property (1.8) for $\beta > 1$ cannot generally hold true with any assumption like (1.6).

Example 3.1. Let $s > 0$, $\beta > 1$, and $1 < p < q < \infty$ satisfy $\frac{q-1}{p-1} = e^{2s}$. If $v_0 = \delta$, then $v_t = \gamma_{\beta(1-e^{-2t})}$ satisfies $\partial_t v = \mathcal{L}^*_\beta v$ on $(t,x) \in (0,\infty) \times \mathbb{R}$ but if one has

$$\partial_t \widetilde{v} \ge \mathcal{L}^*_{\beta_3} \widetilde{v}, \quad x \in \mathbb{R}$$

for some $t > 0$ and $\beta_3 > 0$, then

$$\beta_3 \le \beta_{s,p}, \quad t \le \frac{1}{2} \log \big((1 - \frac{1}{\beta})^{-1} \big).$$

Here as before $\big(\frac{\widetilde{v}}{\gamma_1}\big)^{\frac{1}{q}} := P_s\big[\big(\frac{v}{\gamma_1}\big)^{\frac{1}{p}}\big]$.

In particular, the closure property
(3.1)
$$\partial_t v = \mathcal{L}^*_\beta v, \quad (t,x) \in (0,\infty) \times \mathbb{R} \Rightarrow \exists \beta_3 > 0 : \partial_t \widetilde{v} \ge \mathcal{L}^*_{\beta_3} \widetilde{v}, \quad (t,x) \in (0,\infty) \times \mathbb{R},$$

cannot be true unless $\beta \le 1$.

We give a detail of the proof of Example 3.1. To this end, we corrects formulae that we showed in the above.

(1) Let $\beta > 0$ in general. If $a \in [-\beta, \infty)$, then $v_t := \gamma_{\beta + e^{-2t}a}$ is a Fokker–Planck solution: $\partial_t v_t = \mathcal{L}^*_\beta v_t$ on $(t,x) \in (0,\infty) \times \mathbb{R}$ with the initial data $v_0 = \gamma_{\beta + a}$. In particular, $v_t := \gamma_{\beta(1-e^{-2t})}$ is also Fokker–Planck solution which does not satisfy the log-convexity nor (1.7).
(2) Using Lemma 2.2, one can see that for $v_t := \gamma_{\beta(1-e^{-2t})}$,

(3.2)
$$\begin{aligned}\widetilde{v}_t(x) :=& \gamma_1(x) P_s\big[\big(\frac{v_t}{\gamma_1}\big)^{\frac{1}{p}}\big](x)^q \\ =& \frac{1}{(2\pi)^{\frac{1}{2}}} \big(\beta(1-e^{-2t})\big)^{\frac{q}{2p'}} \big(\beta_{s,p} - \frac{q}{p}\beta e^{-2s} e^{-2t}\big)^{-\frac{q}{2}} e^{-\frac{1}{2(\beta_{s,p} - \frac{q}{p}\beta e^{-2s} e^{-2t})} x^2}.\end{aligned}$$

Proof of Claim 3.1. In view of the formula (3.2), we write $\widetilde{v}_t = \frac{\beta^{\frac{q}{2p'}}}{(2\pi)^{\frac{1}{2}}} V_t$, where

$$V_t(x) := (1-e^{-2t})^a \big(\beta_{s,p} - c e^{-2t}\big)^{-d} e^{-\frac{1}{2(\beta_{s,p} - ce^{-2t})} x^2},$$

and

$$a := \frac{q}{2p'}, \quad c := \frac{q}{p}\beta e^{-2s}, \quad d := \frac{q}{2} > 0.$$

It suffices to investigate if V_t is a supersolution for some $\beta_3 > 0$ or not.

From the direct computations, we have that

$$\partial_x V_t(x) = -x(1-e^{-2t})^a\big(\beta_{s,p} - ce^{-2t}\big)^{-d-1} e^{-\frac{1}{2(\beta_{s,p}-ce^{-2t})}x^2},$$
$$\partial_{xx} V_t(x) = \big(\frac{x^2}{\beta_{s,p} - ce^{-2t}} - 1\big)(1-e^{-2t})^a\big(\beta_{s,p} - ce^{-2t}\big)^{-d-1} e^{-\frac{1}{2(\beta_{s,p}-ce^{-2t})}x^2}$$

from which we have that

$$\begin{aligned}\mathcal{L}^*_{\beta_3} V_t(x) =& \Big(\beta_3\big(\frac{x^2}{\beta_{s,p} - ce^{-2t}} - 1\big) - x^2 + \beta_{s,p} - ce^{-2t}\Big)\\ &\times (1-e^{-2t})^a\big(\beta_{s,p} - ce^{-2t}\big)^{-d-1} e^{-\frac{1}{2(\beta_{s,p}-ce^{-2t})}x^2}.\end{aligned}$$

Also,

$$\begin{aligned}\partial_t V_t(x) =& \Big(\frac{cx^2}{\beta_{s,p} - ce^{-2t}} + 2a\frac{\beta_{s,p} - ce^{-2t}}{1-e^{-2t}} - 2cd\Big)\\ &\times (1-e^{-2t})^a\big(\beta_{s,p} - ce^{-2t}\big)^{-d-1} e^{-\frac{1}{2(\beta_{s,p}-ce^{-2t})}x^2} e^{-2t}.\end{aligned}$$

Hence, we obtain that

$$\partial_t V - \mathcal{L}_{\beta_3} V = \big(A_t x^2 + B_t\big)(1-e^{-2t})^a\big(\beta_{s,p} - ce^{-2t}\big)^{-d-1} e^{-\frac{1}{2(\beta_{s,p}-ce^{-2t})}x^2},$$

where

$$\begin{aligned}A_t &:= -\frac{ce^{-2t} - \beta_3}{ce^{-2t} - \beta_{s,p}} + 1,\\ B_t &:= \big(2a\frac{e^{-2t}}{1-e^{-2t}} - 1\big)(\beta_{s,p} - ce^{-2t}) + \beta_3 - 2e^{-2t}cd.\end{aligned}$$

The sign of $\partial_t V - \mathcal{L}_{\beta_3} V$ is determined by the one of $A_t x^2 + B_t$ since other parts are all positive.

First, notice that $A_t \geq 0$ if and only if $\beta_3 \leq \beta_{s,p}$. So, considering the case $x \to \infty$, we see that $\beta_3 \leq \beta_{s,p}$ is necessary for $\partial_t V - \mathcal{L}_{\beta_3} V \geq 0$. Next, consider B_t which can be further simplified to

$$B_t = \frac{q}{pp'}(1-e^{-2s})e^{-2t}\big(\frac{e^{-2t}}{1-e^{-2t}} - (\beta - 1)\big) - \beta_{s,p} + \beta_3.$$

Since $-\beta_{s,p} + \beta_3 \leq 0$, the first term need to be nonnegative to ensure $B_t \geq 0$, namely

$$\frac{e^{-2t}}{1-e^{-2t}} - (\beta - 1) \geq 0$$

is necessary. It is immediate to see that the last expression is equivalent to

$$t \leq \frac{1}{2}\log\big((1-\frac{1}{\beta})^{-1}\big).$$

By considering $x = 0$, we know that $B_t \geq 0$ is necessary to ensure $\partial_t V - \mathcal{L}_{\beta_3} V \geq 0$. □

References

[1] Y. Aoki, J. Bennett, N. Bez, S. Machihara, K. Matsuura, S. Shiraki, *A supersolutions perspective on hypercontractivity*, Annali di Matematica, **199** (2020), 2105–2116.

[2] D. Bakry, I. Gentil, M. Ledoux, *Analysis and geometry of Markov diffusion operators.* Grundlehren der mathematischen Wissenschaften 348, Springer (2014).

[3] J. Bennett, N. Bez, *Closure properties of solutions to heat inequalities*, J. Geom. Anal. **19** (2009), 584–600.

[4] J. Bennett, N. Bez, *Generating monotone quantities for the heat equation*, J. reine angew. Math. **756** (2019), 37–63.

[5] J. Bennett, A. Carbery, M. Christ, T. Tao, *The Brascamp–Lieb inequalities: finiteness, structure and extremals*, Geom. Funct. Anal. **17** (2008), 1343–1415.

[6] N. Bez, S. Nakamura, *Regularised inverse Brascamp–Lieb inequalities*, arXiv:2110.02841.

[7] N. Bez, S. Nakamura, H. Tsuji, *Regularisation of gaussian hypercontractivity and logarithmic Sobolev inequalities via Fokker–Planck flow*, arXiv:2201.12478v2.

[8] E. Carlen, *Supper additivity of Fisher's information and Logarithmic Sobolev inequalities*, J. Funct. Anal. **101** (1991), 194–211.

[9] E. Carlen, E. Lieb, M. Loss, *A sharp analog of Young's inequality on* $\mathbb{S}^N$ *and related entropy inequalities*, J. Geom. Anal. **14** (2004), 487–520.

[10] W. Faris, ed. *Diffusion, Quantum Theory, and Radically Elementary Mathematics.* (MN-47). PRINCETON; OXFORD: Princeton University Press, 2006. Accessed July 6, 2021. http://www.jstor.org/stable/j.ctt7ztfkx.

[11] R. Eldan, J. Lehec, Y. Shenfeld, *Stability of the logarithmic Sobolev inequality via the Föllmer process*, Ann. Inst. Henri Poincaré Probab. Stat. **56** (2020), 2253–2269.

[12] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. **97** (1975), 1061–1083.

[13] M. Ledoux, *On an integral criterion for hypercontractivity of diffusion semigroups and extremal functions*, J. Funct. Anal. *105* (1992), 445–467.

[14] M. Ledoux, *Heat flows, geometric and functional inequalities*, Proceedings of the International Congress of Mathematicians, *4* (2014), 117–135.

[15] E.H. Lieb, *Gaussian kernels have only Gaussian maximizers*, Inventiones Mathematicae, **102** (1990), 179–208.

[16] E. Nelson, *The free Markov field*, J. Funct. Anal. **12** (1973), 211–227.

(Shohei Nakamura) Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

Email address: srmkn@math.sci.osaka-u.ac.jp