The proximal point algorithm of a resolvent for equilibrium problems in geodesic spaces with negative curvature
曲率上限が負の測地距離空間における均衡問題のリゾルベントと近接点法

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Abstract

In this paper, we consider properties of a resolvent of equilibrium problems. We prove a Δ-convergence theorem with the proximal point algorithm using a resolvent of equilibrium problems in a CAT(-1) space having the convex hull finite property.

1 Introduction

Let K a nonempty set and $f\colon K \times K \to \mathbb{R}$. An equilibrium problem is defined as to find $z_0 \in K$ such that $f(z_0, y) \geq 0$ for $y \in K$. Equilibrium problems were first studied by Blum and Oettli [1]. Equilibrium problems include optimization problems, saddle point problems and fixed point problems, etc. In 2005, Combettes and Hirstoaga introduced the resolvent of equilibrium problems in Hilbert spaces [3].

Theorem 1.1 (Combettes and Hirstoaga [3]). *Let H be a Hilbert space and K a nonempty closed convex subset of H. Let $f\colon K \times K \to \mathbb{R}$ and S_f the set of solutions to the equilibrium problem for f. Suppose the following conditions:*

- *$f(y, y) = 0$ for all $y \in K$;*
- *$f(y, z) + f(z, y) \leq 0$ for all $y, z \in K$;*
- *$f(y, \cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex for every $y \in H$;*

- *$f(\cdot, z)\colon K \to \mathbb{R}$ is upper hemicontinuous for every $z \in K$.*

Then the resolvent operator J_f defined by

$$J_f x = \left\{ z \in K \;\middle|\; \inf_{y \in K} (f(z, y) + \langle z - x, y - z \rangle) \geq 0 \right\}$$

has the following properties:

(i) *$D(J_f) = X$;*
(ii) *J_f is single-valued and firmly nonexpansive;*
(iii) *$F(J_f) = S_f$;*
(iv) *S_f is closed and convex.*

In 2018, Kimura and Kishi [7] introduced a resolvent of equilibrium problems in a complete CAT(0) space having the convex hull finite property. In 2021, Kimura [6] introduced a resolvent of equilibrium problems in an admissible complete CAT(1) space having the convex hull finite property.

In this paper, we propose fundamental properties of a resolvent of equilibrium problems and prove a Δ-convergence theorem with the proximal point algorithm in a complete CAT(-1) space having the convex hull finite property.

2 Preliminaries

Let (X, d) a metric space, and T a mapping of X into itself. The set of all fixed points of T is denoted by $\mathcal{F}(T)$. Let $\{x_n\}$ be a bounded sequence of X. The set of $\mathrm{AC}(\{x_n\})$ of all *asymptotic centers of* $\{x_n\} \subset X$ is defined by

$$\mathrm{AC}(\{x_n\}) = \left\{ x_0 \in X \;\middle|\; \limsup_{n \to \infty} d(x_n, x_0) = \inf_{x \in X} \limsup_{n \to \infty} d(x_n, x) \right\}.$$

A sequence $\{x_n\} \subset X$ is said to be *Δ-convergent to* $x_0 \in X$ if $\mathrm{AC}(\{x_{n_i}\}) = \{x_0\}$ for all subsequence $\{x_{n_i}\}$ of $\{x_n\}$. It is denoted by $x_n \xrightarrow{\Delta} x_0$. Let f be a function of X into $\mathbb{R}$. Then, $\mathrm{Argmin}_{x \in X} f(x)$ is the set of all minimizers of f. Let T be a mapping of X into itself. Then, a mapping T is *hyperbolically nonspreading* if for $x, y \in X$, the inequality

$$2 \cosh d(Tx, Ty) \leq \cosh d(Tx, y) + \cosh d(x, Ty)$$

holds. A mapping T is *quasinonexpansive* if $\mathcal{F}(T)$ is nonempty and the inequality $d(Tx, z) \leq d(x, z)$ holds for $x \in X$ and $z \in \mathcal{F}(T)$. We know that if T is hyperbolically nonspreading and $\mathcal{F}(T)$ is nonempty, T is quasinonexpansive. In fact, for $x \in X$ and $z \in \mathcal{F}(T)$, by hyperbolical nonspreadingness of T, we get

$$2 \cosh d(Tx, z) \leq \cosh d(Tx, z) + \cosh d(x, z)$$

and hence T is quasinonexpansive.

Let $x, y \in X$ and γ_{xy} a mapping of $[0, d(x, y)]$ into X. A mapping γ_{xy} is called *a geodesic with endpoints* x *and* y if $\gamma_{xy}(0) = x$, $\gamma_{xy}(d(x, y)) = y$ and $d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$ for all $s, t \in [0, d(x, y)]$. X is called *a geodesic space* if for all $x, y \in X$, there exists geodesic with endpoints x and y. In what follows, we assume that for $x, y \in X$, X has a unique geodesic with endpoints x and y. The image of geodesic with endpoints x and y is denoted by $\operatorname{Im} \gamma_{xy}$. For $x, y \in X$ and $t \in [0, 1]$, there exists $z \in \operatorname{Im} \gamma_{xy}$ such that $d(x, z) = (1-t)d(x, y)$ and $d(y, z) = td(x, y)$, which is denoted by $z = tx \oplus (1-t)y$.

Let X be a geodesic space, and $\mathbb{H}^2$ the 2-dimensional hyperbolic space. *A geodesic triangle* $\triangle(x, y, z)$ *with vertices* $x, y, z \in X$ is defined by $\operatorname{Im} \gamma_{xy} \cup \operatorname{Im} \gamma_{yz} \cup \operatorname{Im} \gamma_{zx}$. Further, *a comparison triangle* $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$ *to* $\triangle(x, y, z)$ *with vertices* $\bar{x}, \bar{y}, \bar{z} \in \mathbb{H}^2$ is defined by $\operatorname{Im} \gamma_{\bar{x}\bar{y}} \cup \operatorname{Im}_{\bar{y}\bar{z}} \cup \operatorname{Im} \gamma_{\bar{z}\bar{x}}$ with $d(x, y) = d_{\mathbb{H}^2}(\bar{x}, \bar{y})$, $d(y, z) = d_{\mathbb{H}^2}(\bar{y}, \bar{z})$ and $d(z, x) = d_{\mathbb{H}^2}(\bar{z}, \bar{x})$, where $d_{\mathbb{H}^2}(\cdot, \cdot)$ is the hyperbolic metric on $\mathbb{H}^2$. A point $\bar{p} \in \operatorname{Im} \gamma_{\bar{x}\bar{y}}$ is called *a comparison point for* $p \in \operatorname{Im} \gamma_{xy}$ if $d(x, p) = d_{\mathbb{H}^2}(\bar{x}, \bar{p})$. X is a CAT(-1) space if for $p, q \in \triangle(x, y, z) \subset X$ and their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z}) \subset \mathbb{H}^2$, the inequality $d(p, q) \leq d_{\mathbb{H}^2}(p, q)$ holds for all geodesic triangles in X. In general, a CAT(-1) space is a CAT(0) space [2]. In CAT(-1) spaces, the inequality

$$\begin{aligned}
&\cosh d(tx \oplus (1-t)y, z) \sinh d(x, y) \\
&\leq \cosh d(x, z) \sinh td(x, y) + \cosh d(y, z) \sinh(1-t)d(x, y)
\end{aligned}$$

always holds for $x, y, z \in X$ and $t \in [0, 1]$.

The following lemmas are important properties of a CAT(0) space.

Lemma 2.1 (Kirk and Panyanak [9]). *Let* X *be a complete* CAT(0) *space. Then every bounded sequence has a subsequence which is* Δ*-convergent to* $x_0 \in X$.

Lemma 2.2 (Dhompongsa, Kirk and Sims [4]). *Let* X *be a complete* CAT(0) *space and* $\{x_n\}$ *a bounded sequence of* X. *Then the asymptotic center of* $\{x_n\}$ *consists of one point.*

Let X be a geodesic space and f a function of X into $\mathbb{R}$. A function f is said to be *lower semicontinuous* if the inequality

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

holds, wherever $\{x_n\} \subset X$ converges to $x \in X$. If f is continuous, then it is lower semicontinuous. A function f is said to be *convex* if

$$f(\alpha x \oplus (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

holds for all $x, y \in X$ and $\alpha \in]0, 1[$. A function f is said to be *upper hemicontinuous* if the inequality

$$f(x) \geq \limsup_{t \to 0^+} f((1-t)x \oplus ty)$$

holds for all $x, y \in X$.

We will consider the following conditions for a function used by an equilibrium problems.

Condition 2.1. Let X be a geodesic space and K a nonempty closed convex subset of X. We suppose that a bifunction $f\colon K \times K \to \mathbb{R}$ satisfies the following conditions:

- $f(x,x) = 0$ for all $x \in K$;
- $f(x,y) + f(y,x) \le 0$ for all $x, y \in K$;
- for every $x \in K$, $f(x,\cdot)\colon K \to \mathbb{R}$ is lower semicontinuous and convex;
- for every $y \in K$, $f(\cdot,y)\colon K \to \mathbb{R}$ is upper hemicontinuous.

The following theorem is important to show a Δ-convergence theorem with the proximal point algorithm.

Theorem 2.1 (Kajimura and Kimura [5])**.** *Let X be a complete* CAT(−1) *space, $\{z_n\}$ a bounded sequence in X, $\{\beta_n\}$ a sequence of positive real numbers with $\sum_{n=1}^{\infty} \beta_n = \infty$ and*

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \beta_k \cosh d(y, z_k)$$

for $y \in X$. Then, $\operatorname{Argmin}_X g$ *consists of one point.*

The set of solutions to the equilibrium problem for f is denoted by $\operatorname{Equil} f$, that is,

$$\operatorname{Equil} f = \left\{ z \in K \;\middle|\; \inf_{y\in K} f(z,y) \ge 0 \right\}.$$

Let X be a CAT(−1) space and E a nonempty subset of X. Then *a convex hull of* E is defined by

$$\operatorname{co} E = \bigcup_{n=0}^{\infty} X_n,$$

where $X_0 = E$ and $X_n = \{tu_{n-1} \oplus (1-t)v_{n-1} \mid u_{n-1}, v_{n-1} \in X_{n-1}, t \in [0,1]\}$. X has *the convex hull finite property* if every continuous mapping T of $\operatorname{clco} E$ into itself has a fixed point for all finite subsets E of X, where $\operatorname{clco} E$ is the closure of $\operatorname{co} E$; see [11].

In the following theorem shows the properties of a resolvent of equilibrium problems in a CAT(−1) space having the convex hull finite property.

Theorem 2.2 (Kimura and Ogihara [8])**.** *Let X be a complete* CAT(−1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K\times K \to \mathbb{R}$ satisfies* Condition 2.1*. Define a set-valued mapping $L_f\colon X \to 2^K$ by*

$$L_f x = \left\{ z \in K \;\middle|\; \inf_{y\in K} (f(z,y) + \cosh d(x,y) - \cosh d(x,z)) \ge 0 \right\}$$

for all $x \in X$. Put $C_z = \cosh d(z, L_f z)$ for $z \in X$. Then the following hold:

(i) *$D(L_f) = X$;*

(ii) L_f *is single-valued and the inequality*

$$(C_x + C_y)\cosh d(L_f x, L_f y) \le \cosh d(L_f x, y) + \cosh d(x, L_f y)$$

holds for $x, y \in X$*, and thus* L_f *is hyperbolically nonspreading;*

(iii) $\operatorname{Equil} f = \mathcal{F}(L_f)$*, and thus it is closed and convex.*

3 Fundamental properties of resolvents

In this section, we prove the lemmas which is necessary to prove a Δ-convergence theorem in a CAT(-1) space having the convex hull finite property.

Lemma 3.1. *Let* X *be a complete* CAT(-1) *space having the convex hull finite property and* K *a nonempty closed convex subset of* X*. Suppose that* $f\colon K \times K \to \mathbb{R}$ *satisfies* Condition 2.1*. Let* $L_{\lambda f}$ *a resolvent of* λf *for* $\lambda > 0$*. Then the inequality*

$$0 \le f(L_{\lambda f}x, w) + \frac{d(L_{\lambda f}x, w)}{\lambda \sinh d(L_{\lambda f}x, w)}(\cosh d(x, w) - \cosh d(x, L_{\lambda f}x)\cosh d(w, L_{\lambda f}x))$$

holds for $x \in X$ *and* $w \in K$ *with* $w \ne L_{\lambda f}x$.

Proof. Let $x \in X$ and $w \in K$ with $w \ne L_{\lambda f}x$. Put $\tau_t = tw \oplus (1-t)L_{\lambda f}x \in K$ for $t \in \,]0, 1[$. Then, we get

$$\begin{aligned} 0 &\le \lambda f(L_{\lambda f}x, \tau_t) + \cosh d(x, \tau_t) - \cosh d(x, L_{\lambda f}x) \\ &\le \lambda t f(L_{\lambda f}x, w) + \cosh d(x, \tau_t) - \cosh d(x, L_{\lambda f}x) \\ &\le \lambda t f(L_{\lambda f}x, w) + \frac{L(t) - \cosh d(x, L_{\lambda f}x)\sinh d(L_{\lambda f}x, w)}{\sinh d(L_{\lambda f}x, w)} \end{aligned}$$

where

$$L(t) = \cosh d(x, w)\sinh t d(L_{\lambda f}x, w) + \cosh d(x, L_{\lambda f}x)\sinh(1-t)d(L_{\lambda f}x, w).$$

Dividing by λt and letting $t \searrow 0$, we obtain

$$\begin{aligned} 0 &\le f(L_{\lambda f}x, w) + \frac{1}{\lambda \sinh d(L_{\lambda f}x, w)} \lim_{t \searrow 0} \frac{L(t) - \cosh d(x, L_{\lambda f}x)\sinh d(L_{\lambda f}x, w)}{t} \\ &= f(L_{\lambda f}x, w) + \frac{1}{\lambda \sinh d(L_{\lambda f}x, w)} \lim_{t \searrow 0} \frac{d}{dt}(L(t) - \cosh d(x, L_{\lambda f}x)\sinh d(L_{\lambda f}x, w)) \\ &= f(L_{\lambda f}x, w) + \frac{d(L_{\lambda f}x, w)}{\lambda \sinh d(L_{\lambda f}x, w)}(\cosh d(x, w) - \cosh d(x, L_{\lambda f}x)\cosh d(L_{\lambda f}x, w)) \end{aligned}$$

and hence we get the desired result. $\square$

Corollary 3.1. *Let X be a complete* CAT(-1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 2.1*. Let $L_{\lambda f}$ a resolvent of λf for $\lambda > 0$. Then the following inequalities hold:*

$$(\mu C_{\lambda,x} + \lambda C_{\mu,y}) \cosh d(L_{\lambda f}x, L_{\mu f}y) \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y)$$

and

$$(\lambda + \mu) \cosh d(L_{\lambda f}x, L_{\mu f}y) \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y)$$

for all $x, y \in X$ and $\lambda, \mu > 0$, where $C_{\eta,z} = \cosh d(z, L_{\eta f}z)$ for $z \in X$ and $\eta > 0$.

***Proof*.** Let $x, y \in X$ and $\lambda, \mu > 0$ with $D = d(L_{\lambda f}x, L_{\mu f}y) > 0$ and put $C_{\eta,z} = \cosh d(L_{\eta f}z, z)$ for $z \in X$ and $\eta > 0$. By Lemma 3.1, we get

$$0 \leq f(L_{\lambda f}x, L_{\mu f}y) + \frac{D}{\lambda \sinh D}(\cosh d(x, L_{\mu f}y) - C_{\lambda,x} \cosh D).$$

Similarly, it holds that

$$0 \leq f(L_{\mu f}y, L_{\lambda f}x) + \frac{D}{\mu \sinh D}(\cosh d(L_{\lambda f}x, y) - C_{\mu,y} \cosh D).$$

From Condition 2.1, adding these inequalities, we get

$$\begin{aligned} 0 &\leq f(L_{\lambda f}x, L_{\mu f}y) + f(L_{\mu f}y, L_{\lambda f}x) + \frac{D}{\lambda \sinh D}(\cosh d(x, L_{\mu f}y) - C_{\lambda,x} \cosh D) \\ &\quad + \frac{D}{\mu \sinh D}(\cosh d(L_{\lambda f}x, y) - C_{\mu,y} \cosh D) \\ &\leq \frac{D}{\sinh D} \left(\frac{\cosh d(x, L_{\mu f}y) - C_{\lambda,x} \cosh D}{\lambda} + \frac{\cosh d(x, L_{\mu f}y) - C_{\lambda,x} \cosh D}{\mu} \right) \end{aligned}$$

Since $t/(\sinh t) > 0$ for $t > 0$, we get

$$(\mu C_{\lambda,x} + \lambda C_{\mu,y}) \cosh D \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y).$$

Since $\cosh t \geq 1$ for $t \geq 0$, we get

$$(\lambda + \mu) \cosh D \leq \mu \cosh d(x, L_{\mu f}y) + \lambda \cosh d(L_{\lambda f}x, y).$$

If $D = 0$, the inequalities obviously hold. It completes the proof. □

Corollary 3.2. *Let X be a complete* CAT(-1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 2.1 *and that* Equil f *is nonempty. Let $L_{\lambda f}$ a resolvent of λf for $\lambda > 0$. Then the following inequality holds:*

$$\cosh d(x, L_{\lambda f}x) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z)$$

for all $x \in X$ and $z \in$ Equil f.

Proof. Let $x \in X$, $z \in \operatorname{Equil} f$ and $\lambda > 0$. By Corollary 3.1, we get

$$(\cosh d(x, L_{\lambda f}x) + \lambda \cosh d(z, L_f z)) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z) + \lambda \cosh d(L_{\lambda f}x, z)$$

and hence

$$(\cosh d(x, L_{\lambda f}x) + \lambda) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z) + \lambda \cosh d(L_{\lambda f}x, z).$$

Therefore, we have

$$\cosh d(x, L_{\lambda f}x) \cosh d(L_{\lambda f}x, z) \leq \cosh d(x, z)$$

and get the desired result. □

Lemma 3.2. *Let X be a complete* CAT(-1) *space having the convex hull finite property and K a nonempty closed convex subset of X. Suppose that $f\colon K \times K \to \mathbb{R}$ satisfies* Condition 2.1. *Let $\{\lambda_n\} \subset \,]0, \infty[$ such that $\limsup_{n\to\infty} \lambda_n > 0$, $L_{\lambda_n f}$ a resolvent of $\lambda_n f$, and $\{x_n\}$ a bounded sequence of X such that $x_n \overset{\Delta}{\rightharpoonup} x_0 \in X$ and $\lim_{n\to\infty} d(x_n, L_{\lambda_n f}x_n) = 0$. Then $x_0 \in \operatorname{Equil} f$.*

Proof. Put $\lambda_0 = \limsup_{n\to\infty} \lambda_n$. By Corollary 3.1, we get

$$\cosh d(L_{\lambda_n f}x_n, L_f x_0) \leq \frac{\lambda_n}{1+\lambda_n} \cosh d(L_{\lambda_n f}x_n, x_0) + \frac{1}{1+\lambda_n} \cosh d(x_n, L_f x_0).$$

Take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ arbitrarily. For $y \in K$, then, we get

$$d(L_{\lambda_{n_j} f}x_n, y) \leq d(L_{\lambda_{n_j} f}x_{n_j}, x_{n_j}) + d(x_{n_j}, y) \leq 2d(L_{\lambda_{n_j} f}x_{n_j}, x_{n_j}) + d(L_{\lambda_{n_j} f}x_{n_j}, y).$$

Since $d(x_{n_j}, L_{\lambda_{n_j} f}x_{n_j}) \to 0$, letting $j \to \infty$, we get

$$\limsup_{j\to\infty} d(L_{\lambda_{n_j} f}x_{n_j}, y) = \limsup_{j\to\infty} d(x_{n_j}, y).$$

Suppose $\lambda_0 = \infty$. Then we take a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lim_{i\to\infty} \lambda_{n_i} = \infty$. It implies that

$$\begin{aligned}
\limsup_{i\to\infty} (\cosh d(x_{n_i}, L_f x_0)) &= \limsup_{i\to\infty} \left(\cosh d(L_{\lambda_{n_i} f}x_{n_i}, L_f x_0)\right) \\
&\leq \limsup_{i\to\infty} \left(\cosh d(L_{\lambda_{n_i} f}x_{n_i}, x_0)\right) \\
&= \limsup_{i\to\infty} (\cosh d(x_{n_i}, x_0)).
\end{aligned}$$

Since x_0 is an asymptotic center of $\{x_{n_i}\}$, we get $x_0 = L_f x_0$ and hence $x_0 \in \operatorname{Equil} f$. We next suppose $\lambda_0 < \infty$. Then we get

$$
\begin{aligned}
&\limsup_{i\to\infty} (\cosh d(x_{n_i}, L_f x_0)) \\
&= \limsup_{i\to\infty} \big(\cosh d(L_{\lambda_{n_i} f} x_{n_i}, L_f x_0)\big) \\
&\leq \frac{\lambda_0}{1+\lambda_0} \limsup_{i\to\infty} \big(\cosh d(L_{\lambda_{n_i} f} x_{n_i}, x_0)\big) + \frac{1}{1+\lambda_0} \limsup_{i\to\infty} (\cosh d(x_{n_i}, L_f x_0)) \\
&= \frac{\lambda_0}{1+\lambda_0} \limsup_{i\to\infty} (\cosh d(x_{n_i}, x_0)) + \frac{1}{1+\lambda_0} \limsup_{i\to\infty} (\cosh d(x_{n_i}, L_f x_0))
\end{aligned}
$$

and hence

$$
\limsup_{i\to\infty} (\cosh d(x_{n_i}, L_f x_0)) \leq \limsup_{i\to\infty} (\cosh d(x_{n_i}, x_0)).
$$

Since x_0 is an asymptotic center of $\{x_{n_i}\}$, we get $x_0 = L_f x_0$ and hence $x_0 \in \operatorname{Equil} f$. Consequently, we complete the proof. □

4 A Δ-convergence theorem

In this section, we prove a Δ-convergence theorem with the proximal point algorithm in a CAT(−1) space having the convex hull finite property.

Theorem 4.1. *Let X be a complete* CAT(−1) *space having the convex hull finite property, K a nonempty closed convex subset of X, $f\colon K \times K \to \mathbb{R}$ satisfying* Condition 2.1 *and $\{\lambda_n\} \subset \,]0, \infty[$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. For given $x_1 \in X$, define $\{x_n\}$ by*

$$
x_{n+1} = L_{\lambda_n f} x_n = \left\{ z \in K \,\middle|\, \inf_{y \in K} (f(z, y) + \cosh d(x_n, y) - \cosh d(x_n, z)) \geq 0 \right\}
$$

for all $n \in \mathbb{N}$. Then, the following hold:

(i) Equil f *is nonempty if and only if $\{x_n\}$ is bounded;*
(ii) *if* Equil f *is nonempty and* $\liminf_{n\to\infty} \lambda_n > 0$, *$\{x_n\}$ is Δ-convergent to an element of* Equil f.

Proof. (i) We first suppose that Equil f is nonempty and show that $\{x_n\}$ is bounded. Let $u \in \operatorname{Equil} f$. Since $L_{\lambda_n f}$ is quasinonexpansive, we get

$$
d(x_{n+1}, u) = d(L_{\lambda_n f} x_n, u) \leq d(x_n, u)
$$

and hence $\{d(x_n, u)\}$ is nonincreasing and $\{x_n\}$ is bounded for $n \in \mathbb{N}$. We next suppose $\{x_n\}$ is bounded and show that Equil f is nonempty. For $k \in \mathbb{N}$ with $k \leq n$, by Corollary 3.1 we get

$$
(1 + \lambda_k) \cosh d(L_{\lambda_k f} x_k, L_f y) \leq \lambda_k \cosh d(L_{\lambda_k f} x_k, y) + \cosh d(x_k, L_f y)
$$

and hence

$$\lambda_k \cosh d(x_{k+1}, L_f y) \leq \lambda_k \cosh d(x_{k+1}, y) + (\cosh d(x_k, L_f y) - \cosh d(x_{k+1}, L_f y))$$

for all $y \in X$. Adding both sides of the inequality above from $k = 1$ to $k = n$ and dividing both sides by $\sum_{l=1}^{n} \lambda_l$, we get

$$\begin{aligned}
&\frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, L_f y) \\
&\leq \frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, y) + \frac{1}{\sum_{l=1}^{n} \lambda_l} (\cosh d(x_1, L_f y) - \cosh d(x_{n+1}, L_f y)) \\
&\leq \frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, p) + \frac{1}{\sum_{l=1}^{n} \lambda_l} \cosh d(x_1, p).
\end{aligned}$$

By Theorem 2.1, we know that $\operatorname{Argmin}_X g$ consists of one point, where

$$g(z) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \lambda_l} \sum_{k=1}^{n} \lambda_k \cosh d(x_{k+1}, z)$$

for all $z \in X$. Let $p \in \operatorname{Argmin}_X g$. Since $\sum_{l=1}^{\infty} \lambda_l = \infty$, letting $n \to \infty$, we get

$$g(L_f p) \leq g(p) \leq g(L_f p)$$

and hence $p = L_f p$. This implies that $p \in \operatorname{Equil} f$.

(ii) Suppose $\operatorname{Equil} f$ is nonempty and $\liminf_{n\to\infty} \lambda_n > 0$. Let $p \in \operatorname{Equil} f$. Since $L_{\lambda_n f}$ and $\operatorname{Equil} f$ is nonempty, $L_{\lambda_n f}$ is quasinonexpansive. Then, we get

$$0 \leq d(x_{n+1}, p) = d(L_{\lambda_n f} x_n, p) \leq d(x_n, p)$$

and hence $\{d(x_n, p)\}$ is nonincreasing. Then, there exists $\lim_{n\to\infty} d(x_n, p)$. By Corolally 3.1, we have

$$1 \leq \cosh d(x_n, L_{\lambda_n f} x_n) \leq \frac{\cosh d(x_n, p)}{\cosh d(x_{n+1}, p)}.$$

Letting $n \to \infty$, we obtain $d(x_n, L_{\lambda_n f} x_n) \to 0$. Put $\operatorname{AC}(\{x_n\}) = \{x_0\}$ Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Since $\{x_{n_i}\}$ is bounded and by Lemma 2.2, $\operatorname{AC}(\{x_{n_i}\}) = \{y_0\}$. Further, there exists a subsequece $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \xrightarrow{\Delta} z_0 \in X$. Then, $\operatorname{AC}(\{x_{n_{i_j}}\}) = \{z_0\}$. By Lemma 3.2, we get $z_0 \in \operatorname{Equil} f$. Then, we get

$$\limsup_{n\to\infty} d(x_n, z_0) = \limsup_{j\to\infty} d(x_{n_{i_j}}, z_0)$$

$$
\begin{aligned}
&\leq \limsup_{j\to\infty} d(x_{n_{i_j}}, y_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, y_0) \\
&\leq \limsup_{i\to\infty} d(x_{n_i}, x_0) \\
&\leq \limsup_{n\to\infty} d(x_n, x_0) \leq \limsup_{n\to\infty} d(x_n, z_0)
\end{aligned}
$$

and hence $y_0 = x_0 = z_0 \in \operatorname{Equil} f$. Therefore $x_n \overset{\Delta}{\rightharpoonup} x_0 \in \operatorname{Equil} f$. Consequently, we get the desired result. □

Corollary 4.1. *Let X be a complete* CAT(-1) *space having the convex hull finite property, K a nonempty closed convex subset of X, $f\colon K \times K \to \mathbb{R}$ satisfying* Condition 2.1 *and L_f a resolvent of f. Then the following hold:*

(i) Equil f *is nonempty if and only if $\{L_f^n x\}$ is bounded for each $x \in X$;*
(ii) *If* Equil f *is nonempty, $\{L_f^n x\}$ is Δ-convergent to an element of* Equil f *for each $x \in X$.*

Proof. Let $\{\lambda_n\} \subset]0, \infty[$ such that $\lambda_n = 1$ for $n \in \mathbb{N}$. Using Theorem 4.1, we get (i) and (ii). Consequently, we complete the proof. □

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